Stability of Kutta-Merson Method for Solving the Linear Delay Differential Equation

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Abstract

We discuss the practical determination of stability regions when various fixed- step size Kutta-Merson method is applied to the following linear delay differential equation (DDE)

$u'(t) = au(t) + bu(t-1) \qquad t \geq 0 \;..........(1)$

$u(t) = \varphi(t) \qquad t < 0$

Introduction

The basic formula investigated here involve a strategy of adapting on Runge-Kutta (RK) methods (ussing sequence variable step sizes) for the ordinary differential equation:-

$$u'(t) = f(t, u(t))$$
$$u(t_0) = u_0$$

with prescribed initial value. We assume that it is familiar with RK methods (using a sequence of step size h_n) for (2), the concept of the their order and internal stage-orders, and their representation Butcher table of the form [1] :

$$\begin{array}{c|c} C & A \\ \hline & b^T \end{array} \qquad \begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1v} \\ \vdots & \vdots & & \vdots \\ c_v & a_{v1} & \cdots & a_{vv} \\ \hline & b_1 & \cdots & b_v \end{array}$$

Where $\sum_{i=1}^{v} a_{ij} = c_i \qquad i = 1,........v$

We shall adapted the RK method for (2) to solve DDE (1). For this purpose of the present paper the step size are fixed and we restrict our attention to a linear test equation with fixed delay.

The general numerical strategy for (1) reduces to invoking the numerical solution by an RK process of a retarded RK method for solving a DDE involves

i) Choice of RK method.

ii) Choice of interpolation method.

Stability properties[1],[4]

Before we plot the region of stability of Kutta-Merson method, we can assume that mh=1, $m \in I^+$ (positive integer) ....(3)

Definition (1)

Let $b = re^{i\phi}$ and a=0 in (1). A numerical method is said to be Q-Stable if under the condition

i) Re(b)<0

ii) $0 < r < \min\left(\frac{3\pi}{2} - \phi, \phi - \frac{\pi}{2}\right)$

the numerical solution $y_n \to 0$ as $t_n \to \infty$ for all h satisfies (3) ....(2)

Definition (2)

A numerical method applied to (1), is said to be p-stable if under the condition

$$Re(a) < -|b|$$

The numerical solution $y_n \to 0$ as $t_n \to \infty$ for all h satisfies (3)

Definition (3)

1. If a an b are real in (1), then:
   The region Rp(a, b) in a, b-plane is called the p- stability region if for any $(a, b) \in Rp(a, b)$, the numerical solution of (1) satisfies $y_n \to 0$ as $t_n \to \infty$.
2. If a=0 and b are complex in (1), then:
   The region RQ(b) in the b-plane is called the Q-stability region if for any $b \in RQ(b)$, the numerical solution $y_n \to 0$ as $t_n \to \infty$.
   We can use the Kutta-Merson method to plot the region of stability in sence of the definitions above.

Stability of Kutta-Merson Method [1], [2]

If we use, the Kutta-Merson method (with $h = t_{n+1} - t_n$)

$y_1 = y(t_n) + \frac{1}{3} h\, f(t_n, y_n)$

$y_2 = y(t_n) + \frac{1}{6} h\, f(t_n, y_n) + \frac{1}{6} h\, f(t_n + \frac{1}{3} h, y_1),$

$y_3 = y(t_n) + \frac{1}{8} h\, f(t_n, y(t_n)) + \frac{3}{8} h\, f(t_n + \frac{1}{3} h, y_2),$

$y_4 = y(t_n) + \frac{1}{2} h\, f(t_n, y(t_n)) - \frac{3}{2} h\, f(t_n + \frac{1}{3} h, y_2) + 2h\, f(t_n + \frac{1}{2} h, y_3)$

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$y_3 = y(t_n) + \frac{1}{6} f(t_n, y(t_n)) + \frac{2}{3} h\, f(t_n + \frac{1}{2} h, y_2) + \frac{1}{6} h\, f(t_n + h, y_n)$

To advance the numerical solution of DDE (1) to the point t_{n+1}, the Kutta-Merson method yields:

$$y_1 = y(t_n) + \frac{h}{3} y'(t_n)$$

$$y_2 = y(t_n) + \frac{1}{6} h\, y'(t_n) + \frac{1}{6} ha\, y_1 + \frac{1}{6} hbz(t_n + \frac{1}{3} h - 1)$$

$$y_3 = y(t_n) + \frac{1}{8} h\, y'(t_n) + \frac{3}{8} ha\, y_2 + \frac{3}{8} hbz(t_n + \frac{1}{3} h - 1)$$

$$y_4 = (1 + ha + \frac{1}{2} h^2a^2 + \frac{1}{6} h^3a^3 + \frac{1}{24} h^4a^4)\, y(t_n) + \frac{1}{2} hb(1 + \frac{1}{12} h^2a^2 + \frac{1}{12} h^3a^3) z(t_n - 1) - \frac{3}{2} hb(1 + \frac{1}{3} ha + \frac{1}{2} h^2a^2) z(t_n + \frac{1}{3} h - 1) + 2hb\, z(t_n + \frac{1}{2} h - 1)$$

Then:-

$$y(t_{n+1}) = (1 + ha + \frac{1}{2} h^2a^2 + \frac{1}{6} h^3a^3 + \frac{1}{24} h^4a^4 + \frac{1}{144} h^5a^5)\, y(t_n) + \frac{1}{6} hb(1 + ha + \frac{1}{4} h^2a^2 + \frac{1}{8} h^3a^3 + \frac{1}{24} h^4a^4) z(t_n - 1) - \frac{1}{8} h^2a^2(1 - \frac{1}{6} ha) z(t_n + \frac{1}{3} h - 1) + \frac{2}{3} hb(1 + \frac{1}{2} ha)\, z(t_n - \frac{1}{2} h - 1) + \frac{1}{6} hb\, z(t_n - \frac{1}{2} h - 1)$$

where the function z is the approximation of delay terms, which is evaluated using interpolation methods.

Using condition (3), and assuming that the values of the solution and its derivative are stored at earlier mesh points, then using hermite interpolation [3] to evaluate the delay term we get:

$$z(t_n + \frac{1}{3} h - 1) = z(t_{n-m} + \frac{1}{3} h)$$

$$= (\frac{20}{27} + \frac{4}{27} ha) y(t_{n-m}) + (\frac{7}{27} + \frac{2}{27} ha) y(t_{n-m+1}) + \frac{4}{27} hby(t_{n-2m}) - \frac{2}{27} hb\, y(t_{n-2m+1})$$

$$z(t_n + \frac{1}{2} h - 1) = z(t_{n-m} + \frac{1}{2} h)$$

$$= (\frac{1}{2} + \frac{1}{2} ha) y(t_{n-m}) + (\frac{1}{2} - \frac{1}{8} ha) y(t_{n-m+1}) + \frac{1}{8} hb\, y(t_{n-2m}) - \frac{1}{8} hb(t_{n-2m+1})$$

Then (4) becomes:

$$y(t_{n+1}) = (1 + ha + \frac{1}{2} h^2a^2 + \frac{1}{6} h^3a^3 + \frac{1}{24} h^4a^4 + \frac{1}{144} h^5a^5)\, y(t_n) + hb(\frac{1}{2} + \frac{1}{12} ha - \frac{1}{108} h^2a^2 + \frac{5}{1296} h^3a^3 - \frac{1}{648} h^4a^4)\, y(t_{n-m+1}) + hb(\frac{1}{2} + \frac{5}{12} ha + \frac{19}{108} h^2a^2 + \frac{71}{1296} h^3a^3 + \frac{13}{1296} h^4a^4)\, y(t_{n-m}) + h^2b^2(\frac{1}{12} + \frac{1}{24} ha - \frac{1}{108} h^2a^2 + \frac{1}{648} h^3a^3)\, y(t_{n-2m+1}) + h^2b^2(\frac{1}{12} - \frac{1}{24} ha - \frac{1}{54} h^2a^2 - \frac{1}{324} h^3a^3)\, y(t_{n-2m})$$

All the solutions of the difference equation (5) tend to zero as $n \to \infty$ if all the roots of the characteristic equation are in the unit circle.

The characteristic equation of above equation is given by :

$$\lambda^{2m+1} - (1 + ha + \frac{1}{2} h^2a^2 + \frac{1}{6} h^3a^3 + \frac{1}{24} h^4a^4 + \frac{1}{144} h^5a^5)\lambda^{2m} - \frac{1}{2} hb(1 + \frac{1}{6} ha - \frac{1}{54} h^2a^2 - \frac{5}{648} h^3a^3 + \frac{1}{324} h^4a^4) - \lambda^{m+1} - \frac{1}{2} hb(1 + \frac{5}{6} ha - \frac{19}{54} h^2a^2 + \frac{71}{642} h^3a^3 + \frac{13}{642} h^4a^4)\lambda^m + \frac{1}{12} h^2b^2(1 + \frac{1}{2} ha + \frac{1}{9} h^2a^2 + \frac{1}{54} h^3a^3)\lambda - \frac{1}{12} h^2a^2(1 - \frac{1}{2} ha - \frac{2}{9} h^2a^2 - \frac{1}{27} h^3a^3) = 0$$

When a and b are real, we give in figure (1) the p-stability region for m=1, 2,3,4, and compare it with the stability region of the DDE (1) in the a, b plane. All the p-stability regions are closed region, then interest

If a=0 and b is complex, we give in figure (2) the Q-stability region for m=1,2,3,4, and compare it with stability region of the DDE (1) in the b-plane.

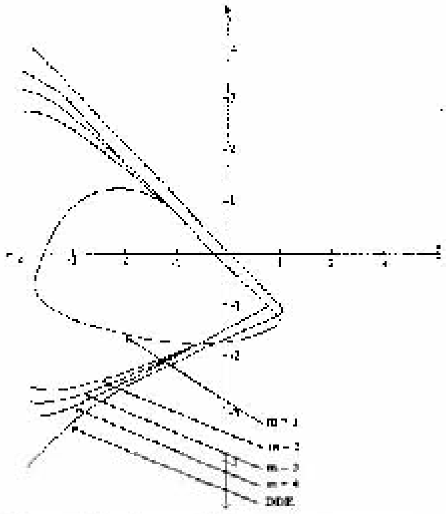

Figure (1): P-stability region of the Kutta-Merson method for solving DDE, with the stability region for the DDE (1) a, b real numbers

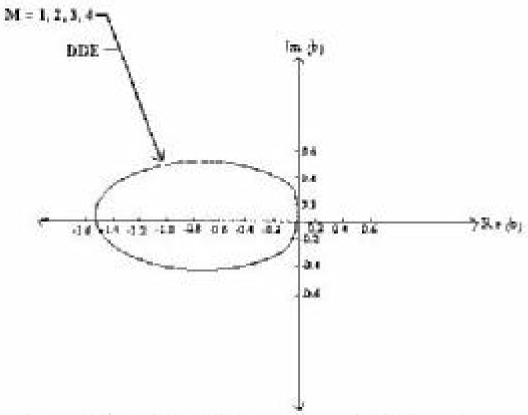

Figure (2): Q-stability region of Kutta-Merson method for solving DDE, with the stability region for DDE (1) a = 0, and b is complex.

References

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الخلاصة

في هذا البحث تم دراسة استقرارية طريقة كوتا- ميرسن لحل المعادلات التفاضلية التباطؤية الخطية والتي بالصورة:

$$u'(t) = au(t) + bu(t-1) \qquad t \geq 0$$
$$u(t) = \phi(t) \qquad t < 0$$

كما تم استحداث طريقة لرسم منطقة الأستقرارية لهذه الطريقة.