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ANALYTICAL SOLUTION OF PARTIAL FRACTIONAL ORDER DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we extended a theory for fractional derivatives of many variables based on the definitions of the fractional calculus of single variables. The implementation of the proposed approach for solving the partial fractional differential equation is presented.

Introduction

Partial differential equations are the heart of many, if not most, computational analysis or simulations of continuous physical systems, such as fluids, electromagnetic fields, the human body ,and so on Indeed PDE 's are usually classified as elliptic, hyperbolic or parabolic according to the form of the equation and the form of the subsidiary conditions which must be assigned to produce a well–posed problem, for example the Laplace equation…etc. but we often numerically due to the fact that analytic (closed form) solutions are usually not available

The differential equation involving derivatives of non-integer order have shown to be adequate models for various physical phenomena in areas like damping laws, diffusion processes, etc. The use of fractional calculus of modeling physical system has been widely considered in the last decades [1].

There are many numerical algorithms for the solution of partial fractional differential equation (PFDE's) with constant coefficients such as, generalization of the definition of Green's function, and fractional difference method, which are based on the approximation of the two definitions of fractional derivatives, Granwald-Letnikov (GL) definition, and the Remann-Lioville (RL) definition, in which all these methods are complicated and time consuming , see[1] , [4]&[5] . In this paper, we are constructing an extended some properties and theories of PFDE's, for many variables, based on the theory of the fractional calculus of single variable, see [2] [3] & [6], to convert PFDE's to non fractional partial differential equations.

Generalization Theory

In [6], it had been considered that a function of two variables f (x,t) for $-\infty<x<\infty$, $t \geq 0$ to the complex numbers, in which its fractional derivatives in x based on the Fourier transform, which is equivalent to the RL-function derivative[8].

$$\frac{\partial^\alpha}{\partial x^\alpha} f(x,t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^{\infty} \frac{f(y,t)dy}{(x-y)^{\alpha-n+1}} \quad \dots (1)$$

Where n= [α] +1, [α] is the largest integer not greater than α.

We begin by declaring that the set of functions to which we shall apply "partial fractional differential operator" is the set of all real valued function on

$$\Omega = \{(x_1 \dots x_n) : a_i < x_i \leq b_i, -\infty < a_i < \infty, -\infty < b_i < \infty, i = 1,\dots,n\}$$

Now, we are extended the definition of equation (1) which based on the theory in [1], [4] into following:

Definition (1):

The partial fractional derivative ${}_0D^\alpha_{x_i}$ of order α > 0 of absolutely continuous function u ($\bar{x}$) is defined as

$$\begin{aligned}
\frac{\partial^\alpha}{\partial x_i^\alpha} u(\bar{x}) &= \tfrac{1}{\Gamma(n-\alpha)} \int_0^{x_i} (x_t - t)^{-\alpha} u_{x_i}(x_1, \dots x_{i-1}, t, \dots, x_n) dt \\
&= I_{x_t}^{n-\alpha} \frac{\partial^n}{\partial x_i^n} u(\bar{x}) \\
&= \frac{x_i^{n-\alpha-1}}{\Gamma(n-\alpha)} u_{x_i}(\bar{x}), for\ x \geq 0, \\
&= 0,\ for\ x \leq 0
\end{aligned} \quad \dots\dots\dots\dots (2)$$

where $\bar{x} = (x_1 \dots x_n)$, and the operator ${}_0D^\alpha_{x_i}$ is a mapping from
C (Ω) "the set of all continuous real valued function on Ω" into C (Ω).
The following two properties are needed.
Property (1):

The operator ${}_0D^{\alpha}_{x_i}$, α > 0, is linear.

Proof :

Let u, v ∈ C (Ω), $r_1, r_2 \in R$, n=[α]+1

$${}_0D^{\alpha}_{x_i}(r_1u(\bar{x})+r_2v(\bar{x})) = {}_0I^{n-\alpha}_{x_i}\frac{\partial^n}{\partial x_i^n}(r_1u(\bar{x})+r_2v(\bar{x}))$$

$$= {}_0I^{n-\alpha}_{x_i}(r_1u_{x_i}(\bar{x})+r_2v_{x_i}(\bar{x}))$$

(since $I^{n-\alpha}_{x_i}$ Is linear see [6])

$$= r_1\,{}_0I^{n-\alpha}_{x_i}u_{x_i}(\bar{x}) + r_2\,{}_0I^{n-\alpha}_{x_i}v_{x_i}(\bar{x})$$

$$= r_1\,{}_0D^{\alpha}_{x_i}u(\bar{x}) + r_2\,{}_0D^{\alpha}_{x_i}v(\bar{x}).$$

<u>Property (2)</u>:

If u ∈ C (Ω) `with 0 < α < 1 and 0<β< 1, then

$${}_0D^{\alpha}_{x_i}\,{}_0D^{\beta}_{x_i}u(\bar{x}) = {}_0D^{\beta}_{x_i}\,{}_0D^{\alpha}_{x_i}u(\bar{x})$$

$$= {}_0D^{\alpha+\beta}_{x_i}u(\bar{x})$$

Proof :

$${}_0D^{\alpha}_{x_i}\,{}_0D^{\beta}_{x_i}u(\bar{x}) = {}_0D^{\alpha}_{x_i}({}_0I^{1-\beta}_{x_i}u_{x_i})$$

$$= {}_0I^{1-\alpha}_{x_i}(\frac{\partial}{\partial x_i}({}_0I^{1-\beta}_{x_i}u_{x_i}))$$

$$= {}_0I^{1-\alpha}_{x_i}({}_0I^{1-\beta}_{x_i}\frac{\partial}{\partial x_i}u_{x_i})$$

$$= {}_0I^{1-\alpha}_{x_i}({}_0I^{1-\beta}_{x_i}u_{x_ix_i})$$

$$= {}_0I^{1-\alpha+1-\beta}_{x_i}u_{x_ix_i}$$

$$= {}_0I^{2-\alpha-\beta}_{x_i}$$

$${}_0D^{\beta}_{x_i}\,{}_0D^{\alpha}_{x_i}u(\bar{x}) = {}_0D^{\beta}_{x_i}({}_0I^{1-\alpha}_{x_i}u_{x_i})$$

$$= {}_0I^{1-\beta}_{x_i}(\frac{\partial}{\partial x_i}({}_0I^{1-\alpha}_{x_i}u_{x_i}))$$

$$= {}_0I^{1-\beta}_{x_i}({}_0I^{1-\alpha}_{x_i}\frac{\partial}{\partial x_i}u_{x_i})$$

$$= {}_0I^{1-\beta}_{x_i}({}_0I^{1-\alpha}_{x_i}ux_ix_i)$$

$$= {}_0I^{1-\beta+1-\alpha}_{x_i}u_{x_ix_i}$$

$$= {}_0I^{2-\beta-\alpha}_{x_i}u_{x_ix_i}$$

Now we can state the following proposition.

<u>Proposition</u>:

Let $u(\bar{x})$ & $v(\bar{x}) \in C(\Omega), D^{\alpha}_{X_I}$ be the partial fractional derivative of order $\alpha \in (0,1], and \lambda$ is any real number. Then

(i) $D^{\alpha}_{x_i}(u+v)(\bar{x}) = D^{\alpha}_{x_i}u(\bar{x}) + D^{\alpha}_{x_i}v(\bar{x})$

(ii) $D^{\alpha}_{x_i}(\lambda u)(\bar{x}) = \lambda D^{\alpha}_{x_i}u(\bar{x})$

(iii) $D^{\alpha}_{x_i}(uv)(\bar{x}) = v(\bar{x})D^{\alpha}_{x_i}u(\bar{x}) + u(\bar{x})D^{\alpha}_{x_i}v(\bar{x})$

Proof of (i) and (ii) are trivial from property (1) .

Proof (iii): using definition (1)

$$D^{\alpha}_{x_i}(uv)(\bar{x}) = \frac{x_i^{-\alpha}}{\Gamma(1-\alpha)}\frac{\partial}{\partial x_i}(uv)(\bar{x})$$

$$= \frac{x_i^{-\alpha}}{\Gamma(1-\alpha)}[v(\bar{x})u_{x_i}(\bar{x}) + u(\bar{x})v_{x_i}(\bar{x})]$$

$$= v(\bar{x})\frac{x_i^{\alpha}}{\Gamma(1-\alpha)}u_{x_i}(\bar{x}) + u(\bar{x})\frac{x_i^{-\alpha}}{\Gamma(1-\alpha)}v_{x_i}(\bar{x})$$

$$= v(\bar{x})D^{\alpha}_{x_i}u(\bar{x}) + u(\bar{x})D^{\alpha}_{x_i}v(\bar{x})$$

From the above propositions and definitions, one can be calculate the following:

Let U ≡ U(x,y), α >0, β >0

$$D^{\alpha}_{xx}u(x,y) = I^{n-\alpha}(\frac{\partial^n}{\partial x^n}u(x,y)), n=[\alpha+1]$$

$$= \frac{x^{n-\alpha-1}}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}u(x,y) \quad ....(1)$$

Similarly

$$D^{\beta}_{xx}u(x,y) = I^{m-\beta}\frac{\partial^m}{\partial y^m}u(x,y), m=[\beta]+1$$

$$= \frac{y^{m-\beta-1}}{\Gamma(m-\beta)}\frac{\partial^m}{\partial y^m}u(x,y) \quad ....\ (2)$$

if 0 <α , β < 1

$$\frac{\partial^{\alpha,\beta}}{\partial x^{\alpha}\partial y^{\beta}} = \frac{\partial^{\alpha}}{\partial x^{\alpha}}(\frac{y^{-\beta}u_y}{\Gamma(1-\beta)})$$

$$= I^{\alpha-1}(\frac{\partial}{\partial x}(\frac{1}{\Gamma(1-\beta)}y^{-\beta}u_y))$$

$$= I^{\alpha-1}(\frac{1}{\Gamma(1-\beta)}y^{-\beta}u_{xy})$$

$$= \frac{x^{-\alpha}}{\Gamma(1-\alpha)}(\frac{1}{\Gamma(1-\beta)}y^{-\beta}u_{xy})$$

$$= \frac{x^{-\alpha}y^{-\beta}}{\Gamma(1-\alpha)\Gamma(1-\beta)}u_{xy}$$

..........(3)

Implementation:

Let us consider the following partial fractional differential equation

$$D_{xx}^{2\alpha}U(x,y) - D_y^{\beta}U(x,y) = 0, with \alpha,\beta \in (0,1] \quad ....(4)$$

With known given conditions:

Solution

$$Let U(x,y) = X(x)Y(y), then$$

$$D_x^{\alpha}U(x,y) = Y(y)D_x^{\alpha}X(x),$$

$$D_{xx}^{2\alpha}U(x,y) = D_x^{\alpha}(D_x^{\alpha}U(x,y)) = Y(y)D_{xx}^{2\alpha}X(x)$$

$$D_y^{\alpha}U(x,y) = X(x)D_y^{\alpha}Y(y)$$

Substitute them in the FPDE of equation (4); we have

$$Y(y)D_{xx}^{2\alpha}X(x) - X(x)D_y^{\beta}Y(y) = 0$$

$$\frac{D_x^{2\alpha}X(x)}{X(x)} = \frac{D_y^{\beta}Y(y)}{Y(y)} = k$$

Where k ≡ +ve integer

Therefore, we have two ordinary linear fractional differential equations

$$D_{xx}^{2\alpha}X(x) - kX(x) = 0..................................(4)$$

$$D_y^{\beta}Y(y) - kY(y) = 0...................................(5)$$

Now, in order to find the solution X(x)

Let X(x) = e^{rx} (see [7])

Substitute in (4), we obtain

$$\frac{x^{n-2\alpha-1}}{\Gamma(n-2\alpha)} r^n e^{rx} - Ke^{rx} = 0,$$

$$e^{rx}(\frac{x^{n-2\alpha-1}}{\Gamma(n-2\alpha)}.r^n - K) = 0$$

$$e^{rx}(\frac{r^n x^{n-2\alpha-1} - K\Gamma(n-2\alpha)}{\Gamma(n-2\alpha)}) = 0$$

Since $e^{rx} \neq 0$

$$r^n x^{n-2\alpha-1} - K\Gamma(n-2\alpha) = 0$$

if n =1 , $r = \frac{K\Gamma(n-2\alpha)}{x^{n-2\alpha-1}}$, then

$$X(x) = \exp(\frac{K\Gamma(n-2\alpha)}{x^{n-2\alpha-1}}).x$$

and if n=2, $r = \pm\sqrt[2]{\frac{K\Gamma(n-2\alpha)}{x^{n-2\alpha-1}}}$, then

$$X(x) = \exp(\pm\sqrt[2]{\frac{K\Gamma(n-2\alpha)}{x^{n-2\alpha-1}}}).x$$

Let Y(y) = e^{sy} , using definition (1):

$$\frac{\partial^{\beta}}{\partial y^{\beta}}e^{sy} = I^{1-\beta}(\frac{\partial}{\partial y}e^{sy}) = I^{1-\beta}(se^{sy}) = se^{sy}\frac{y^{-\beta}}{\Gamma(1-\beta)}$$

Substitute in (5)

$$e^{sy}(\frac{sy^{-\beta}}{\Gamma(1-\beta)} - K) = 0$$

Since

$$e^{sy} \neq 0 \Rightarrow (\frac{sy^{-\beta}}{\Gamma(1-\beta)} - K) = 0 \Rightarrow s = y^{\beta}K\Gamma(1-\beta)$$

$$\therefore Y(y) = \exp(K\Gamma(1-\beta)y^{\beta+1})$$

U=X(x).Y(y)

If n=1 , then

$$U = (\exp(\frac{K\Gamma(n-2\alpha)}{x^{n-2\alpha-1}}).x).\exp(K\Gamma(1-\beta)y^{\beta+1})$$

and if n=2 , then

$$U = (\exp(\pm\sqrt[2]{\frac{K\Gamma(n-2\alpha)}{x^{n-2\alpha-1}}}).x).\exp(K\Gamma(1-\beta)y^{\beta+1}$$

The value of K can be obtained by substituting the initial conditions.

References

[1]AGRAWAL, O.P.AND MACHADO, J.A.T.(eds) (2003);Fractional derivatives and their applications. American society of Mechanical Engineers.

[2] DAVISION, M.AND ESSEX, C. (1998). Fractional differential equations and initial value problems. Math. Scientist 23,108-116.

[3] I. Podlubny (1999); Fractional differential equations, Academic Press.

[4] S.Samko ,A.Kilbas ,and O Marichev (1993); Fractional integrals and Derivative Theory and Applications, Gordon & Breach, London /New York.

[5] Watanabe (1931); Notes on the generalized derivative of Riemann-Liouville And its application to Leibntz's formula,I I, Tokoku Math.J.34 ,8-41.

[6] X, Aorang Li. (2003);Fractional differential equations and stable distributions. Applied Probability.

[7] N.Saad (2004); "Some result in fractional calculas" Ph.D. Thesis, Baghdad University. Ibn-Al Haythim Education College.

[8] XIORA GLI, 2003, fractional differential equations and stable distribution, applied probability trust, No. 18.