LOCAL SOLVABILITY AND CONTROLLABILITY OF SEMILINEAR INITIAL VALUE CONTROL PROBLEMS VIA SEMIGROUP APPROACH

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Abstract

The aim of this paper is to prove the local existence, uniqueness and the exact controllability of the mild solutions of semilinear initial value control problems in suitable Banach spaces using semigroup theory "compact semigroup" and Schauder fixed point theorem.

Keywords: Local existence, uniqueness, exact controllability, mild solution, fixed point theorem and Semigroup theory of control problems.

List of symbols

$\mathcal{L}(X)$ Banach algebra {the set of bounded linear operators}

$\{T(t)\}_{t\geq 0}$ Family of bounded linear operators

U Open set

O, X, Y Banach space

B Bounded linear operator

w Control function

$L^p([0, r):O)$ Banach space of p-integrable functions with its domain [0,r) into O such that

$$\|f\|_p = \left(\int_{t=0}^{r} |f(t)|^p \, dt\right)^{\frac{1}{p}} < \infty$$

u_w^n Sequence of continuous function depend on w

$\Im_F(E)$ The set of all continuous function define from E into F

$\mathcal{B}_\rho(u_0)$ Closed ball with center u_0 and radius ρ

$\bar{\bar{u}}_w, \bar{u}_w, u_w$ Continuous function depend on w

W Linear operator

O/kerW Quotient space

$[w(t)]$ An equivalent classes of $w(t)$

$\underline{w}$ Control function belong to $[w(t)]$

$\phi_{\underline{w}}$ Mapping depend on the control function $\underline{w}$

$u_{\underline{w}}^n$ Sequence of continuous function depend on the control function $\underline{w}$

V Subspace of X

$\tilde{W}^{-1}$ The inverse linear operator of $\tilde{W}$

$\|\cdot\|_Y = \sup_{0\leq t\leq t_1} \|\cdot(t)\|_X$

$\|[\cdot]\|_{O/\,\mathrm{KerW}} = \inf_{\cdot\in[\cdot]} \|\cdot\|_O$.

1. Introduction

Byszewski in 1991 [1], has study the local existence and uniqueness of the mild solution to the semilinear initial value problem:

$$\left.\begin{aligned} &\frac{du}{dt} + Au(t) = f(t, u(t)) \\ &u(0) = u_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Where A is the infinitesimal generator of a C_0 semigroup defined from $D(A)\subset X$ into X and f is a nonlinear continuous map define from $[0,r)\times X$ into X.

A continuous function u is said to be a mild solution to the semilinear initial value problem (1) given by [2]

$$u(t)=T(t)u_0+\int_{s=0}^{t} T(t-s)f(s,u(s))ds \ldots\ldots\ldots (2)$$

Where T(t) is bounded linear operator for t>0.

Bahuguna.D in 1997 [3], has studied the local existence without uniqueness of the mild solution to the semilinear initial value problem:

$$\left.\begin{aligned} &\frac{du}{dt} + Au(t) = f(t, u(t)) + \int_{s=0}^{t} h(t-s)g(s,u(s)\,ds \quad t>0 \\ &u(0) = u_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3)$$

where A is the infinitesimal generator of a C_0 semigroup defined from $D(A)\subset X$ into X and f and g are a nonlinear continuous maps defined from $[0,r)\times X$ into X and h is the real valued continuous function defined from [0,r) into R where R is the real number.

Pavel in 1999 [4] has studied the uniqueness of the mild solution to the semilinear initial value problem given by (3).

A continuous function u is said to be a mild solution to the semilinear initial value problem (3) given by [3]:

$$u(t)=T(t)u_0+\int_{s=0}^{t} T(t-s)\left[f(s,u(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u(\tau))d\tau\right]ds \quad ....(4)$$

Our work is concerned the semilinear initial value control problem:

$$\left.\begin{array}{l}\frac{du}{dt}+Au(t)=f(t,u(t))+\int_{s=0}^{t} h(t-s)g(s,u(s)ds+Bw(t), t>0\\ u(0)=u_0\end{array}\right\} \quad ....(5)$$

where A is the infinitesimal generator of a C_0 semigroup defined from $D(A)\subset X$ into X and f and g are a nonlinear continuous maps defined from $[0,r)\times X$ into X, h is the real valued continuous function defined from [0,r) into R where R is the real number and B is a bounded linear operator define from O into X. Where O is a Banach space and $w(.)$ be the arbitrary control function is given in $L^p([0, r):O)$, a Banach space of control functions with $\|w(t)\|_O \le k_1$, for $0 \le t < r$.

The mild solution will be developed as follow:

A continuous function u_w will be called a mild solution of (5), given by:

$$u_w(t)=T(t)u_0+\int_{s=0}^{t} T(t-s)\left[\begin{array}{l}Bw(s)+f(s,u_w(s))+\\ \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau\end{array}\right]ds \quad ....(6)$$

For every given $w\in L^p([0, r):O)$.

The local existence, uniqueness and controllable of the mild solution defined in (6) to the semilinear initial value control problem defined in (5) have been developed.

2. Preliminaries

2.1 <u>Definition:</u> A family $\{T(t)\}_{t\ge 0}$ of bounded linear operators on a Banach space X is called a (one-parameter) semigroup on X if it satisfies the following conditions:

$T(t+s)=T(t)T(s),\ \forall\ t,s\ge 0$

$T(0)=I$

2.2 <u>Definition [5]</u>

A semigroup $\{T(t)\}_{t\ge 0}$ on a Banach space X is called strongly continuous semigroup of bounded linear operators or (C_0 semigroup) if

The map $R^+ \ni t \longrightarrow T(t) \in \mathcal{L}(X)$, satisfies the following conditions:

1. $T(t + s) = T(t)T(s),\ \forall\ t, s \in R^+$.
2. $T(0) = I$.
3. $\lim_{t\downarrow 0} \| T(t)x-x \| = 0$, for every $x\in X$.

2.3 <u>Definition [6]</u>

A semigroup $\{T(t)\}_{t\ge 0}$ is said to be compact if T(t) is a compact operator for each $t > 0$.

2.4 <u>Definition [5],[7]</u>

Given any two points u_0 , $u_\gamma \in X$ (X is a Banach space) , we say that the mild solution given by (6) to the semilinear initial value control problem given by (5) is exactly controllable on $J_0 = [0,\gamma]$, if there exist a control $\underline{w} \in L^p(J_0:O)$ such that the mild solution $u_w(.)$ of equation (5) satisfy the following conditions $u_{\underline{w}}(0)=u_0$ and $u_{\underline{w}}(\gamma)=u_\gamma$.

2.5 <u>Precompact set</u>

Let X be a Banach space, a subset S of X is said to be precompact if for each $\varepsilon>0$, there exists some finite set $S = \{x_1,\ldots,x_n\}$ in X such that S is contained in $\bigcup_{i=1}^{n}\beta(x_i,\varepsilon)$, where $\beta(x_i,\varepsilon) = \{y \in X : \|y-x_i\|<\varepsilon\}$.

2.6 <u>Equicontinuous set</u>

A subset S of C[a,b] is said to be equicontinuous, for each $\varepsilon>0$, there is a $\delta>0$, such that:

$$|x-y|<\delta \text{ And } u \in M \text{ imply}$$

$$\|u(x)-u(y)\|_{C[a,b]}<\varepsilon$$

2.7<u>Arzela-Ascoli's theorem</u>

Suppose F is a Banach space and E is a compact metric space. In order that a subset H of the Banach space $\Im_F(E)$ be relatively compact, if and only if H be equicontinuous and that, for each $x \in E$, the set $H(x) = \{f(x): f \in H\}$ be relatively compact in F.

2.8 Schauder fixed point theorem

Let M be a nonempty closed, bounded, convex subset of a Banach space X and the map T: M ⟶ M is compact then T has a fixed point.

2.9 Compact map

Let S, M be two sets, a map T: S ⟶ M is said to be compact if the following conditions are hold:

1-T is continuous map.

2-For each bounded subset of S, T(S) is relatively compact set in M.

2.10 Remark [2]:

A semigroup $\{T(t)\}_{t\geq 0}$ is called a continuous in the uniform operator topology if:

(1) $\|T(t+\Delta)x - T(t)x\| \longrightarrow 0$, as $\Delta \longrightarrow 0$, $\forall x \in X$.

(2) $\|T(t)x - T(t-\Delta)x\| \longrightarrow 0$, as $\Delta \longrightarrow 0$, $\forall x \in X$.

3. Main results

It should be notice that the local existence and uniqueness of a mild solution of (5) have been developed, by assuming the following assumptions:

1. A be the infinitesimal generator C_0 compact semigroup $\{T(t)\}_{t\geq 0}$, where A defined from $D(A) \subset X$ into X and X is a Banach space.
2. Let ρ>0 such that $B_\rho(u_0) = \{x \in X \mid \|x - u_0\| \leq \rho\}$, (where $u_0 \in U$ and U is an open subset of X), The nonlinear maps f, g defined from [0, r) ×U into X, satisfy the locally Lipschitze condition with respect to second argument, i.e. $\|f(t, v_1) - f(t, v_2)\|_X \leq L_0\|v_1 - v_2\|$ and $\|g(t, v_1) - g(t, v_2)\|_X \leq L_1\|v_1 - v_2\|$
   For $0 \leq t < r$ and $v_1, v_2 \in B_\rho(u_0)$ and L_0, L_1 are Lipschitze constant.
3. h is continuous function which at least $h \in L1([0,r):R)$, where R is the set real numbers.
4. Let $t' > 0$ such that $\|f(t, v)\|_X \leq N_1$, $\|g(t, v)\|_X \leq N_2$, for $0 \leq t \leq t'$ and $v \in B_\rho(u_0)$.
   Also let $t''>0$ such that $\|T(t)u_0 - u_0\|_X \leq \rho'$ for $0 \leq t \leq t''$ and $u_0 \in U$, where ρ′ is a positive constant such that $\rho' < \rho$.
5. w(.) be an arbitrary given control function is given in Lp([0, r): O), a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X with $\|w(t)\|_O \leq k1$, for $0 \leq t < r$.
6. Let t1>0 such that:
   $t_1 = \min\{r, t', t''\}$ and satisfy the following conditions

   (6.i) $t_1 \leq \dfrac{\rho - \rho'}{(K_0K_1 + N_1 + h_{t_1}N_2)M}$

   (6.ii) $t_1 < \dfrac{1}{M(L_0 + L_1 h_{t_1})}$

Theorem (1)

Assume the hypotheses (1)-(6) are hold. Then for every $u_0 \in U$, there exist a fixed number t_1, $0 < t_1 < r$, such that the initial value control problem (5) has a unique local mild solution $u_w \in C([0, t_1]:X)$, for every control function $w(.) \in L^p([0, r): O)$.

Proof

Without loss of generality, we may suppose $r<\infty$, because we are concerned here with the local existence only.

Since T(t) is a bounded linear operator on X, there exist $M \geq 0$ such that $\|T(t)\| \leq M$, $0 \leq t \leq r$.

Let ρ > 0 be such that $B_\rho(u_0) = \{v \in X \mid \|v - u_0\| \leq \rho\} \subset U$ { since U is an open subset of X}. Assume

$$h_r = \int_{s=0}^{r} |h(s)|\,ds \quad \text{.............................. (7)}$$

Set Y= C([0,t1]:X), where Y is a Banach space with the sup-norm defined as follows:

$$\|y\|_Y = \sup_{0\leq t\leq t_1} \|y(t)\|_X \text{, and define}$$

$S_w = \{u_w \in Y \mid u_w(0) = u_0, u_w(t) \in B_\rho(t_0)$, for a given $w \in L^p([0,t_1]:O), 0 \leq t \leq t_1\}$..................... (8)

It is clearly S_w is bounded, convex and closed subset of Y.

Define a map $F_w : S_w \longrightarrow Y$, by:

$$(F_w u_w)(t) = (t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s, u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau, u_w(\tau))d\tau\right]ds + \int_{s=0}^{t} T(t-s)Bw(s)ds,$$

For arbitrary $w(.) \in L^p([t_0, r): O)$....... (9)

Let u_w be arbitrary element in S_w such that $F_w u_w \in F_w(S_w)$, we must prove $F_w u_w \in S_w$.
From (8), and the definition of the map F_w,notice that $F_w u_w \in Y$.
and $(F_w u_w)(0) = u_0$ by (9), to prove $(F_w u_w)(t) \in \mathcal{B}_\rho(u_0)$, for any $u_w \in S_w$
From the definition of the closed ball $\mathcal{B}_\rho(u_0)$, notice that $(F_w u_w)(t) \in X$ { the definition of the Banach space Y} and

$$\|(F_w u_w)(t) - u_0\|_X = \|T(t)u_0 - u_0 + \int_{s=0}^{t} T(t-s)Bw(s)\,ds + \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right]ds\|$$

By using the conditions (4), (5) and (7), we get:

$$\|(F_w u_w)(t) - u_0\|_X \le \rho' + MK_0K_1t_1 + MN_1t_1 + Mh_{t_1}N_2\,t_1 \Rightarrow$$

$$\|(F_w u_w)(t) - u_0\|_X \le \rho' + (K_0K_1 + N_1 + h_{t_1}N_2)\,M\,t_1$$

,By using the assumption (6.i),we get:
$\|(F_w u_w)(t) - u_0\|_X \le \rho$, for $0 \le t \le t_1$, i.e., $(F_w u_w)(t) \in \mathcal{B}_\rho(u_0)$, for $0 \le t \le t_1$
Hence $F_w u_w \in S_w$, for arbitrary $u_w \in S_w$, which implies that $F_w : S_w \longrightarrow S_w$
So one can select the time t_1 such that:

$$t_1 = \min\left\{t', t'', r, \frac{\rho-\rho'}{(K_0K_1 + N_1 + h_{t_1}N_2)M}\right\}$$

To complete the poof, we have to show that $F_w : S_w \longrightarrow S_w$ is a continuous map:
Given $\|u_w^n - u_w\|_Y \longrightarrow 0$, as $n \longrightarrow \infty$, to prove $\|F_w u_w^n - F_w u_w\|_Y \longrightarrow 0$, as $n \longrightarrow \infty$
Notice that:

$$\|F_w u_w^n - F_w u_w\|_Y = \sup_{0\le t\le t_1} \|(F_w u_w^n)(t) - (F_w u_w)(t)\|_X \Rightarrow$$

$$\|F_w u_w^n - F_w u_w\|_Y = \sup_{0\le t\le t_1} \|T(t)u_0 + \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} T(t-s)\left[f(s,u_w^n(s)) + \int_{\tau=t_0}^{s} h(s-\tau)g(\tau,u_w^n(\tau))\,d\tau\right]ds - T(t)u_0 - \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds - \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right]ds\|$$

After simple calculatios and using the conditions (4), (5) and (7), we get:

$$\|F_w u_w^n - F_w u_w\|_Y \le M\left[L_0 + h_{t_1}L_1\right]\|u_w^n - u_w\|_Y\, t_1,$$

since $\|u_w^n - u_w\|_Y \to 0$, as $n \to \infty \Rightarrow$

$\lim_{n\to\infty} \|F_w u_w^n - F_w u_w\|_Y = 0$,i.e.

$\|F_w u_w^n - F_w u_w\|_Y \to 0$, as $n \to \infty$

Now, assume that $\tilde{S} = F_w(S)$, and for fixed t $\in [0, t_1]$, let $\tilde{S}(t) = \{(F_w u_w)(t) : u_w \in S_w\}$.
To show that $\tilde{S}(t)$ is a precompact set for every fixed $t \in [0, t_1]$,
For $t = 0 \Rightarrow \tilde{S}(0) = \{(F_w u_w)(0) : u_w \in S_w\} = \{u_0\}$ which is a precompact set in X. Now for $t > 0$, $0 < \varepsilon < t$, define:

$$(F_w^\varepsilon u_w)(t) = T(t)u_0 + \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right]ds + \int_{s=0}^{t-\varepsilon} T(t-s)(Bw)(s)\,ds,$$

For arbitrary $u_w \in S_w$
$\Rightarrow$

$$(F_w^\varepsilon u_w)(t) = T(t)u_0 + T(\varepsilon)\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[\begin{array}{l} f(s,u_w(s)) + \\ \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau + Bw(s)\end{array}\right]ds \quad \text{.......... (10)}$$

From the compactness of the semigroup $\{T(t)\}_{t\ge0}$ and use equation (10) one can get for every ε, $0 < \varepsilon < t$,
Set $\tilde{S}_\varepsilon(t) = \{(F_w^\varepsilon u_w)(t) : u_w \in S_w\}$is precompact set .
Moreover for any $u_w \in S_w$, we have:

$$\|(F_w u_w)(t) - (F_w^\varepsilon u_w)(t)\|_X = \|T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right]ds + \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds - T(t)u_0 - \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))\,d\tau\right]ds - \int_{s=0}^{t-\varepsilon} T(t-s)(Bw)(s)\,ds\|$$

After simple calculatios and using the conditions (4), (5) and (7), we get:

$$\|(F_w u_w)(t) - (F_w^\varepsilon u_w)(t)\|_X \le (N_1 + h_{t_1}N_2 + K_0K_1)M\varepsilon \Rightarrow$$

$\|(F_w u_w)(t) - (F_w^\varepsilon u_w)(t)\|_X \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, i.e., $\lim_{\varepsilon\to\infty} (F_w^\varepsilon u_w)(t) = (F_w u_w)(t)$

Which imply that $\tilde{S}(t)$ is precompact set in X, for every fixed $t > 0$ {see [3], [7]}.

To prove that $\tilde{S} = F_w(S_w)$ is an equicontinuous family of functions, we have:

$\|(F_w u_w)(r_1)-(F_w u_w)(r_2)\|_X \le \|(T(r_1) - T(r_2))u_0\|_X + M\, K_0 K_1 (r_1 - r_2) + M(N_1 + h_{t_1} N_2)(r_1 - r_2)$

Since $\{T(t)\}_{t\ge 0}$ is a compact semigroup which implies T(t) is continuous in the uniform operator topology for $t > 0$, therefore the right hand side of the above inequality tends to zero as $r_1 - r_2$ tends to zero. Thus $\tilde{S}$ is equicontinuous family of functions. It follows from the theorem "Arzela-Ascoli's theorem" that is $\tilde{S} = F_w(S)$ be relatively compact in Y and by Applying "Schauder fixed point theorem", which implies $F_w : S_w \longrightarrow S_w$ has a fixed point, i.e., $F_w u_w = u_w$, Hence the initial value control problem given by equation (5) has a local mild solution $u_w \in C([0,t_1]:X)$.

To show the uniqueness,

Let $\bar{\bar{u}}_w(t)$, $\bar{u}_w(t)$ be two local mild solutions of the initial value control problem given by equation (5) on the interval $[0, t_1]$. We must prove:

$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X = 0$,

Assume $\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \ne 0$, notice that:

$$\|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X = \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} T(t-s)\left[f(s,\bar{\bar{u}}_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{\bar{u}}_w(\tau))\,d\tau\right] ds - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,\bar{u}_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\bar{u}_w(t))\,d\tau\right] ds - \int_{s=0}^{t} T(t-s)(Bw)(s)\,ds \Bigg\|_X$$

$\Rightarrow \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X \le M(L_0 + h_{t_1} L_1)\, \|\bar{\bar{u}}_w - \bar{u}_w\|_Y\, t_1$, By using assumption (6.ii)

$$\Rightarrow \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X < M(L_0 + h_{t_1} L_1) * \frac{1}{M(L_0 + h_{t_1} L_1)} \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$$

$\Rightarrow \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X < \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$

By taking the suprumun over $[0, t_1]$ of the both sides of the above inequality, we get:

$\|\bar{\bar{u}}_w - \bar{u}_w\|_Y < \|\bar{\bar{u}}_w - \bar{u}_w\|_Y$, which implies to a contradiction

$\Rightarrow \|\bar{\bar{u}}_w(t) - \bar{u}_w(t)\|_X = 0 \Rightarrow \bar{\bar{u}}_w(t) = \bar{u}_w(t)$ for $0 \le t \le t_1$.

Hence we have a unique local mild solution $u_w \in C([0, t_1]:X)$, for arbitrary $w(.) \in L^p([0, t_1): O)$.

So one can select $t_1 > 0$ such that:

$$t_1 = \min\left\{t', t'', r, \frac{\rho - \rho'}{(K_0 K_1 + N_1 + h_{t_1} N_2)M}, \frac{1}{M(L_0 + L_1 h_{t_1})}\right\}$$

It should be notice that the controllable of the local mild solution to the semilinear initial value control problem equation (5) will be developed by using the following assumptions:

1. A be the infinitesimal generator C_0 compact semigroup $\{T(t)\}_{t\ge 0}$, where A defined from $D(A) \subset X$ into X. where X be a Banach space.
2. For $\rho > 0$, we define $\mathcal{B}_\rho(u_0) = \{x \in X \mid \|x - u_0\|_X \le \rho\}$, where $u_0 \in U$ (open subset of X),The nonlinear maps f, g define from $[0,r) \times U$ into X, satisfy the local Lipschitz condition with respect to the second arguments , i.e.
   $\|f(t, v_1) - f(t, v_2)\|_X \le L_0\|v_1 - v_2\|$ and
   $\|g(t, v_1) - g(t, v_2)\|_X \le L_1\|v_1 - v_2\|$
   For $0 \le t \le t_1$ and $v_1, v_2 \in \mathcal{B}_\rho(u_0)$ and L_0, L_1 is a Lipschitz constants.
3. h is continuous function which at least $h \in L^1([0,t_1):R)$,Where R is the real number.
4. Let $t' > 0$ such that $\|f(t,a)\|_X \le N_1$, $\|g(t,a)\|_X \le N_2$, for $0 \le t \le t'$ and $a \in \mathcal{B}_\rho(u_0)$. Also let $t'' > 0$ such that $\|T(t)u_0 - u_0\|_X \le \rho'$ for $0 \le t \le t''$ and $u_0 \in U$, where ρ' is a positive constant such that $\rho' < \rho$.
5. $w(.)$ be an arbitrary control function is given in $L^p([0,t_1]:O)$, a Banach space of control function with O as a reflexive Banach space and here B is a bounded linear operator from O into X.
6. The linear operator G from O into X defined by:
   $Gw(\gamma) = \int_{s=0}^{\gamma} T(\gamma - s)Bw(s)\,ds, \ \forall\, w(.) \in L^p([0,\gamma):O)$.
   Induces an invertible operator $\tilde{G}$ defined on O/kerG.

7. There exist a positive constant I1, such that $\|\tilde{G}-1\| \le I1$.
8. Let $\gamma=\min\{t', t'', t1\}$ and satisfy the following conditions: (8.i) $\gamma \le \dfrac{\rho-\rho'-I_0I_1(\|v\|+M\|u_0\|)}{(1+I_0I_1)M(N_1+h_\gamma N_2)}$ and

(8.ii) $\gamma < \dfrac{1}{(L_0+h_\gamma L_1)(1+I_0I_1)M}$

Remark

The condition (6) in our assumptions can be satisfied {see appendix}.

Theorem (2)

Assume that the hypotheses (1)-(8) are hold. Then for every u_0, $v_0 \in V \subseteq U$, there exists a fixed number,γ, $0<\gamma<t_1$, such that (6) is exactly controllable on $J_0 = [0,\gamma]$.

Proof

Using the condition (6), define the control:

$$\underline{w}(t)=\tilde{G}^{-1}\left[v_0-T(t)u_0-\int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u(\tau))d\tau\right]ds\right] \quad ....(11)$$

Define the following map, given by:

$$(\phi_w u_w)(t) = T(t)u_0+\int_{s=0}^{t} T(t-s)\left[(Bw)(s)+f(s,u_w(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds, \ \forall w(.)\in L^p([0, t_1): O)$$

....(12)

By using (11) and (12), we have to show that $\phi_w u_w$ has a fixed point.

We can rewrite equation (12) as follows:

$$(\phi_w u_w)(t)=T(t)u_0+\int_{s=0}^{t} T(t-s)\left[f(s,u_w(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_w(\tau))d\tau\right]ds + L(t)Bw(t) \quad ,\forall w(.) \in L^p([0, t_1): O)$$

..........(13)

By using (11) and (13), we obtain:

$$(\phi_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds$$
$$+L(t)B\tilde{G}^{-1}\left[v_0-T(t)u_0-\int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds\right],$$

For $\underline{w} \in L^p([0, t_1):O)$.

There exist $M\ge 0$, such that $\|T(t)\| \le M$, for $0\le t\le t_1$ {since T(t) is a bounded linear operator on X}.

Let $\rho > 0$ be such that $\mathcal{B}_\rho(u_0) = \{x\in X : \|x - u_0\|_x \le \rho\} \subset U$ {since U is an open subset of X}.To guarantee the fixed point property, we have done as follow:

Assume $h_r = \int_{s=0}^{r} |h(s)|\, ds$

Set $Z=C(J_0 :X)$, where Z is a Banach space with the supremum defined as follows:

$$\|z\|_Z = \sup_{0\le t\le\gamma} \|z(t)\|_x$$

And define $Z_0 = \{u_{\underline{w}} \in Z : u_{\underline{w}}(0) = u_0$, $u_{\underline{w}}(t)\in\mathcal{B}_\rho(u_0)$, for $0\le t \le \gamma\}$

It is clearly Z_0 is bounded, Closed and convex subset of Z.

Define a nonlinear map $\phi_{\underline{w}} : Z_0 \longrightarrow Z$, by:

$$(\phi_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds$$
$$+ (t)B\tilde{G}^{-1}\left[v_0-T(t)u_0-\int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds\right]$$

, for $\underline{w} \in L^p([0, t_1):O$....................... (14)

Let $u_{\underline{w}}$ be arbitrary element in Z_0 such that $\phi_{\underline{w}} u_{\underline{w}} \in \phi_{\underline{w}}(Z_0)$, to prove $\phi_{\underline{w}} u_{\underline{w}} \in Z_0$ for arbitrary element $u_{\underline{w}} \in Z_0$. from the definition of Z_0 , notice that $\phi_{\underline{w}} u_{\underline{w}} \in Z$ {the definition of $\phi_{\underline{w}}$ } and $(\phi_{\underline{w}} u_{\underline{w}})(0) = u_0$ {by equation(14)}, to prove $(\phi_{\underline{w}} u_{\underline{w}})(t)\in \mathcal{B}_\rho(u_0)$, for $0\le t \le \gamma$. From the definition of the closed ball $\mathcal{B}_\rho(u_0)$, notice that $(\phi_{\underline{w}} u_{\underline{w}})(t)\in X$ and

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X = \Big\| T(t)u_0 - u_0 + \int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds + L(t)B\tilde{G}^{-1}\Big[v_0-T(t)u_0-\int_{s=0}^{t} T(t-s)\Big[f(s,u_{\underline{w}}(s))+\int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\Big]ds\Big]\Big\|_X$$

After simple calculations and using conditions (4) and (7), we get:

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho' + [1+I_0I_1](N_1+h_\gamma N_2)M\gamma + I_0I_1[M\,\|u_0\| + \|v_0\|]$$

By using the condition (8.i), we get:

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - u_0\|_X \le \rho$$

Therefore $\phi_{\underline{w}} u_{\underline{w}} \in Z_0$, for any $u_{\underline{w}} \in Z_0 \Rightarrow$

$\phi_{\underline{w}} : Z_0 \longrightarrow Z_0$

So, one can select $\gamma > 0$, such that:

$$\gamma = \text{Min}\left\{t', t'', t_1, \frac{\rho - \rho' - I_0I_1(\|v_0\| + M\|u_0\|)}{(1+I_0I_1)M(N_1 + h_\gamma N_2)}\right\}$$

To complete the proof, to show that $\phi_{\underline{w}} : Z_0 \longrightarrow Z_0$ is a continuous map

Given $\|u^n_{\underline{w}} - u_{\underline{w}}\|_Z \longrightarrow 0$, as $n \longrightarrow \infty$, To prove $\|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z \longrightarrow 0$, s $n \longrightarrow \infty$

Notice that:

$$\|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z = \sup_{0\le t\le\gamma} \|(\phi_{\underline{w}} u^n_{\underline{w}})(t) - (\phi_{\underline{w}} u_{\underline{w}})(t)\|_X$$

$$\Rightarrow \|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z = \sup_{0\le t\le\gamma} \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u^n_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u^n_{\underline{w}}(\tau))d\tau\right]ds + L(t)B\tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,u^n_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u^n_{\underline{w}}(\tau))d\tau\right]ds\right] - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds - L(t)B\tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds\right]\Bigg\|_X$$

After simple calculations and using conditions (2), (4) and (7), we get:

$$\|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z \le (1+I_0I_1)M\int_{s=0}^{\gamma}\left[L_0\|u^n_{\underline{w}} - u_{\underline{w}}\|_Z + h_\gamma L_1\|u^n_{\underline{w}} - u_{\underline{w}}\|_Z\right]ds$$

$$\Rightarrow \|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z \le (1+I_0I_1)(L_0 + h_\gamma L_1)\|u^n_{\underline{w}} - u_{\underline{w}}\|_Z\,\gamma$$

Since $\|u^n_{\underline{w}} - u_{\underline{w}}\|_Z \to 0$, as $n \to \infty$, $\Rightarrow$

$$\lim_{n\to\infty} \|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z = 0,$$

i.e. $\|\phi_{\underline{w}} u^n_{\underline{w}} - \phi_{\underline{w}} u_{\underline{w}}\|_Z \to 0$, as $n \to \infty$.

Assume $\tilde{R} = \phi_{\underline{w}}(Z_0)$, let $\tilde{R}(t) = \{(\phi_{\underline{w}} u_{\underline{w}})(t) : u_{\underline{w}} \in Z_0\}$, to show that $\tilde{R}(t)$ is a precompact set in X, for every fixed $t \in J_0$, when $t = 0$

$\Rightarrow \tilde{R}(0) = \{(\phi_{\underline{w}} u_{\underline{w}})(0) : u_{\underline{w}} \in Z_0\} = \{u_0\}$ which is a precompact set in X.

Now, for $t > 0$, $0 < \varepsilon < t$, define:

$$(\phi^{\varepsilon}_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds + \int_{s=0}^{t-\varepsilon} T(t-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds$$

$$\Rightarrow (\phi^{\varepsilon}_{\underline{w}} u_{\underline{w}})(t) = T(t)u_0 + T(\varepsilon)\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds + T(\varepsilon)\int_{s=0}^{t-\varepsilon} T(t-s-\varepsilon)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds \quad (15)$$

From the compactness of the semigroup $\{T(t)\}_{t\ge0}$ and equation (15) which implies that for any ε, $0<\varepsilon< t$.

The set $\tilde{R}_{\varepsilon}(t) = \{(\phi^{\varepsilon}_{\underline{w}} u_{\underline{w}})(t) : u_{\underline{w}} \in Z_0\}$ is precompact set in X.

Moreover for any $u_{\underline{w}} \in Z_0$, notice that:

$$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - (\phi^{\varepsilon}_{\underline{w}} u_{\underline{w}})(t)\|_X = \Bigg\| T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds + \int_{s=0}^{t} T(t-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds - T(t)u_0 - \int_{s=0}^{t-\varepsilon} T(t-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds - \int_{s=0}^{t-\varepsilon} T(t-s)B\tilde{G}^{-1}\left[v_0 - T(s)u_0 - \int_{\theta=0}^{s} T(s-\theta)\left[f(\theta,u_{\underline{w}}(\theta)) + \int_{\tau=0}^{\theta} h(\theta-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]d\theta\right]ds\Bigg\|_X$$

After simple calculations and using conditions (2), (4) and (7), we get:

$\|(\phi_{\underline{w}} u_{\underline{w}})(t) - (\phi_{\underline{w}}^{\varepsilon} u_{\underline{w}})(t)\|_X \le M(N_1+h_\gamma N_2)\,\varepsilon + MK_0 I_1[\|v_0\| + M\|u_0\|]\,\varepsilon + M^2K_0I_1(N_1+h_\gamma N_2)\left(t\varepsilon-\frac{\varepsilon^2}{2}\right)$

$\Rightarrow \|(\phi_{\underline{w}} u_{\underline{w}})(t) - (\phi_{\underline{w}}^{\varepsilon} u_{\underline{w}})(t)\|_X \longrightarrow 0$, as $\varepsilon \longrightarrow 0$.

Which implies that $\tilde{R}(t)$ is precompact set in X for every fixed t > 0 {see [3], [7]}.

To prove that $\tilde{R} = \phi_{\underline{w}}(Z_0)$ is an equicontinuous family of functions.

Notice that: $\|(\phi_{\underline{w}} u_{\underline{w}})(r_1) - (\phi_{\underline{w}} u_{\underline{w}})(r_2)\|_X \le$

$\|(T(r_1)-T(r_2))u_0\|_X + M(N_1+h_{r_1}N_2)(r_1-r_2) + MK_0I_1[\|v_0\|+M\|u_0\|](r_1-r_2) + \frac{M^2K_0I_1(N_1+h_{t_1}N_2)}{2}(r_1^2-r_2^2)$

Since $\{T(t)\}_{t\ge 0}$ is a compact semigroup, which implies T(t) is continuous in the uniform operator topology for t > 0, therefore the right hand side tends to zero as $r_1 - r_2$ tends to zero.Thus $\tilde{R}$ is equicontinuous family of functions.It follows from the theorem "Arzela-Ascoli's theorem" that is $\tilde{R} = \phi_{\underline{w}}(Z_0)$ be relatively compact in Z.

By applying "schauder fixed point theorem", which implies $\phi_{\underline{w}}$ has a fixed point, i.e.

$\phi_{\underline{w}} u_{\underline{w}} = u_{\underline{w}}$ .To verifies that the uniqueness:

Let $\bar{\bar{u}}_{\underline{w}}(t)$ and $\bar{u}_{\underline{w}}(t)$ be two mild solution of equation (5) on the interval J_0, we must prove that $\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X = 0$.Assume That

$\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X \ne 0$. Notice that:

$$\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X = \Big\| T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[f(s,\bar{\bar{u}}_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau)(\tau))d\tau\right]ds + L(t)B\tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,\bar{\bar{u}}_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{\bar{u}}_{\underline{w}}(\tau))d\tau\right]ds\right] - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,\bar{u}_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{u}_{\underline{w}}(\tau))d\tau\right]ds - L(t)B\int_{s=t_1}^{t} T(t-s)\left[(Bw)(s) + f(s,v_w(s))\right] + L(t)B\tilde{G}^{-1}\left[v_0 - T(t)u_0 - \int_{s=0}^{t} T(t-s)\left[f(s,\bar{u}_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,\bar{u}_{\underline{w}}(\tau))d\tau\right]ds\right]\Big\|_X$$

After simple calculations and using conditions (2), (4) and (7), we get:

$\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X \le (L_0+h_\gamma L_1)(1+I_0I_1)M\,\|\bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}}\|_Z\,\gamma$

Take the spremum over $0\le t\le\gamma$ of the above inequality, we obtain:

$\sup_{0\le t\le\gamma}\|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X \le (L_0+h_\gamma L_1)(1+I_0I_1)M\,\|\bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}}\|_Z\,\gamma$

$\|\bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}}\|_Z \le (L_0+h_\gamma L_1)(1+I_0I_1)M\,\|\bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}}\|_Z\,\gamma$

By using the condition (8.ii), we get:

$\|\bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}}\|_Z < \|\bar{\bar{u}}_{\underline{w}} - \bar{u}_{\underline{w}}\|_Z$, which get a contradiction

$\Rightarrow \|\bar{\bar{u}}_{\underline{w}}(t) - \bar{u}_{\underline{w}}(t)\|_X = 0 \Rightarrow \bar{\bar{u}}_{\underline{w}}(t) = \bar{u}_{\underline{w}}(t)$, for $0\le t\le\gamma$

Therefore, we have a unique local mild solution $u_{\underline{w}} \in C(J_0:X)$

So one can select the time γ Such that:

$\gamma = \mathrm{Min}\left\{t', t'', t_1, \frac{\rho-\rho'-I_0I_1(\|v_0\|+M\|u_0\|)}{(1+I_0I_1)M(N_1+h_\gamma N_2)}, \frac{1}{(L_0+h_\gamma L_1)(1+I_0I_1)M}\right\}$

Notice that $(\phi_{\underline{w}} u_{\underline{w}})(0) = u_0$ and

$$(\phi_{\underline{w}} u_{\underline{w}})(\gamma) = T(\gamma)u_0 + \int_{s=0}^{\gamma} T(\gamma-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds + L(\gamma)B\tilde{G}^{-1}\left[v_0 - T(\gamma)u_0 - \int_{s=0}^{\gamma} T(\gamma-s)\left[f(s,u_{\underline{w}}(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,u_{\underline{w}}(\tau))d\tau\right]ds\right]$$

$\Rightarrow (\phi_{\underline{w}} u_{\underline{w}})(\gamma) = v_0$, Thus equation (6) is exactly controllable on J_0.

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Appendix

(construction of $\tilde{W}$) {[9], [10]}:

Define a linear operator $\tilde{W}$: O/Ker W $\longrightarrow$ X, by:

$\tilde{W}[w(t)] = Ww(t), w(t) \in [w(t)]$

$\tilde{W}$ is one-to-one

Since $\tilde{w}[\bar{\bar{w}}(t)] = \tilde{w}[\bar{w}(t)], \forall[\bar{\bar{w}}(t)], [\bar{w}(t)] \in O/\,KerW$

$\Rightarrow W\bar{\bar{w}}(t) = W\bar{w}(t), \forall \bar{\bar{w}}(t) \in [\bar{\bar{w}}(t)], \bar{w}(t) \in [\bar{w}(t)]$

$\Rightarrow W\bar{\bar{w}}(t) - W\bar{w}(t) = 0 \Rightarrow$

$W(\bar{\bar{w}}(t) - \bar{w}(t)) = 0$

$\Rightarrow \bar{\bar{w}}(t) - \bar{w}(t) \in KerW$

$\Rightarrow \bar{\bar{w}}(t) \in [\bar{w}(t)]$ {Since $[\bar{w}(t)] =$

$\{\bar{\bar{w}}(t) \in O: \bar{\bar{w}}(t) - \bar{w}(t) \in KerW\}$

$\Rightarrow [\bar{\bar{w}}(t)] = [\bar{w}(t)]$

So, there exist $\tilde{W}^{-1}$ defined from V into O/Ker W.

To prove Range W=V is a Banach spaces via the norm defined as follow:

$$\|v\|_V = \|\tilde{W}^{-1}v\|_{O/\,KerW}$$

Notice that:

$\|Ww(t)\|_V = \|\tilde{W}^{-1}Ww(t)\|_{O/\,KerW} = \|\tilde{W}^{-1}W[w(t)]\|_{O/\,ker\,W}, \forall w(t) \in [w(t)]$

$=$

$\|[w(t)]\|_{O/\,KerW} = \inf_{w(t)\in[w(t)]} \|w(t)\|_O \le \|w(t)\|_O, \forall w(t) \in O$

So, W is a bounded linear operator for $0 \le t \le \gamma$.

And $\|\tilde{W}[w(t)]\|_X = \|Ww(t)\|_X, \forall w(t) \in [w(t)]$

$\Rightarrow \|\tilde{W}[w(t)]\|_X \le \|W\|\|w(t)\|_O, \forall w(t) \in [w(t)]$

$\Rightarrow$

$\|\tilde{W}[w(t)]\|_X$

$\le \|W\| \inf_{w(t)\in[w(t)]} \|w(t)\|_O = \|W\|\|[w(t)]\|_{O/\,ker\,W}$

$\Rightarrow \|\tilde{W}[w(t)]\|_X \le \|W\|\|[w(t)]\|_{O/\,ker\,W}$.

(1) Assistance proof in mathematics and computer applications, Department of mathematics, College of Science, AL–Nahrain University, Baghdad, Iraq.

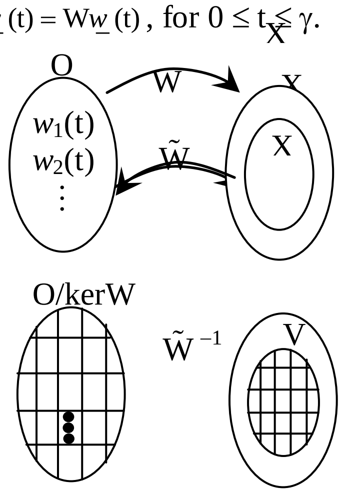
Since $\tilde{W}$ is bounded and D($\tilde{W}$) = O/KerW is closed which implies that $\tilde{W}^{-1}$ is closed

Since $\tilde{W}^{-1}$ is closed operator and by the norm $\|v\|_V = \|\tilde{W}^{-1}v\|_{O/\,KerW}$, which implies that

V = RangeW a Banach space {[9]}.Since O is reflexive Banach space and KerW is weakly closed, So the infimum is actually attained, we can choose a control function $\underline{w}(t) \in [w(t)]$ such

that $\underline{w}(t) = \tilde{W}^{-1}W\underline{w}(t)$,{see[9],[10]}.

$\Rightarrow \tilde{W}\underline{w}(t) = W\underline{w}(t)$, for $0 \le t \le \gamma$.



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المستخلص :

الهدف من هذا البحث هو أثبات وجود, وحدانية وقابلية السيطرة للحل العام (محلي) لمسألة سيطرة شبه خطية ذات قيمة أبتدائية في فضاء باناخ مناسب بأستخدام منهج شبه الزمرة (شبه الزمرة المتراصة) و نظرية النقطة الثابتة لشويدر