Numerical Solution of the Sugeno Fuzzy Integrals

Radhi A. Zboon, Meena M. Mahmood

Department of Mathematics & Computer Applications, College of Science, AL-Nahrain University, Baghdad, IRAQ.

Abstract

An interesting theorem regarding the numerical solution of Sugeno fuzzy integrals have been presented and developed.

Numerical procedures supported by optimization technique as well as some numerical methods have also been developed and presented in order to solve numerically the Sugeno fuzzy integrals.

Some examples have been illustrated using the proposed computational procedures.

Introduction

One of the first concepts of a fuzzy integrals was put forward by [Sugeno, 1974], who consider fuzzy measures and suggested a definition of a fuzzy integral which is a generalization of Lebesgue integrals. Fuzzy integrals are merely a kind of nonlinear functionals, while Lebesgue integrals are linear ones. [Sugeno, 1977]

The ordinary measures in the theory of Lebesgue integral have additive property, while Sugeno consider the fuzzy measures as a set function with monotonic but not always with additivity. Also, he proposed Sugeno fuzzy measure or (λ–fuzzy measure) as a special form of fuzzy measure. [Sugeno, 1977].

[Sugeno, 1974] introduced and investigated the fuzzy measure and fuzzy integral for real- valued functions between 0 and 1, [Ralescu, 1980] generalized the definition of the fuzzy measures and fuzzy integrals for real values functions between 0 and ∞. An alternative expression to the fuzzy integral was later suggested by [Kandel, 1978], and investigated several interesting inequalities related to fuzzy integrals by [Dinola, 1987].

The term "fuzzy integral" actually applies to a large class of non-linear functionals that are based on the concept of a non-additive (fuzzy) measures [Keller, 1999]. The application of fuzzy integral can be found in image processing and pattern recognition. [Tizhoosh, 1997]

Sugeno fuzzy integration[Dubois 1980]

Now, by using fuzzy measure [Dubois 1980], let us define Sugeno fuzzy integrals, which are very similar to Lebesgue integrals. Let h be a function from X to [0,1], the Sugeno fuzzy integral over the non-fuzzy set $A\subseteq X$ of the function h with respect to a fuzzy measure g is defined as follows.

$$\int_A h(x)\circ g(\cdot) = \sup_{\alpha\in[0,1]} \min[\alpha, g(A\cap H_\alpha)] \qquad (1)$$

Where H_α being an α-cut of h, $H_\alpha = \{x\in X | h(x)\geq \alpha\}$ and the symbol $\int$ denotes the Sugeno fuzzy integral.

The Sugeno fuzzy integral is a nonlinear functional processing; the following on some useful properties see [Sugeno, 1977]:

1. $0 \leq \int h(x)\circ g(\cdot) \leq 1$ (2)
2. $\int_A a\circ g(\cdot) = a \qquad \forall a \in [0,1]$ (3)
3. $\forall x, h(x) \leq h'(x)$ then $\int_A h(x)\circ g(\cdot) \leq \int_A h'(x)\circ g(\cdot)$ (Monotonic) (4)
4. g(A) (5) whether A is fuzzy or not, where μ_A stands for the membership function or characteristic function respectively.
5. if $A\subseteq B$, then $\int_A h(x)\circ g(\cdot) \leq \int_B h(x)\circ g(\cdot)$ (6)

Let P be a probability measure, then both integrals Lebesgue and fuzzy are defined with respect to P:

$$\left|\int_0^1 h(x)dP - \int_0^1 h(x)\circ g(\cdot)\right| \leq \frac{1}{4} \qquad (7)$$

for more details[Sugeno, 1974]

Now, let $\mu_A(x)$ be the membership function of a fuzzy set A. Then the fuzzy measure of a fuzzy set A is defined as follows:

$g(A)=\int_X \mu_A(x)\circ g(\cdot)$

Where $\circ$ stands for the fuzzy integral with respect to fuzzy measure g(.). Thus, the Sugeno fuzzy integral of a function h from X to [0,1] over a fuzzy set A with respect to fuzzy measure g is defined as follows:

$$\int_A h(x)\circ g(\cdot) = \int_X \min(\mu_A(x), h(x))\circ g(\cdot) \qquad (8)$$

Assume $X = \{x_1, x_2, \ldots\ldots, x_n\}$ and the function h be a monotonically increasing sequence say: $h(x_1) \leq h(x_2) \leq \ldots \leq h(x_n)$, then we have the Sugeno fuzzy integral is defined as follows

$$\int_X h(x)\circ g(\cdot) = \max_{1\leq i\leq n} \min(h(x_i), g(H_i)) \qquad (9)$$

Where H_i consists of a set of 'n', the greatest element of X

$H_i = \{x_i, x_{i+1}, \ldots\ldots, x_n\}$ and $g(H_n) = 1$

When the fuzzy measure g(.) is continuous, the Sugeno fuzzy integral is defined as follows: [Kandel, 1978]

$$\int_A h(x)\circ g(\cdot) = \inf_{\alpha\in[0,1]} \max[\alpha, g(A\cap H_\alpha)] \qquad (10)$$

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By noting that g $(A \cap U_\alpha)$ is a non-increasing function of α.

Numerical solution of Sugeno fuzzy integral:

In this section, we present a general numerical procedure to calculate the Sugeno fuzzy integral.

Theorem 1: [Kandel, 1996]

Let f(x) be a non-negative monotonic increasing Liphschitz continuous function defined over the interval [a,b], that means..

1. $f(x) \geq 0$ for all $a \leq x \leq b$.
2. If $x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$ for all $a \leq x_1, x_2 \leq b$
3. $|f(x_2) - f(x_1)| \leq L\,|x_2 - x_1|$ for all $a \leq x_1, x_2 \leq b$

and for same constant $L > 0$

Then the Sugeno fuzzy integral of f(x) is defined as:

$$\int_a^b f(x)\,dx = \sup_{\alpha}[\min[\alpha, m(F_\alpha)]] \quad (11)$$

Where $F_\alpha = \{x : f(x) \geq \alpha\}$ and m is the Lebesgue measure. Kandel has proved that, if s be a numeral between a and b for which f(s) = b – s, then

EMBED Equation.3 f(s) (12)

In this paper, the following theorem have been developed and proved to insure that the Sugeno fuzzy integral solution s∈[a,b] of (11) and (12), must exist and unique.

Theorem 2:

Let the assumptions of Equation (1) be valid, then the solution of f(x) = b-x, for all x∈[a,b], exist and unique (and hence the solution of (11) and (12) is exist and unique).

Proof:

Let f(x) = b – x, x∈[a,b]

f(a) = b – a, if a is not equal b, then there are two cases:

Either f(a) > b-a or f(a) < b-a

Case 1:

If f(a) < b-a (13)

Then define g(x) = f(x) – b + x

Since f(x) is non-negative function (i.e. $f(x) \geq 0$, for all x∈[a,b]), and from equation (13), we have:

g(b) = f(b) > 0, and:

g(a) = f(a) – b + a < 0 (14)

Since g is continuous on x∈[a,b], and g(x) satisfy (14) ∀x∈[a,b] using intermediate value theorem, then there exist s∈[a,b], such that g(s) = 0 exists, and thus f(s) = b – s.

Now, to prove the solution of f(s) = b - s is unique, suppose $\exists\, s_1, s_2 \in [a,b]$, such that:

$g(s_1) = f(s_1) - b + s_1 = 0$ (15)

$g(s_2) = f(s_2) - b + s_2 = 0$ (16)

Since g(x) is continuous differentiable on [a,b], and from (15) and 16) exists, and applies Rolle's theorem, there is a number s in [a,b] such that:

$g'(s) = 0$

and from (13):

$f'(s) + 1 = 0 \Rightarrow f'(s) = -1$ (17)

Since f(x) is monotonic increasing, and by definition of derivative:

$$f'(x) = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{for } x_2 > x_1,\ x_2, x_1 \in [a,b]$$

We get $f'(x) \geq 0$, for all x∈[a,b] but from equation (17), we have: $f'(s) = -1$

Therefore, this contradicts the Rolle's theorem

Thus s (if exists) is not belong to the interval [a,b], and therefore:

$s_1 = s_2$

Case2:

If f(a) > b-a (18)

Since f(x) is monotonic increasing function on x∈[a,b], and since:

$$\int_a^b f(x)\,dx = \sup_{\alpha}[\min[\alpha, m(F_\alpha)]] \quad (19)$$

Where $F_\alpha = \{x \mid f(x) \geq \alpha\}$

Now, since a < x, ∀ x∈[a,b], and f(x) is monotonic increasing function that means:

f(a) < f(x) ∀ x∈[a,b]

and $F_\alpha = \{x \mid f(x) \geq \alpha\}$, and thus:

$f(x) \geq f(a)$, ∀ x∈[a,b]

From equation (18), we have: f(x) > f(a) > b – a

Then, $f(x) \geq b - a$ (20)

From equation (20), and f(x) = b – a

There is only one solution x∈[a,b], at x = a:

Then $\int_a^b f(x)\,dx = f(a) = b - a$.

The following computational procedures for finding numerical Sugeno fuzzy integration have been developed and as follows:

Computational procedure for solving sugeno integral

Now, we shall discuss some numerical methods in detail to find the numerical solution of sugeno fuzzy integration.

1 (Newton-Raphson method): [Burden, 2001]

Newton's method is one of the most powerful and well-known numerical method for finding a root of a suitable equation. By virtue of theorems (1) and (2), the solution of the Sugeno integral becomes finding the root of:

F(x) = f(x) – b + x

Where a root s is unique, f(x) is the integrand function satisfies:

$$\int_a^b f(x)\,dx = \sup_{\alpha}[\min[\alpha, m(F_\alpha)]]$$

Where $F_\alpha = \{x \mid f(x) \geq \alpha\}$, and is equivalent to find the root of f(x) – b + x.

In order to find such a root s, we assume $f(x) \in C^2[a,b]$ and the initial value is sufficiently close to the solution of:

$F(x) = x + f(x) - b = 0$ and $F'(x) \neq 0$ ∀ x∈[a,b]

Newton-Raphson iteration procedure for finding such a root is given by:

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

$$= x_n - \frac{x_n + f(x_n) - b}{1 + f'(x_n)} \text{ for some } n \in N \tag{21}$$

Since f(x) is monotonic increasing function, we have $f'(x) \geq 0$, and the denominator in equation (21) is always bounded below by 1.

Now it should be noted that:

1. For any x_1, x_2 which satisfy $a \leq x_1 \leq x_2 \leq b$:
   $F(x_2) - F(x_1) = (x_2 - f(x_2) - b) - (x_1 - f(x_1) - b)$
   $= (x_2 - x_1) + (f(x_2) - f(x_1))$ (22)
   Since f(x) is monotonic increasing Liphschitz continuous function defined over the interval [a,b], then we have:
   $F(x_2) - F(x_1) \geq 0$ for all $a \leq x_1 \leq x_2 \leq b$
   That is F(x) is monotonic increasing, also from equation (22), we have:
   $| F(x_2) - F(x_1) | \leq (1 - L) |x_2 - x_1|$, for some $0 < L < 1$
   Where L is Liphschitz constant for f(x).
   Therefore, F(x) has Liphschitz constant, $L^* = (1 - L)$.
2. In case 2, we have $f(a) \leq b\text{-}a$ and $f(b) \geq 0$
   This implies that:
   $F(a)=f(a) + a - b < 0$ and $F(b) = f(b) + b - b > 0$
   That means F(a) . F(b) < 0, then by using the intermediate value theorem, we have $p \in [a,b]$, such that $F(p) = 0$ (existence of such a solution)
3. If the initial point is "too far" from a solution, the procedure may diverge see [Burdin, 2001].

So, the bisection method can be applied to find an initial approximation x_0 to the solution of the equation (13), and then jump to apply Newton-Raphson method of the quadratic convergence to get an accurate root.

4. In bisection method, we generate a sequence $\{p_n\}_{n=1}^{\infty}$ approximating to zero p of F(x), $x \in [a,b]$ *and we have the following standard* equation: [Burdin, 2001]; where:
   $$| p_n - p | \leq \frac{b - a}{2^n} \text{ when } n \geq 1$$
5. We can use another approach to find a suitable initial point x_0 approximation to s.
   Consider $f \in c^2[a,b]$, let $x_0 \in [a,b]$ be an approximation to s, such that:
   $f'(x_0) \neq 0$ and (s - x), is small.
   Consider the first three trms of Taylor polynomial for f(x) extended about x:
   $$f(x) = f(\overline{x}) + (x - \overline{x})f'(x) + \frac{x + \overline{x}}{2} f''(\zeta(x))$$
   Where $\zeta(x)$ lies between x and $\overline{X}$, since f(x) is defined to be (f(x) = b − x), and using the mean value theorem $f'(\overline{X}) = \frac{f(b) - f(a)}{b - a}$
   letting $\overline{X} = a$, then:

$$b - x = f(a) + (x - a) * \frac{f(b) - f(a)}{b - a} - \frac{(x - a)^2}{2} * f''(\zeta(x))$$

Since (x - a) is small, the term involving $(x - a)^2$ is much smaller, therefore:

$b - x \approx f(a) + (x - a) * A$

where $A = \frac{f(b) - f(a)}{b - a}$

$$x \approx \frac{aA - f(a) + b}{1 + A} \tag{25}$$

This x is the exact s if f(x) is linear as can be easily verified.

Remark 1:

If f(x) is a non-negative monotonic decreasing, we simply define:

$f_1(x) = f(a + b - x)$,

for all $a \leq x \leq b$

Then the new function will satisfy the required assertion for the theorems (1), and theorem (2).

Thus from above one can develop the following algorithm:

Algorithm A.1:

Input: the interval end points a,b and the tolerance ε_1(accuracy for initial point using bisection method), ε_2(accuracy of a root of Newton-Raphson)

Output: The Sugeno fuzzy integral $\int_a^b f(x)\,dx$.

Step 1:

if $f(a) \geq b\text{-}a$, Output $\int_a^b f(x)\,dx = b\text{-}a$, and stop. Otherwise go to step 2.

Step 2:

Find an initial approximation x_0 to the root of F(x) = f(x) − x − b and as follows: (using bisection method)

Step 2.1:

set N_0 = round $\left(\frac{-log_{10}(\varepsilon_1)}{\log_{10}2 + \log_{10}(b - a)} \right)$

Step 2.2:

set i = 1 and gA = F(a) = f(a) + a - b

Step 2.3:

while $i \leq N_0$ do

Set $x_0 = a + (b\text{-}a)/2$ $gx_0 = F(x_0)$

Step 2.3-1:

if $F(a).F(x_0) > 0$, then set $a = x_0$ and $FA = Fx_0$ else set $b = x_0$

Step 2.4:

set i = i + 1

Step 2.5:

output x_0 " initial approximate value for Newton-Raphson."

Step 3:
repeat

Step 3.1:
$x_0 = x_1$, $F(x_0)$, $F'(x_0)$

Step 3.2:
$$x_1 = x_0 - \frac{F(x_0)}{F'(x_0)}$$

Step 3.3:
until ($|x_1 - x_0| < \varepsilon_2$)

Step 4:
x = root (output root of $F(x) = f(x) - x - b$)

Step 5:
$y = b - x$ (output the Sugeno fuzzy integral $\int_a^b f(x)\,dx$)

The following illustrations have been applied using the computational algorithm1:

Illustration A.1:

Consider the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dx$.

Note that $f(x) = \frac{2}{\sqrt{\pi}} e^{-x^2}$ and $x \in [0,2] = [a,b]$. Let $\varepsilon = 10^{-5}$

step0:
Define $F(x) = f(x) + x - b$
$$= \frac{2}{\sqrt{\pi}} e^{-x^2} + x - 2$$
then $\quad F'(x) = -\frac{4}{\sqrt{\pi}} x e^{-x^2} + 1$

step1:

Note that compute $f(a) = \frac{2}{\sqrt{\pi}} < 2 = b - a$, then case 1 of theorem 2 is applied.

step2:
Find initial point using approximation method from equation (25), we get:
$$x_0 = \frac{aA - f(a) - b}{1 + A}$$
where $\quad A = \frac{f(b) - f(a)}{b - a}$
then: $x_0 = 1.953676414$
$x_0 \approx 2$

step3:
Now, using Newton-Raphson method to find the root of
$$F(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} + x - 2$$
$$F'(x) = -\frac{4}{\sqrt{\pi}} x e^{-x^2} + 1, \; F'(x) \neq 0 \; \forall x \in [0,2]$$

step4:

The iterative process is: $x_{i+1} = x_i - \frac{F(x_i)}{F'(x_i)}$

With condition $|x_{i+1} - x_i| < \varepsilon$
The numerical results are shown in table (1):

Table (1)

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
1	2.0000	1.97747	2.252944848IE-02
2	1.97747	1.97738	8.238834379 9E-05
3	1.97738	1.97738816	1.151420292 4E-09

iterations = 3
The root is $x = 1.97738816$

step5
Sugeno fuzzy integral is:
$\int_a^b f(x)\,dx = b - x = 0.02261184$
but the ordinary integration from the tables where $a = 0$, $b = 2$ one can found:
$$\int_a^b f(x)dx = \int_0^2 \frac{2}{\sqrt{\pi}} e^{-x^2} dx = 0.9953223$$

Illustration A. 2
Consider the crisp function: EMBED Equation.3 .

Step 0:
Define $F(x) = f(x) + x - b$
EMBED Equation.3
Then:
EMBED Equation.3

Step 1:
Compute $f(a) = 1 < 8 = b - a$, then case 1 applies.

Step 2:
Find initial point using bisection method.

Step 2.1:
Check $F(a) \cdot F(b) < 0$
at $0 \le x \le 1$, for $F(x) = x$ ($F(a) \cdot F(b)$) < 0
at $1 < x < 2$, for $F(x) = 2x - 2$ ($F(a) \cdot F(b)$) < 0
at $2 < x \le 3$, for $F(x) =$ EMBED Equation.3
$F(a) \cdot F(b) < 0$
Step 2.2: Set $\varepsilon = 10^{-1}$
$$n > \frac{-\log_{10}(\varepsilon_1)}{\log_{10} 2 + \log_{10}(b - a)} = 4.3219$$
$n = 4$

Step 2.3 Let $a = 2$, $b = 8$, $x_1 = \frac{a+b}{2}$

Then, the initial point $x_0 = 4.625$

Table (1)

i	a_n	b_n	x_n
1	2	8	5
2	2	5	3.5
3	3.5	5	4.25
4	4.25	5	4.625

The numerical results as shown in table (3.3)

Step 3

Now, using Newton-Raphson method to find the root of

$$F(x) = \sqrt{2x} + x - 8 \qquad x \in [0,8]$$

and satisfy the required value of F(a) . F(b) < 0

$$F'(x) = \frac{1}{\sqrt{2x}} + 1, \quad F'(x) \neq 0 \qquad \forall x \neq 0.$$

Then, the iterative process is : $x_{i+1} = x_i - \dfrac{F(x_i)}{F'(x_i)}$

Table (3)

i	x_i	x_{i+1}	$\lvert x_{i+1} - x_i \rvert$
1	4.625	4.8760680639	2.5106806394E-01
2	4.8760680639	4.8768943659	8.2630195539E-04
3	4.8768943659	4.8768943744	8.48376578E-09

With condition $\lvert x_{i+1} - x_i \rvert \le \varepsilon_2 \qquad (\varepsilon_2 = 10^{-5})$

The numerical results as shown in table (3.4):

Iterations = 3

The root is x = 4.87689437

Step 4:

Sugeno fuzzy integration is: $\int_a^b f(x)\,dx = b - x = 3.12310563$

(5.B) Computational Procedure using Optimization method:

The following equivalent are needed in the

but the ordinary integration is:

$$\int_a^b f(x)dx = \int_0^8 f(x)dx = 2.39685$$

numerical solution of Sugeno fuzzy integral:

Remark 2:

Let f(x) has a unique root in [a,b] and f(a) . f(b) < 0, then one can write an equivalent form as follows:

$f(x) = 0 \Leftrightarrow \min (f(x))^2$ exists $x \in [a,b]$

From This remark, we can use the optimization method to find the solution of Sugeno fuzzy integral. So, In this paper the steepest-descent method to find the numerical Sugeno fuzzy integral has been considered.

5.B.1 The steepest-descent method:

Suppose that $F(x) = f(x) + x - b$, where f(x) denoted a non-negative monotonic increasing Lipschitz-continuous function defined over the interval [a,b], we noted that:

1. F(x) be a continuous function over the interval [a,b].
2. $F'(x) \neq 0$, furthermore $F'(x) = f'(x) + 1$.
3. F(a) . F(b) < 0 as mentioned in section (3.7.1), than from the remark 3, we get:

$$F(x) = 0 \Leftrightarrow \min (F(x))^2$$
$$x \in [a,b] \qquad\qquad x \in [a,b]$$

Hence, using the steepest-descent method of unconstrained minimization technique, it uses the negative of the gradient vector as a direction of minimization. The general idea is to generate successive point starting from a given initial point in the direction of the fastest decrease (minimization) of the function. This technique is known as the gradient method. Termination of the gradient method is effected at the point where the gradient vector becomes null, this is only a necessary condition for optimality.

Let $g(x) = [F(x)]^2$, is a quadratic function to be minimized, and let x_0 be the initial point, define $\nabla g(x_i)$,(i.e.: $\nabla g(x_m) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})_{x_m}$) as the gradient of the function g(x) at the ith point x_i. The idea of the method is to determine a particular path g along which $-\frac{\partial g}{\partial x}$ is minimized at a given point.

This method is an iterative process, at stage i, we have an approximation x_i for the minimum point. Our next approximation is:

$$x_{i+1} = x_i - \lambda_i \nabla g(x_i) \qquad (26)$$

Where λ_i is a parameter called the optimal step size and defined $h(\lambda_i)$ such that:

$$h(\lambda_i) = g[x_i - \lambda_i \nabla g(x_i)] \qquad (27)$$

Where λ_i is the value of λ that minimizing h(λ).

Remark 3:

Let $x_{i+1} = x_i + \lambda_i^* S_i = x_i - \lambda_i \nabla g_i \;\; i \in n$

Where λ_i^* is the optimal step length.

Define $g(x) = [f(x) + x - b]^2$

We need to find the optimal step length λ_i^* that minimizing:

$$g(x_i + \lambda_i^* s_i) = [f(x_i + \lambda_i^* s_i) + (x_i + \lambda_i^* s_i) - b]^2 \; i \in n$$

$$\frac{\partial g}{\partial \lambda^*} = 2[f(x_i + \lambda_i^* s_i) + (x_i + \lambda_i^* s_i) - b].\{\frac{\partial f}{\partial \lambda^*} + s_i\}$$

set $\frac{\partial g}{\partial \lambda^*} = 0$, this implies that:

$$2\{\frac{\partial f}{\partial \lambda^*} + s_i\} * [f(x_i + \lambda_i^* s_i) + (x_i + \lambda^* s_i) - b] = 0$$

is a non-linear algebraic equation in parameter λ^*, so we can solved it by using any suitable numerical method to find its root.

Algorithm B.1:

Input: initial point x_0 and a termination criteria ε (accuracy).

Output: the solution of Sugeno integral.

Step 1: set i = 0.

Step 2: Find the search direction $s_i = -\nabla f_i$.

Step 3: find λ_i^* to minimize $g[x_i + \lambda s_i]$.

Step 4: set $x_{i+1} = x_i + \lambda_i^* s_i$.

Step 5: If $\|\nabla g(x_{i+1})\| \le \varepsilon$, then the output is the solution x_{i+1}

Step 6: set $i = i + 1$, and go to step 2.

Step 7: x_{i+1} is the root.(using any suitable method)

Step 8: $y = b - x_{i+1}$ (output solution of sugeno fuzzy integral.)

Illustration B.1:

Consider the error function $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}dx$, note that $f(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}$ where $x \in [0,2]$. Given $\varepsilon = 10^{-4}$

step0:

Define $F(x) = f(x) + x - b$

$= \frac{2}{\sqrt{\pi}}e^{-x^2} + x - 2$ Let $g(x) = (F(x))^2$

$= (\frac{2}{\sqrt{\pi}}e^{-x^2} + x - 2)^2$

step1:

Minimize $g(x) = (\frac{2}{\sqrt{\pi}}e^{-x^2} + x - 2)^2$

Starting from the initial point $x = 0$, $x \in [0,2]$

step2:

The gradient of g is given by:

$\nabla g = \frac{\partial g}{\partial x} = 2 \quad (2e^{-x^2} + \sqrt{\pi}x - 2\sqrt{\pi}) \quad *$

$(\frac{4xe^{-x^2} - \sqrt{\pi}}{\pi})$

$\nabla g_1 = \nabla g(x_1) = -1.743$

Let $s_1 = -\nabla g_1 = 1.743$

step3:

To find x_2, we need to find the optimal step length λ^*, and for this, we minimize:

$g(x_1 + \lambda^* s_1) = g(1.743\lambda_1) = (\frac{2}{\sqrt{\pi}}e^{-(1.743\lambda_1)^2} + 1.743\lambda_1 - 2)^2$

With respect to λ_1

$\frac{dg}{d\lambda_1} = 0 \Rightarrow \lambda^* = 1.134$

step4: we obtain:

$x_2 = x_1 + \lambda_1^* s_1 = 1.977$

as $\nabla g_2 = \nabla g(x_2) = -0.00005436 \cong 0 \Rightarrow x_2$ is optimum

$\therefore$ The root $x = 1.977$

Then, Sugeno fuzzy integral is $\int_a^b f(x)\,dx = b - x =$ 0.023

but the ordinary integration from the table one where $a = 0$, $b = 2$:

$\int f(x)dx = \int_0^2 \frac{2}{\sqrt{\pi}}e^{-x^2}dx = 0.9953223$

We see there is a difference between the fuzzy integration and the ordinary integration.

Illustration B.2:

Consider the crisp function $f(x) = \sin(x)$ to be integrated using Sugeno fuzzy integration discussed above over the interval $x \in [0, \frac{\pi}{2}]$.

Let $\varepsilon = 10^{-4}$

step0:

Define $F(x) = f(x) + x - b$

$= \sin(x) + x - \frac{\pi}{2}$

Let $g(x) = (F(x))^2$

$= (\sin(x) + x - \frac{\pi}{2})^2$

step1

Minimize $g(x) = (\sin(x) + x - \frac{\pi}{2})^2$

Starting from the initial point $x_1 = 0$, $x \in [0, \frac{\pi}{2}]$

step2:

The gradient of g is given by:

$\nabla g = \frac{\partial g}{\partial x}$

$= 2\sin(x)\cos(x) + 2\sin(x) + 2x\cos(x) + 2x - \pi\cos(x) - \pi$

$\nabla g_1 = \nabla g(x_1) = -2\pi = -6.283$

Let $s_1 = -\nabla g_1 = 2\pi = 6.283$

step3:

To find x_2, we need to find the optimal step length λ_1^*, and for this, we minimize:

$g(x_1 + \lambda_1^* s_1) = g(2\pi(\lambda_1)) = (\sin(2\pi\lambda_1) + 2\pi\lambda_1 - \frac{\pi}{2})^2$

With respect to λ_1:

$\frac{dg}{d\lambda} = 0 \Rightarrow \lambda^* = 0.123$

We obtain:

$x_2 = x_1 + \lambda_1^* s_1 = 0.829$

Since

$\nabla g_2 = \nabla g(x_2) = -0.015 \neq 0 \Rightarrow x_2$ is not optimum

$s_2 = -\nabla g_2 = 0.015$

So, we have to minimize the following

$$g(x_2 + \lambda_2^* s_2) = g(0.829 + 0.015\lambda_2)$$

$$= (\sin(0.829 - 0.015\lambda_2) - (0.829 - 0.015\lambda_2)\lambda_1 - \frac{\pi}{2})^2$$

With respect to $\hat{\lambda}_2$ to compute the new iteration x_3:

$$\frac{dg}{d\lambda_2} = 0 \Rightarrow \lambda_2^* = 0.181$$

we obtain:

$$x_3 = x_2 + \hat{\lambda}_2^* s_2 = 0.832$$

Since the gradient at x_3 is:

$\nabla g_3 = \nabla g(x_3) = -0.012 \approx 0 \Rightarrow x_3$ is optimum (for a required accuracy)

The root = 0.832

step4:

Sugeno fuzzy integral is: $\int_0^1 f(x)\,dx = 0.738796$ and the ordinary integration for this function is 1.

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الخلاصة

لقد قدمت وطورت نظرية تتعلق بالحل العددي لتكامل سوجينو الضبابي . كذلك قدمت أساليب عددية مستندة على بعض طرق الأمثلية وبعض الطرق العددية لحل تكاملات سوجينو الضبابية عدديا. لقد نفذت بعض الأمثلة باستخدام الأساليب العددية المقترحة.