On the Stability of Most General Exact Solution for Isentropic Superdense Stars

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Abstract

Most general exact solutions of Einstein's field equation for isentropic fluid spheres have been investigated. Spherically symmetric model based on most general exact solution by Gupta and Jasim, 2000 have been analyzed for the stability performance. The analysis yields a strong indication that the model is stable with respect to infinitesimal radial pulsation. We also find that the adiabatic speed of sound is smaller than unity inside the fluid sphere if and only if the radius of the sphere is larger than 1.46 times its Schwarzschild radius. Besides that the solution for case K=-11 (case study), have been tested for the stability and found that, it is stable for least admissible value of $\lambda = \frac{\rho_R}{\rho_0} = 0.3$

Introduction

The difficulty of obtaining relativistic models of spherical fluid spheres based on exact solution of Einstein's field equation, which is describing the spherical distributions of the matter. Due to the non-linear nature of the Einstein's field equation is a consequence of the self- interaction of the gravitational field this makes its difficult [4].

The actual properties in the central region of a relativistic compact star are not precisely known and so assumptions of general nature to obtain exact solutions of Einstein's field equations become necessary and the solution so obtained should be physically plausible and the same time simple in form.

Several authors have considered the problem with some special attempt [1], [5], [6], [7]. Here we have investigated the gravitational significance of space time whose physical space obtained as t = constant section has the geometry of 3-spheroid, which Gupta-Jasim [2000] its most general solutions already have been obtained [3]. So, with this regard the authors have been discussed its stability and found that for k<0 the fluid spheres is stable. In addition to that for k=-11 (study case) have been analyzed with respect to the reality conditions as a study case.

Description of Field Equations

A 3-pseudo spheroid immersed in the 4-dimensional space with metric

$$ds^2 = dx^2 + dy^2 + dz^2 + dt^2 \qquad (1)$$

will have the Cartesian equation

$$\frac{t^2}{a^2} - \frac{x^2 + y^2 + z^2}{R^2} = 1 \qquad (2)$$

where a and R are constants

The section t = constant of 3-pseudo spheroid are pseudo spheres, while sections x, y and z are constants represents respectively hyperboloids of two sheets [6].

The parameterization

$$x = r\sin\theta\cos\phi,$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta \qquad (3)$$

$$t = a\left(1 - \frac{r^2}{R^2}\right)^{1/2}$$

of 3-pseudo spheroid leads to

$$ds^2 = -e^{\lambda}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}dt^2 \qquad (4)$$

Assuming that the physical three-space in a relativistic fluid spheres has the geometry of a 3-spheroid we should have [7].

$$e^{\lambda} = \frac{1 - K\left(\frac{r^2}{R^2}\right)}{1 - \frac{r^2}{R^2}} \qquad (5)$$

and e^{ν} such that the line element ds^2 describes perfect fluid distribution.

The above metric (4) with (5) is regular and positive definite at all points r<R.

For case k<1 the perfect fluid distribution can be expressed (assumes the form) as

$$(1 - k + kx^2)\frac{d^2y}{dx^2} - kx\frac{dy}{dx} + k(k-1)y = 0$$

Where

$$y = e^{\nu/2} \qquad (6)$$

$$\text{and } x = \sqrt{1 - \frac{r^2}{R^2}} \qquad (7)$$

The Einstein's Field Equation

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\left(\frac{8\pi G}{c^4}\right)T_{ij} \qquad (8)$$

leads to [Jasim 2000]

$$8\pi\rho - \frac{2-(1-k)\cos^2\theta}{R^2(1-k)\cos^4\theta} \tag{9}$$

and

$$8\pi p = \frac{-2\cos(m\theta+\alpha)\sin\theta}{R^2\cos^2\theta\ [illegible]} \ [illegible] \tag{10}$$

$$\text{where } k = 2 - m^2 \text{ with } 1 < m < \sqrt{2} \tag{11}$$

Dynamical Stability of Fluid Sphere

The basic method for examining whether a relativistic fluid sphere is stable with respect to infinitesimal radial adiabatic pulsation has been developed by Chandrasekhar, 1964. A normal mode of radial oscillation leads for an equilibrium configuration [2], i.e.

$$dr = \xi(r)e^{i\sigma t} \tag{12}$$

is stable (periodic oscillation) if its frequency σ is real and unstable (exponential growth) if σ is imaginary.

Gupta-Jasim (2000) have obtain their general solution by demanding that the energy-momentum tensor was that of a perfect fluid. However, the same static solution is in fact obtained assuming only that the static energy-momentum tensor is given by

$$T_i^{\,j} = (-p, -p, -p, \rho) \tag{13}$$

Now, when investigating stability one is studying a dynamical object, and to describe its behavior one needs to know the non-static and energy momentum tensor [4].

To perform the stability analysis we restrict our examination to the case where the fluid is isentropic under static conditions and our analysis is valid for stars where temperature is essentially at absolute zero (white dwarfs, neutron stars).

Following Barden et al, 1966 and Chandrasekhar's, 1964 pulsation equation for the line element (4) is given by [4].

$$\sigma^2 \int_{center}^{boundary} e^{\frac{3\lambda+\nu}{2}} \frac{(\rho+p)u^2}{r^2}\,dr$$

$$= \int_{center}^{boundary} e^{\frac{\lambda+3\nu}{2}} \left\{\left[\frac{-2}{r}\frac{d\nu}{dr} - \frac{1}{4}\left(\frac{d\nu}{dr}\right)^2\right] + 8\pi p e^{\lambda}\right\} \ [illegible] + \frac{\gamma p}{r^2}\left(\frac{du}{dr}\right)^2 \ [illegible]\, dr \tag{14}$$

with

$$u = \xi_{(r)} r^2 e^{-\nu/2} \tag{15}$$

$$\text{and } \lambda = \ln\left(\frac{1-11r^2/R^2}{1+r^2/R^2}\right) \tag{16}$$

with (k=-11) as a study case.

Hence we have used the relativistic adiabatic index γ given by

$$\gamma = \frac{p+\rho}{p}\frac{dp}{d\rho} \tag{17}$$

The adiabatic index γ should be larger than unity for temperature away from the center, or even by larger than 4/3 to prevent instability under radial perturbation. The later condition is however only a necessary but not sufficient condition to obtain a dynamically stable model.

After some calculations and with using computer programming algebra we find the pulsation equation for Gupta-Jasim model (test case: k= 11), the following table shows the values of γ inside the star at λ 0.3

x	γ
1	Very large no.
0.9	7.3615
0.8	4.2091
0.7	3.1425
0.6	2.6064
0.5	2.2811
0.4	2.0653
0.3	1.9179
0.2	1.8205
0.1	1.7646
0	1.7463

Equation (14) with reference to (15) and (16) have been integrated numerically for different values of $\frac{a^2}{R^2} < \frac{1}{2}$, to have the weak energy conditions for different values of $\frac{a^2}{R^2} < \frac{1}{2}$, to have the weak energy condition ρ>p fulfilled at the origin r=0. So, we have evaluated the integral in (14) as

$\frac{a^2}{R^2}$	Values of integral
0.1	0.1515×10^{-1}
0.2	0.2342×10^{-1}
0.3	0.2344×10^{-1}
0.4	0.1888×10^{-1}
0.5	0.1325×10^{-1}

This analysis indicates that these models with λ≥0.3 will be stable and found that the integral admits positive values also the adiabatic speed of sound is smaller than unity inside the fluid sphere if and only if the radius of the sphere is larger than 1.46 times its Schwarzschild radius.

The space-time with pseudo spheroid geometry for its spatial sections t = constant this may admit the possibilities of describing interiors of the superdense fluid stars in equilibrium.

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الخلاصة

تمت دراسة الفضاء الزمن – مكاني (space-time) والموصوف بمجموعة التي تتسم 3مجاميع امثالية مع مجموعتين تبسيطيتين بفضاء متكور ومكي مكافئة بموديل البحث للموائع التامة و شروطها الفيزيائية.

لقد ابدت طرائق مونجي دورا مهما في دراسة هذه المسائل ، فقد تم ايجاد حلول جديدة أكثر عمومية ، كذلك تم التطرق الى دراسة الحالات امثالية لما يتطلب الاهتمام لكثافة الطاقة المتغيرة والثابتة.