# SOLUTION OF A SYSTEM OF LINEAR VOLTERA-INTEGRO DIFFERENTIAL EQUATIONS BY WEIGHTED RESIDUAL METHODS

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**Abstract**

In this paper we consider system of linear Volterra integro-differential equations of the second kind. Two methods are used to solve this system, collocation method and partition method. A comparison between approximate and exact results for two numerical examples depending on the least-square error is given, to show the accuracy of the results obtained by using these methods. For solving examples, we use MatLab program version 6.5.

**Keywords: collocation method, partition method, Volterra integro-differential equation**

## 1-Introduction

One of the uses of approximating functions is to replace complicated functions by some simpler functions so that many operations such as integration can be easily performed. Here, we approximate the unknown functions $u_i(x)$, $i=1, 2, \ldots, m$ by $S_{iN}(x)$ where

$$S_{iN}(x) = \sum_{k=0}^{N} c_{ik}\phi_k(x),\ i=1, 2, \ldots, m.$$

The unknown then being the expansion coefficients $c_{i0}, c_{i1}, \ldots, c_{iN}$, $i=1, 2, \ldots, m$ (which depends on N, as may the $\phi_k(x)$). An algorithm based on the above approximation is an expansion method. The prescribed basis functions $\phi_k(x)$ are important in expansion method. Moreover, it is natural to choose functions $\phi_k(x)$, $k=0, 1, \ldots, N$ which are linearly independent to insure that $S_{iN}(x)$ which is a linear combination of $\phi_k(x)$, then determines uniquely it's expansion coefficients $c_{i0}, c_{i1}, \ldots, c_{iN}$, $i=1, 2, \ldots, m$.

In order to consolidate the expansion method, some error minimizing technique to determine the coefficients $c_{i0}, c_{i1}, \ldots, c_{iN}$, $i=1, 2, \ldots, m$ are needed, one of the most popular minimizing techniques is the weighted residual methods (WRM's) which include the {collocation method (CM) and partition method (PM)} .

Expansion method using weighted residual technique to find parameters $c_{i0}, c_{i1}, \ldots, c_{iN}$, $i=1, 2, \ldots, m$ has been considered by many authors and researchers, Delves and Walsh [10], Davis [9], Hall and Watt [12], Jain [13], Boyd [5] and Chapra and Canale [8].

On the other hand Chambers [7] use this method to solve first and second kind integral equations, approximated solution to non-linear VIE of the first kind and integro-differential equation of Fredholm type respectively, while Al-Rawi [2], Al-Asadi [1] and Kareem [14] applied this method to treat first kind integral equation of convolution type, non-linear VIE's of first kind and linear VIDE's respectively.

In this paper, we use the WRM's for a first time to find the solution of a system of linear Volterra integro-differential equations of the second kind (VIDEK2) of order n

$$[D^n + \sum_{s=0}^{n-1} p_{vs}(x)D^s]\,u_i(x) = f_i(x) + \sum_{j=1}^{m}\int_a^x k_{vj}(x,t)u_j(t)dt,$$

$i=1, 2, \ldots, m,\ x \in I = [a,b];\ a,b \in R$ ............... **(1)**

with initial conditions:

$$u_i(a) = u_{i0},$$

$$u_i'(a) = u_{i1}, \ldots, u_i^{(n-1)}(a) = u_{i(n-1)};\ i=1, 2, , m.$$

.................. **(2)**

where $m \in \mathrm{N}$; $f_i$, i=1, 2, …, m are continuous functions on I and $k_{ij}$, $i, j=1, 2, \ldots, m$ denotes given continuous functions, while $u_i(x)$, $i=1, 2, \ldots, m$ are the unknown functions to be determined. This work is organized as follows:

In section 2, WRM's reformulated to be suitable for above systems.

In section 3, solution of a system of linear VIDEK2 has been proposed, using WRM's.

In section 4, for each of the CM and PM, we choose two basis functions (power function and Chebyshev polynomial), for solving a system of linear VIDEK2.

In section 5, examples are given for illustrations, and comparison between the methods and basis functions has been made depending on the least square errors.

Finally, section 6 includes a conclusion for this work.

## 2-Weighted Residual Methods (WRM's)

The method of weighted residuals has its roots in the calculus of variations. It forms the basis for analytical techniques such as the methods (CM and PM), [5].

All of these techniques are described to create relatively algebraic functions (often polynomials) that can either (1) approximate the solution to functional equations whose exact solutions are unknown or are overly complicated, or (2) give approximate relationships to fit a function through a series of data points.[5]

When using these techniques we realize that there will be a discrepancy between the approximating function and the exact solution to the functional equation being solved or the relationship passing exactly through the data points being treated. The discrepancy is quantified by residual usually defined at several selected points in the domain of the function. If the functions are being used to fit a relationship through a series of data points, the residuals are usually evaluated at the data points themselves. If the functions are to be approximations to solutions of functional equations, the residuals are evaluated at locations distributed conveniently over the domain of the problem.

The analyst's goal is to achieve the best possible agreement by minimizing the residuals. This can be achieved in two ways:

- The approximation can be improved by increasing the complexity of the algebraic function.
- The analyst can optimize the function's constants to improve the fit.

In this section we try to reformulate the WRM to solve a system of linear VIDEK2 as follows:

Consider the functional equation given by:

$T_i[u_i(x)] = f_i(x)$, $x \in D$, $i=1, 2, , m$, **..........(3)**

where $T_i$, $i=1, 2, ..., m$ are given operators which maps a set $U (u_i \in U)$ into a set $F$ ( $f_i \in F$ are given), and $D$ is the domain of $T_i$.

To find an approximate solution of the equation (3), we assume an approximations $S_{iN}(x)$ to the exact solutions $u_i(x)$, such that

$$S_{iN}(x) = \sum_{k=0}^{N} c_{ik}\phi_k(x), \quad \textbf{.............................. (4)}$$

Where the parameters $c_{i0}, c_{i1}, \ldots, c_{iN}$ to be determine and the functions $\phi_k(x)$, $k=0, 1, \ldots, N$ are prescribed basis functions to be chosen.

Now, by substituting the approximate solutions $S_{iN}(x)$ given by (4) into equation (3), we get the residue

$R_{iN}(x) = T_i[S_{iN}(x)] - f_i(x), i=1, 2,$ **...........(5)**

The residue $R_{iN}(x)$ depends on $x$ as well as on the way that the parameters $c_{i0}, c_{i1}, \ldots, c_{iN}$; $i=1, 2, \ldots, m$ are chosen, therefore equation (5) can be written on the form

$$R_{iN}(x,\bar{c}) = T_i[S_{iN}(x)] - f_i(x), i=1,2,\ldots,m \quad \textbf{..................(6)}$$

where $\bar{c} = ( c_{i0}, c_{i1}, \ldots, c_{iN} )$, $i=1, 2, \ldots, m$.

It is obvious that when $R_{iN}(x,\bar{c}) \equiv 0$, then the exact solution is obtained, therefore and throughout this section we shall try to minimize $R_{iN}(x,\bar{c})$ in some sense. In this work, we set the weighted integral of $R_{iN}(x,\bar{c})$ equal to zero, i.e.

$$\int_D w_i(x) R_{iN}(x,\bar{c})dx = 0 \quad \textbf{,....................(7)}$$

where $w_i(x)$ are a prescribed weighted function, the technique described by (5) is call weighted residual methods, by which the optimal values of $\bar{c}$'s that minimize $R_{iN}(x,\bar{c})$, is determine. We now describe a few well-known methods of the weighted residual methods to be determine the arbitrary parameters $c_{i0}$, $c_{i1}$, ..., $c_{iN}$ ; $i$=1, 2, …, $m$ in (4).

## 2.1 Collection Method(CM), [4,8,9,13]:

This method can be used to calculate the parameters $c_{i0}$, $c_{i1}$, ..., $c_{iN}$ ; $i$=1, 2, …, $m$ which minimize $R_{iN}(x,\bar{c})$, $i$=1, 2, ...,$m$. The main idea behind the collocation method is the parameters $c_{i0}$, $c_{i1}$, ..., $c_{iN}$ ; $i$=1, 2, …, $m$ are to be found by foreseeing that the residual $R_{iN}(x,\bar{c})$ vanishes at given set of $N+1$ points in the domain D. Mathematically, this can be described as follows: Let us choose $N+1$ distinct points $x_0$, $x_1$, ..., $x_N \in D$ and define the weighted functions as $w_j(x)=\delta(x-x_j)$, $j$=0, 1, …, $N$ where $\delta$ represent the unit impulse function which vanishes everywhere except at $x=x_j$, $j$=0, 1, …, $N$. This means that

$$\delta(x-x_j)=\begin{cases}0 & \text{if} \quad x\neq x_j\\ 1 & \text{if} \quad x=x_j\end{cases}$$

for $j$=0, 1, …, $N$.

The collocation equations become

$$\int_D \delta(x-x_j)R_{iN}(x,\bar{c})dx=0,$$

this can be written as

$$R_{iN}(x_j,\bar{c})=0,\ i=1,2,m;\ j=0,1,\ldots,N. \quad \textbf{.................\ (8)}$$

This criterion is thus equivalent to putting $R_{iN}(x,\bar{c})$ equal to zero at $N$ points in the domain D. Moreover, the distribution of the collocation points on $D$ is arbitrary; however in practice we describe the collocation points uniformly on $D$.

The equation (8) will provide us by $m\times(N+1)$ simultaneous equations to determine the parameters $c_{i0}$, $c_{i1}$, ..., $c_{iN}$ ; $i=1, 2, ..., m$.

## 2.2 Partition Method (PM):[4, 13]

In this method the domain $D$ is divided into $N+1$ non-overlapping sub-domains $D_j$, $j$=0, 1, ..., $N$ with the weighted functions is chosen as follows:

$$w_j(x)=\begin{cases}1 & \text{if} \quad x\in D_j\\ 0 & \text{if} \quad x\notin D_j\end{cases} \quad \text{for } j=0, 1, ..., N.$$

Hence the equation (3) is satisfied in each of the sub-domains $D_j$, therefore equations (4) become

$$\int_{D_j} R_{iN}(x,\bar{c})dx=0,\ j=0, 1, \ldots, N \quad \textbf{..... (9)}$$

We note that the size of one or more sub-domains decrease as N is increase with the result that the equation (5) is satisfied on the average in smaller and smaller sub-domains, and hence the residue in equation (6) approaches zero as $N\to\infty$.

## 3 Solution of a System of Linear VIDEK2

In this section, we apply the weighted residual methods described in section 4.2 to find an approximate solution of the equation 1.

Using operator's forms, this system can be written as in equation (5), where the operators $T_i$ are defined as:

$$T_i[u_i(x)]=[D^n+\sum_{s=0}^{n-1}p_{is}(x)D^s]u_i(x)-\sum_{j=1}^{m}\int_a^x k_{ij}(x,t)u_j(t)dt,\ i=1, 2, ...,m \quad \textbf{...... (10)}$$

The unknown functions $u_i(x)$ is approximated by $S_{iN}(x)$ which is given by equation (4). Now the approximate solution (4) substituting in the system (10) to obtain equation (6), where

$$T_i[S_{iN}(x)]=[D^n+\sum_{s=0}^{n-1}p_{is}(x)D^s]\sum_{k=0}^{N}c_{ik}\phi_k(x)-\sum_{j=1}^{m}\int_a^x(k_{ij}(x,t)\sum_{k=0}^{N}c_{jk}\phi_k(t))dt\ .$$

From equation (6) we have the following residual equations

$$R_{iN}(x,\bar{c})=[D^n+\sum_{s=0}^{n-1}p_{is}(x)D^s]\sum_{k=0}^{N}c_{ik}\phi_k(x)-\sum_{j=1}^{m}\int_a^x(k_{ij}(x,t)\sum_{k=0}^{N}c_{jk}\phi_k(t))dt - f_i(x),$$

$z_i$=1, 2, …, m; (N>n if a≠0)**. ......... (11)**

Here, we use only one equation in the system which contains all the unknowns $u_1(x)$, $u_2(x)$, …, $u_m(x)$ to find the unknowns (if there is no such equation, we collect any number of equations in the system to obtain the desired equation).

If we chose equation number v from equation (11) with letting $c_{i0}=f_i(a)$, i=1, 2, …, m if a≠0 and $c_{id}=u_i^{(d)}(0)$, i=1, 2, …, m; d=0, 1, …, n-1 if a=0 then we get

$$R_{vN}(x,\bar{c})=[D^n+\sum_{s=0}^{n-1}p_{vs}(x)D^s]$$

$$\sum_{k=\tau}^{N}c_{vk}\phi_k(x)-\sum_{j=1}^{m}\int_a^x k_{vj}(x,t)\sum_{k=\tau}^{N}c_{jk}\phi_k dt-f_v(x)$$

$$+\sum_{z=1}^{m}G_z(x,t)=$$

$$\sum_{k=\tau}^{N}c_{vk}\left[(D^n+\sum_{s=0}^{n-1}p_{vs}(x)D^s)\phi_k(x)-\int_a^x k_{vv}(x,t)\phi_k(t)dt\right]$$

$$-\sum_{\substack{j=1\\j\neq v}}^{m}\sum_{k=\tau}^{N}c_{jk}\int_a^x k_{vj}(x,t)\phi_k(t)dt-f_v(x)+$$

$$\sum_{z=1}^{m}G_z(x,t)\ \textbf{.....................................}\ \textbf{(12)}$$

where

$$G_z(x,t)=\begin{cases}\left.\begin{array}{ll}f_v(a)\left\{\left(D^n+\sum_{s=0}^{n-1}p_{vs}(x)D^s\right)\phi_0(x)-\int_a^x k_{vv}(x,t)\phi_0(t)dt\right\} & \text{if } z=v\\ -f_z(a)\left\{\int_a^x k_{vz}(x,t)\phi_0(t)dt\right\} & \text{if } z\neq v\end{array}\right\}\text{for a}\neq 0\\ \left.\begin{array}{ll}\sum_{d=0}^{n-1}u_v^{(d)}(0)\left\{\left(D^n+\sum_{s=0}^{n-1}p_{vs}(x)D^s\right)\phi_d(x)-\int_0^x k_{vv}(x,t)\phi_d(t)dt\right\} & \text{if } z=v\\ -\sum_{d=0}^{n-1}u_z^{(d)}(0)\left\{\int_0^x k_{vz}(x,t)\phi_d(t)dt\right\} & \text{if } z\neq v\end{array}\right\}\text{for a}=0\end{cases}$$

and

$$\tau=\begin{cases}1 & \text{if } a\neq 0\\ n & \text{if } a=0\end{cases}.$$

To show that the system (11) has a unique solution, we must find the Wronskian *W(x)* of the equation (12) where

$W(x)=\left|C_1'\ \ C_2'\ \ \ldots\ \ C_N'\ \ C_{N+1}'\ \ \ldots\ \ C_M'\right|_{M\times M}$,

$$C'_{(v-1)\mathrm{x}(N+1-\tau)+l}=\begin{bmatrix}(D^n-\sum_{s=0}^{n-1}p_{vs}(x)D^s)\phi_l(x)-\int_a^x k_{vv}(x,t)\phi_l(t)dt\\ \frac{d}{dx}\left((D^n-\sum_{s=0}^{n-1}p_{vs}(x)D^s)\phi_l(x)-\int_a^x k_{vv}(x,t)\phi_l(t)dt\right)\\ \vdots\\ \frac{d^{M-1}}{dx^{M-1}}\left((D^n-\sum_{s=0}^{n-1}p_{vs}(x)D^s)\phi_l(x)-\int_a^x k_{vv}(x,t)\phi_l(t)dt\right)\end{bmatrix}$$

$l$=1, 2, …, $N$,

$$C'_{(q-1)\mathrm{x}(N+1-\tau)+q_1}=\begin{bmatrix}-\int_a^x k_{vq}(x,t)\phi_{q_1}(t)dt\\ \frac{d}{dx}\left(-\int_a^x k_{vq}(x,t)\phi_{q_1}(t)dt\right)\\ \vdots\\ \frac{d^{M-1}}{dx^{M-1}}\left(-\int_a^x k_{vq}(x,t)\phi_{q_1}(t)dt\right)\end{bmatrix},$$

$q$=1, 2, ..., $v$-1,$v$+1, ..., $m$; $q_1$=1, 2, …, $N$, and

$$M=m\times(N+1-\tau).$$

If $W(x)\neq 0$, then the system has a unique solution.

Now, the problem is how to find the optimal values of $c_{i\tau}$, $c_{i,\tau+1}$, ..., $c_{iN}$; $i$=1, 2, …, $m$ which minimize the residual $R_{vN}(x,\bar{c})$ in (12), this can be achieve by using the WRM's.

### 3.1 CM:

Apply the same idea in subsection 2.1; we get the following linear system of equations from equation (12)

$$\sum_{k=\tau}^{N}c_{vk}\left[\begin{array}{l}(D^n+\sum_{s=0}^{n-1}p_{vs}(x)D^s)\phi_k(x)|_{x=x_\alpha}\\ -\int_a^{x_\alpha}k_{vv}(x_\alpha,t)\phi_k(t)dt\end{array}\right]$$

$$-\sum_{\substack{j=1\\j\neq v}}^{m}\sum_{k=\tau}^{N}c_{jk}\int_a^{x_\alpha}k_{vj}(x_\alpha,t)\phi_k(t)dt=f_v(x_\alpha)$$

$$-\sum_{z=1}^{m}G_z(x_\alpha,t)\ \textbf{.......................................(13)}$$

where $x_\alpha=\alpha h$, $\alpha$=1, 2, …, $M$, and $h$ is to be chosen.

Solve the resulting linear system by using Gauss elimination method to find $c_{i\tau}$, $c_{i,\tau+1}$, ..., $c_{iN}$, $i$=1, 2, …, $m$.

### 3.2 PM:

As in subsection 2.2, we get the following linear system of equations from equation (12)

$$\sum_{k=\tau}^{N} c_{vk} \int_a^{x^\alpha} \left[ (D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s) \phi_k(x) - \int_a^x k_{vv}(x,t)\phi_k(t)dt \right] dx - \sum_{\substack{j=1 \\ j\neq v}}^{m} \sum_{k=\tau}^{N} c_{jk} \int_a^{x_\alpha} \left[ \int_a^x k_{vj}(x,t)\phi_k(t)dt \right] dx = \int_a^{x_\alpha} f_v(x)dx - \sum_{z=1}^{m} \int_a^{x_\alpha} G_z(x,t)dx, \text{ .................................... (14)}$$

where $x_\alpha=\alpha h$, $\alpha$=1, 2, ..., $L$, and $h$ is to be chosen.

Solve above linear system for $c_{i\tau}$, $c_{i,\tau+1}$, ..., $c_{iN}$, $i$=1, 2, …, $m$ by using Gauss elimination.

## 4 Choices of Basis Functions:

### 4.1 Solution Technique for System of Linear VIDEK2 Using Power Functions:

Let $\phi_k(x) = x^k$, for $k$=0,1, …, $N$, in equation (4). Substitute these values in equations (13) and (14) respectively, where the integrals in this approaches have been evaluated numerically using composite trapezoid rule or composite Simpson method, we get:

#### 4.1.1 CM:

$$\sum_{k=\tau}^{N} c_{vk} \left[ (D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s) x^k \big|_{x=x_\alpha} - \int_a^{x_\alpha} k_{vv}(x_\alpha,t) t^k dt \right] - \sum_{\substack{j=1 \\ j\neq v}}^{m} \sum_{k=\tau}^{N} c_{jk} \int_a^{x_\alpha} k_{vj}(x_\alpha,t) t^k dt = f_v(x_\alpha) - \sum_{z=1}^{n} G_z(x_\alpha,t), \ \alpha=1, 2, \dots, M, \text{ ..........(15)}$$

where

$$G_z(x_\alpha,t) = \begin{cases} \left.\begin{array}{ll} f_v(a)\left\{ p_{v0}(x_\alpha) - \int_a^{x_\alpha} k_{vv}(x_\alpha,t)dt \right\} & \text{if } z=v \\ -f_z(a)\left\{ \int_a^{x_\alpha} k_{vz}(x_\alpha,t)dt \right\} & \text{if } z\neq v \end{array}\right\} \text{for } a\neq 0 \\ \left.\begin{array}{ll} \sum_{d=0}^{n-1} u_v^{(d)}(0)\left\{ \left( D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s \right) x^d \Big|_{x=x_\alpha} - \int_0^{x_\alpha} k_{vv}(x_\alpha,t) t^d dt \right\} & \text{if } z=v \\ -\sum_{d=0}^{n-1} u_z^{(d)}(0)\left\{ \int_0^{x_\alpha} k_{vz}(x_\alpha,t) t^d dt \right\} & \text{if } z\neq v \end{array}\right\} \text{for } a=0 \end{cases}$$

#### 4.1.2 PM:

$$\sum_{k=\tau}^{N} c_{vk} \int_a^{x_\alpha} \left[ (D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s) x^k - \int_a^x k_{vv}(x,t) t^k dt \right] dx - \sum_{\substack{j=1 \\ j\neq v}}^{m} \sum_{k=\tau}^{N} c_{jk} \int_a^{x_\alpha} \left[ \int_a^x k_{vj}(x,t) t^k dt \right] dx = \int_a^{x_\alpha} f_v(x)dx - \sum_{z=1}^{m} \int_a^{x_\alpha} G_z(x,t)dx, \ \alpha=1, 2, ..., M. \text{ ................(16)}$$

**Note:** In subsection 4.1, the involved integral appears computed numerically by using composite trapezoid method or composite Simpson method, {for composite trapezoid and Simpson methods, see [3, 6 and 15].

### 4.2 Solution Technique for System of Linear VIDEK2 Using Chebyshev polynomials:

Let $\phi_k(x) = T_k(\xi(x,b))$, $k$=0,1, …, $N$, in equation (4) for all a≤x≤b where $T_k(x)$'s are the Chebyshev polynomials defined in chapter one, subsection 1.4.1. Substitute $\phi_k(x) = T_k(\xi(x,b))$ in equations (13) and (14) respectively, we get

#### 4.2.1 CM:

$$\sum_{k=\tau}^{N} c_{vk} \left[ \begin{array}{l} (D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s) T_k(\xi(x,b)) \big|_{x=x_\alpha} \\ - \int_a^{x_\alpha} k_{vv}(x_\alpha,t) T_k(\xi(t,x_\alpha)) dt \end{array} \right] - \sum_{\substack{j=1 \\ j\neq v}}^{m} \sum_{k=\tau}^{N} c_{jk} \int_a^{x_\alpha} k_{vj}(x_\alpha,t) T_k(\xi(t,x_\alpha)) dt = f_v(x_\alpha) - \sum_{z=1}^{m} G_z(x_\alpha,t), \ \alpha=1, 2, \dots, M \text{ ........ (17)}$$

**Using open Gauss-Chebyshev formula** [3, 6 and 15] to calculate the integrals in equation (17) we get:

$$\sum_{k=\tau}^{N} c_{vk} \left[ \begin{array}{l} (D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s) T_k(\xi(x,b)) \big|_{x=x_\alpha} \\ -\dfrac{\pi}{l} \sum_{r=0}^{l-1} V_r k_{vv}(x_\alpha,t) T_k(\xi(x_r,x_\alpha)) \end{array} \right] -$$

$$\sum_{\substack{j=1\\ j\neq v}}^{m}\sum_{k=\tau}^{N} c_{jk}\frac{\pi}{l}\sum_{r=0}^{l-1} V_r k_{vj}(x_\alpha, x_r) T_k\big(\xi(x_r, x_\alpha)\big) =$$

$$f_v(x_\alpha) - \sum_{z=1}^{m} G_v(x_\alpha, t),\ \alpha=1, 2, \ldots, M \ldots\ldots \textbf{(18)}$$

where

$$G_z(x_\alpha,t) = \begin{cases} \left.\begin{array}{ll} f_v(a)\left\{p_{v0}(x_\alpha) - \dfrac{\pi}{l}\displaystyle\sum_{s=0}^{l-1} V_s k_{vv}(x_\alpha, x_s)\right\} & \text{if } z=v \\ -f_z(a)\left\{\dfrac{\pi}{i}\displaystyle\sum_{s=0}^{l-1} V_s k_{vz}(x_\alpha, x_s)\right\} & \text{if } z\neq v \end{array}\right\} \text{for } a\neq 0 \\ \left.\begin{array}{ll} \displaystyle\sum_{d=0}^{n-1} u_v^{(d)}(0)\left\{\left(D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s\right) x^d\Big|_{x=x_\alpha} - \dfrac{\pi}{l}\sum_{s=0}^{l-1} V_s k_{vv}(x_\alpha, x_s) x_s^d\right\} & \text{if } z=v \\ -\displaystyle\sum_{d=0}^{n-1} u_z^{(d)}(0)\left\{\dfrac{\pi}{i}\sum_{s=0}^{l-1} V_s k_{vz}(x_\alpha, x_s) x_s^d\right\} & \text{if } z\neq v \end{array}\right\} \text{for } a=0 \end{cases}$$

$$\ldots\ldots\ldots\ldots \textbf{(19)}$$

$$V_s = \sqrt{(x_s - a)(x_\alpha - x_s)},$$

and

$$V_r = \sqrt{(x_r - a)(x_s - x_r)}.$$

**Using closed Gauss-Chebyshev formula** [3, 6 and 15] to calculate the integrals in equation (17) we get

$$\sum_{k=\tau}^{N} c_{vk}\left[\begin{array}{l}\left(D^n + \displaystyle\sum_{s=0}^{n-1} p_{vs}(x) D^s\right) T_k\big(\xi(x,b)\big)\Big|_{x=x_\alpha} \\ -\dfrac{\pi}{l}\displaystyle\sum_{r=0}^{l}{}'' V_r k_{vv}(x_\alpha, t) T_k\big(\xi(x_r, x_\alpha)\big)\end{array}\right] -$$

$$\sum_{\substack{j=1\\ j\neq v}}^{m}\sum_{k=\tau}^{N} c_{jk}\frac{\pi}{l}\sum_{r=0}^{l}{}'' V_r k_{vj}(x_\alpha, x_r) T_k\big(\xi(x_r, x_\alpha)\big) = f_v(x_\alpha)$$

$$-\sum_{z=1}^{m} G_v(x_\alpha, t),$$

$$\alpha=1, 2, \ldots, M \ldots\ldots\ldots\ldots \textbf{(20)}$$

where

$$G_z(x_\alpha,t) = \begin{cases} \left.\begin{array}{ll} f_v(a)\left\{p_{v0}(x_\alpha) - \dfrac{\pi}{l}\displaystyle\sum_{s=0}^{l}{}'' W_s k_{vv}(x_\alpha, x_s)\right\} & \text{if } z=v \\ -f_z(a)\left\{\dfrac{\pi}{l}\displaystyle\sum_{s=0}^{l}{}'' W_s k_{vz}(x_\alpha, x_s)\right\} & \text{if } z\neq v \end{array}\right\} \text{for } a\neq 0 \\ \left.\begin{array}{ll} \displaystyle\sum_{d=0}^{n-1} u_v^{(d)}(0)\left\{\left(D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s\right) x^d\Big|_{x=x_\alpha} - \dfrac{\pi}{l}\sum_{s=0}^{l}{}'' W_s k_{vv}(x_\alpha, x_s) x_s^d\right\} & \text{if } z=v \\ -\displaystyle\sum_{d=0}^{n-1} u_z^{(d)}(0)\left\{\dfrac{\pi}{l}\sum_{s=0}^{l}{}'' W_s k_{vz}(x_\alpha, x_s) x_s^d\right\} & \text{if } z\neq v \end{array}\right\} \text{for } a=0 \end{cases}$$

$$\ldots\ldots\ldots\ldots \textbf{(21)}$$

and $V_r$, $W_s$ defined above.

**4.2.2 PM:**

$$\sum_{k=\tau}^{N} c_{vk}\int_a^{x_\alpha}\left[\begin{array}{l}\left(D^n + \displaystyle\sum_{s=0}^{n-1} p_{vs}(x) D^s\right) T_k\big(\xi(x, x_\alpha)\big) - \\ \displaystyle\int_a^x k_{vv}(x,t) T_k\big(\xi(t,x)\big)dt\end{array}\right] dx -$$

$$\sum_{\substack{j=1\\ j\neq v}}^{m}\sum_{k=\tau}^{N} c_{jk}\int_a^{x_\alpha}\left[\int_a^x k_{vj}(x,t) T_k\big(\xi(t,x)\big)dt\right]dx = \int_a^{x_\alpha} f_v(x)dx -$$

$$\sum_{z=1}^{n}\int_a^{x_\alpha} G_z(x,t)dx, \ldots\ldots\ldots\ldots \textbf{(22)}$$

$\alpha=1, 2, \ldots, M.$

**Using open Gauss-Chebyshev formula** to calculate the integrals in equation (22) we get

$$\sum_{k=\tau}^{N} c_{vk}\frac{\pi}{l}\sum_{r=0}^{l-1} V_r\left[\begin{array}{l}\left(D^n + \displaystyle\sum_{s=0}^{n-1} p_{vs}(x) D^s\right) T_k\big(\xi(x, x_\alpha)\big)\Big|_{x=x_\alpha} \\ -\displaystyle\int_a^{x_r} k_{vv}(x_r,t) T_k\big(\xi(t, x_r)\big)dt\end{array}\right] -$$

$$\sum_{\substack{j=1\\ j\neq v}}^{m}\sum_{k=\tau}^{N} c_{jk}\frac{\pi}{l}\sum_{r=0}^{l-1} V_r\int_a^{x_r} k_{vj}(x_r,t) T_k\big(\xi(t,x_r)\big)dt =$$

$$\frac{\pi}{l}\sum_{r=0}^{l-1} V_r f_v(x_\alpha) - \sum_{z=1}^{m}\frac{\pi}{l}\sum_{r=0}^{-1} V_r G_z(x_r, t),$$

$$\alpha=1, 2, \ldots, M, \ldots\ldots\ldots\ldots \textbf{(23)}$$

where

$$G_z(x_r,t) = \begin{cases} \left.\begin{array}{ll} f_v(a)\left\{p_{v0}(x_r) - \dfrac{\pi}{l}\displaystyle\sum_{s=0}^{l-1} W_s k_{vv}(x_r, x_s)\right\} & \text{if } z=v \\ -f_z(a)\left\{\dfrac{\pi}{l}\displaystyle\sum_{s=0}^{l-1} W_s k_{vz}(x_r, x_s)\right\} & \text{if } z\neq v \end{array}\right\} \text{for } a\neq 0 \\ \left.\begin{array}{ll} \displaystyle\sum_{d=0}^{n-1} u_v^{(d)}(0)\left\{\left(D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s\right) x^d\Big|_{x=x_r} - \dfrac{\pi}{l}\sum_{s=0}^{l-1} W_s k_{vv}(x_r, x_s) x_s^d\right\} & \text{if } z=v \\ -\displaystyle\sum_{d=0}^{n-1} u_z^{(d)}(0)\left\{\dfrac{\pi}{l}\sum_{s=0}^{l-1} W_s k_{vz}(x_r, x_s) x_s^d\right\} & \text{if } z\neq v \end{array}\right\} \text{for } a=0 \end{cases}$$

$$\ldots\ldots\ldots\ldots \textbf{(24)}$$

and $V_r$ and $W_r$ defined in subsection 4.2.1.

**Using closed Gauss-Chebyshev formula** to calculate the integrals in equation (22) we get:

$$\sum_{k=\tau}^{N} c_{vk} \frac{\pi}{l} \sum_{r=0}^{l}{}'' V_r \left[ \left. \left(D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s\right) T_k\left(\xi(x, x_\alpha)\right) \right|_{x=x_r} - \int_a^{x_r} k_{vv}(x_r, t) T_k\left(\xi(t, x_r)\right) dt \right] -$$

$$\sum_{\substack{j=1\\ j\neq v}}^{m} \sum_{k=\tau}^{N} c_{jk} \frac{\pi}{l} \sum_{r=0}^{l}{}'' V_r \int_a^{x_r} k_{vj}(x_r, t) T_k\left(\xi(t, x_r)\right) dt =$$

$$\frac{\pi}{l} \sum_{r=0}^{l}{}'' V_r f_v(x_r) - \sum_{z=1}^{m} \frac{\pi}{l} \sum_{r=0}^{l}{}'' V_r G_z(x_r, t),$$

$$\alpha = 1, 2, \ldots, L, \ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathbf{(25)}$$

where

$$G_z(x_r,t) = \begin{cases} \left.\begin{array}{ll} f_v(a)\left\{p_{v0}(x_r) - \frac{\pi}{l}\sum_{s=0}^{l}{}'' W_s k_{vv}(x_r, x_s)\right\} & \text{if } z = v \\ -f_z(a)\left\{\frac{\pi}{l}\sum_{s=0}^{l}{}'' W_s k_{vz}(x_r, x_s)\right\} & \text{if } z \neq v \end{array}\right\} \text{for } a \neq 0 \\ \left.\begin{array}{ll} \sum_{d=0}^{n-1} u_v^{(d)}(0)\left\{\left.\left(D^n + \sum_{s=0}^{n-1} p_{vs}(x) D^s\right) x^d\right|_{x=x_r} - \frac{\pi}{l}\sum_{s=0}^{l}{}'' W_s k_{vv}(x_r, x_s) x_s^d\right\} & \text{if } z = v \\ -\sum_{d=0}^{n-1} u_z^{(d)}(0)\left\{\frac{\pi}{l}\sum_{s=0}^{l}{}'' W_s k_{vz}(x_r, x_s) x_s^d\right\} & \text{if } z \neq v \end{array}\right\} \text{for } a = 0 \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots \mathbf{(26)}$$

and $V_r$ and $W_r$ defined in subsection 4.2.1.

## 5 Numerical Examples:

Here, we present two examples for a system of linear VIDEK2's solved by WRM's (CM and PM), and we use Matlab version 6.5 for finding a solution.

### Example 5.1:

Consider the following linear system of Volterra integro-differential equations of the second kind, with initial solutions $u_1(0) = 0$ and $u_2(0) = 0$:

$$u_1'(x) + u_1(x) = 1 + 3x + x^2 - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^7}{3} + \int_0^x t u_1(t) dt + \int_0^x x t^3 u_2(t) dt$$

$$u_2'(x) + u_2(x) = 4x + 2x^2 - \frac{x^3}{2} - \frac{x^4}{3} - \frac{2x^5}{5} + \int_0^x x t u_1(t) dt + \int_0^x x t^3 u_2(t) dt$$

**Solution:**

Assume that the approximate solutions are in the form

$$S_{iN}(x) = \sum_{k=0}^{2} c_{ik} \phi_k(x), \; i = 1, 2,$$

Where $\phi_k(x) = x^k$, $T_k(x)$ (Chebyshev polynomial of the second kind), $k$=0,1,2, and from initial solutions we obtain $c_{10}$=0, $c_{20}$=0.

After solving this system by all above methods, it can be found the coefficients as: $c_{11}$=$c_{12}$=*1*, $c_{21}$=0, $c_{22}$=2.

Thus, the solution of this system is $u_1(x) = c_{10} + c_{11}x + c_{12}x^2 = x + x^2$ and $u_2(x) = c_{20} + c_{21}x + c_{22}x^2 = 2x^2$.

This is the exact solutions to the Example 5.1.

### Example 5.2:

Consider the following linear system of Volterra integro-differential equations of the second kind, with the initial solutions $u_1(0) = 1$, $u_2(0) = -1$, $u_1'(0) = 1$ and $u_2'(0) = 0$:

$$u_1''(x) + x u_1'(x) + \sin(x) u_1(x) = f_1(x) + \int_0^x e^{x-2t} u_1(t) dt + \int_0^x x t^3 u_2(t) dt$$

$$u_2''(x) + x^2 u_2'(x) + \cos(x) u_1(x) = f_2(x) + \int_0^x t^2 u_1(t) dt + \int_0^x (x^2 + x t^2) u_2(t) dt,$$

where

$$f_1(x) = x e^x + e^x \sin(x) + 1 - \frac{1}{4}x^5 + 2x^4 \sin(x) + 6x^3 \cos(x) - 12x\cos(x) - 12x^2 \sin(x) + 12x,$$

and

$$f_2(x) = 3\cos(x) + 4x^2 \sin(x) - 2\cos^2(x) - x^2 e^x + 2x e^x - 2e^x + 2 - x^3 - \frac{1}{3}x^4 + 2x^3 \sin(x) - 4x\sin(x) + 4x^2 \cos(x).$$

The exact solutions for this system are $u_1(x) = e^x$ and $u_2(x) = 1 - 2\cos(x)$.

**Solution:**

**1-** Assume that the approximate solutions are in the form

$$S_{iN}(x) = \sum_{k=0}^{3} c_{ik} \phi_k(x), \; i = 1, 2,$$

where $\phi_k(x) = x^k$, $k$=0, 1, 2, 3 and from initial solutions we obtain $c_{10}$=1, $c_{11}$=1, $c_{20}$=-1, $c_{21}$=0.

After solving this system by above methods, we get the following solutions of the system:

**(i) Using CM:**

$$u_1(x) \approx 1 + x + \frac{491}{1019}x^2 + \frac{493}{2008}x^3,$$

$$u_2(x) \approx -1 - \frac{4}{29741}x^2 + \frac{10}{184793}x^3.$$

**(ii) Using PM:**

$$u_1(x) \approx 1 + x + \frac{18}{33}x^2 + \frac{281}{1682}x^3,$$

$$u_2(x) \approx -1 - \frac{9}{27532}x^2 + \frac{9}{78557}x^3..$$

**Note:** for comparison between exact solutions and approximate solutions of **Example 5.2** where $\phi_k(x) = x^k$, $k$=0, 1, 2, 3 see **Tables (5.1)** and **(5.2).**

**Table (5.1)**
**Show a comparison between the exact solutions $e^x$ and the numerical solution $u_1(x)$ of two types, which depends on least square error.**

| x | $u_1(x)$ | | |
|---|---|---|---|
| | Exact | CM | PM |
| 0 | 1 | 1 | 1 |
| 0.1 | 1.105170918 | 1.105063967 | 1.105621608 |
| 0.2 | 1.221402758 | 1.221237941 | 1.223154686 |
| 0.3 | 1.349858808 | 1.349995029 | 1.353601611 |
| 0.4 | 1.491824698 | 1.492808339 | 1.497964761 |
| 0.5 | 1.648721271 | 1.651150978 | 1.657246514 |
| 0.6 | 1.822118800 | 1.826496053 | 1.832449249 |
| 0.7 | 2.013752707 | 2.020316673 | 2.024575343 |
| 0.8 | 2.225540928 | 2.234085945 | 2.234627175 |
| 0.9 | 2.459603111 | 2.469276976 | 2.463607124 |
| 1 | 2.718281828 | 2.727362874 | 2.712517566 |
| L.S.E. | | $3.1824\times10^{-4}$ | $4.8333\times10^{-4}$ |

**Table (5.2)**
**Show a comparison between the exact solutions 1-2cos(x) and the numerical solution u2(x) of two types, which depends on least square error.**

| x | $u_2(x)$ | | |
|---|---|---|---|
| | Exact | CM | PM |
| 0 | -1 | -1 | -1 |
| 0.1 | -0.999996954 | -1.000001291 | -1.000003154 |
| 0.2 | -0.999987815 | -1.000004947 | -1.000012159 |
| 0.3 | -0.999972584 | -1.000010643 | -1.000026327 |
| 0.4 | -0.999951261 | -1.000018056 | -1.000044971 |
| 0.5 | -0.999923846 | -1.000026859 | -1.000067402 |
| 0.6 | -0.999890339 | -1.000036729 | -1.000092935 |
| 0.7 | -0.999850739 | -1.000047341 | -1.000120881 |
| 0.8 | -0.999805048 | -1.000058370 | -1.000150553 |
| 0.9 | -0.999753265 | -1.000069491 | -1.000181264 |
| 1 | -0.999695390 | -1.000080380 | -1.000212326 |
| L.S.E. | | $3.8900\times10^{-7}$ | $7.1700\times10^{-7}$ |

**2-** Assume that the approximate solutions are in the form

$$S_{iN}(x) = \sum_{k=0}^{3} c_{ik}\phi_k(x),\ i=1,\ 2,$$

where $\phi_k(x) = T_k(\xi(x,b))$, $k$=0, 1, 2, 3 and from initial solutions we obtain $c_{10}$=1, $c_{11}$=1, $c_{20}$=-1, $c_{21}$=0.

After solving this system by above methods, we get the following solutions of the system:

**(i) Using CM:**

$$u_1(x) \approx 1 + x + \frac{491}{1019}x^2 + \frac{493}{2008}x^3,$$

$$u_2(x) \approx -1 - \frac{4}{29741}x^2 + \frac{10}{184793}x^3.$$

**(ii) Using PM:**

$$u_1(x) \approx 1 + x + \frac{18}{33}x^2 + \frac{281}{1682}x^3,$$

$$u_2(x) \approx -1 - \frac{9}{27532}x^2 + \frac{9}{78557}x^3.$$

**Note:** for comparison between exact solutions and approximate solutions of **Example 5.2** where $\phi_k(x) = T_k(\xi(x,b))$, $k$=0, 1, 2, 3 see **Tables (5.3)** and **(5.4).**

**Table (5.3)**

**Show a comparison between the exact solutions $e^x$ and the numerical solution $u_1(x)$ of two types, which depends on least square error.**

| x | $u_1(x)$ | | |
|---|---|---|---|
| | **Exact** | **CM** | **PM** |
| **0** | 1 | 1 | 1 |
| **0.1** | 1.105170918 | 1.105063967 | 1.105621608 |
| **0.2** | 1.221402758 | 1.221237941 | 1.223154686 |
| **0.3** | 1.349858808 | 1.349995029 | 1.353601611 |
| **0.4** | 1.491824698 | 1.492808339 | 1.497964761 |
| **0.5** | 1.648721271 | 1.651150978 | 1.657246514 |
| **0.6** | 1.822118800 | 1.826496053 | 1.832449249 |
| **0.7** | 2.013752707 | 2.020316673 | 2.024575343 |
| **0.8** | 2.225540928 | 2.234085945 | 2.234627175 |
| **0.9** | 2.459603111 | 2.469276976 | 2.463607124 |
| **1** | 2.718281828 | 2.727362874 | 2.712517566 |
| **L.S.E.** | | $3.1824\times10^{-4}$ | $4.8333\times10^{-4}$ |

**Table (5.4)**

**Show a comparison between the exact solutions $1-2\cos(x)$ and the numerical solution $u_2(x)$ of two types, which depends on least square error**

| x | $u_2(x)$ | | |
|---|---|---|---|
| | **Exact** | **CM** | **PM** |
| **0** | -1 | -1 | -1 |
| **0.1** | -0.999996954 | -1.000001291 | -1.000003154 |
| **0.2** | -0.999987815 | -1.000004947 | -1.000012159 |
| **0.3** | -0.999972584 | -1.000010643 | -1.000026327 |
| **0.4** | -0.999951261 | -1.000018056 | -1.000044971 |
| **0.5** | -0.999923846 | -1.000026859 | -1.000067402 |
| **0.6** | -0.999890339 | -1.000036729 | -1.000092935 |
| **0.7** | -0.999850739 | -1.000047341 | -1.000120881 |
| **0.8** | -0.999805048 | -1.000058370 | -1.000150553 |
| **0.9** | -0.999753265 | -1.000069491 | -1.000181264 |
| **1** | -0.999695390 | -1.000080380 | -1.000212326 |
| **L.S.E.** | | $3.8900\times10^{-7}$ | $7.1700\times10^{-7}$ |

## 6 Conclusions:

In this paper, we use WRM's (CM and PM) for solving a system of linear VIDEK2; also we solve two examples by WRM's (CM and PM). In practices, we conclude the following remarks:

- In system of linear VIDEK2, if $f_i(x)$, $i$=1, 2, …,m is a polynomials, we get the exact solution.
- If $f_i(x)$ is not a polynomial, we see that the approximate solution by power functions give a better results than the approximate solution by Chebyshev polynomial for a system of linear VIDEK2.

In general, CM gives better results than the PM.

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**الخلاصة:**

يتعامل هذا ال بحثِ مع نظامَ معادلات فولتيرا التكاملية–التفاضلية الخطية من النّوعِ الثّاني، وفيه تم استخدام طريقتين لحل هذا النظام ، طريقة التجميعية وطريقةِ التجزئة. وفي هذه الدراسة تم اجراء المقارنة بين النَّتائجِ التّقريبيةِ والمضبوطةِ للمثالين العدديين وذلك اعتماداًعلى الخطأ المربّعِ الأقلِّ. و تهدف هذه المقارنة الى اظهار دقة النَّتائجِ التي حَصلنا عليها بإسْتِعْمالِ هذه الطّرق. كما اعتمدت الدراسة الحالية على استخدام برنامجِ ماتلاب إصدار 6.5 لحل الأمثلة حول ما تقدم.