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FULL-ORDER OBSERVER DESIGN FOR NON-LINEAR DYNAMIC CONTROL SYSTEM

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Abstract

The main theme of this paper is to design a full-order non-linear dynamic observer for estimating the state space estimator from its non-linear input-output dynamic control system.

The quadratic Lyapunov function stabilization approach has been developed and the sufficient conditions for existence the dynamic observer for some class of non-linear input-output dynamic system have been presented and discussed. Illustrations are presented to demonstrate the validity of the of the presented procedure.

Introduction

Sometimes all state space variables not available for measurements or it is not practical to measure all of them, or it is too expensive to measure all state space variables.[5]

The concept of observability allong with controllability first introduced by Kalman [2] plays an important role in both theoretical and practical concepts of modern control.

Observability of a system as conceptualized by Kalman fundamentally answers the question whether it is possible to identify any state $x(t_\circ)$ at time $t_\circ$ in the state space by observing the output $y(t_\circ,t_f)$ to a given input $u(t_\circ,t_f)$ over a finite time interval $t_f - t_\circ (t_f > t_\circ)$.

Luenberger observer [5] is essentially a full-order observer, the state variables of which give a one-to-one estimation of the linear dynamical system state variables.

In [1] the observer provides a smooth velocity estimate to be used by a trajectory tracking controller. The observer, controller and manipulator from a system where the observer error as well as the position and velocity tracking error tend to zero asymptotically.

In [7], the authors have consider a class of uncertain non-linear control system, and its controllability and stablizabilty

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x) \;\ldots\ldots\; (1)$$

Where $x(t)$ is n-state space, $u(t)$ is m-control signals which belong to a class of piecewise continuous function.

$f_i(x): R^n \to R^n$ to be continuous vector valued function, $q_i \in R^r$ is a vector of uncertain parameter, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$; $i = 0,1,\ldots,r$ are constant matrices.

Consequently [7], the Lyapunove function that stabilize a class of a non-linear system can be obtained easily from the result of nominal linear dynamic system and under some assumption both Lyapunove function of linear and non-linear dynamic system are obtained to be identical.

The work of this paper, focuses on the full order dynamic state observer for the uncertain non-linear control system

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x)$$

$$y(t) = \sum_{j=1}^{p} c_j x_j + g(x) \quad \textit{such that} \sum_{j=1}^{p} c_j x_j = c^T x$$

where $x(t)$ is n-state vector, y(t) is n –output vector $u(t)$ is m-control signals which belongs to a class of piecewise continuous function $g(x)$, $f_i(x)$; $i = 1,2,\ldots,r$ are non-linear vector functions, respectively of dimensions n and p

$A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$ and $c^T \in R^{1\times n}$; $i = 0,1,\ldots,r$ are constant matrices.

The following definitions and theorems are needed later on.

Definition

The system

$$\dot{x} = Ax$$
$$y = Cx$$

where x is n-state vector, y is m-output vector, $A \in R^{n\times n}$, $C \in R^{m\times n}$ is constant matrix, is said to be observable if every state $x(t_\circ)$ can be determined from the observation of $y(t)$ over a finite time interval $t_\circ \le t \le t_1$.

Definition

A (vector-valued) function f(x) is said to be Lipschitz function if there exists a constant L such that for all $x_1, x_2 \in R^n$,the following inequality holds $\| f(x_1) - f(x_2)\| \le \text{L} \|x_1 - x_2\|$

In this case L is said to be the Lipschitz constant of f .

Theorem

The system is described by

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Where (A,C) is observable means the dynamic system is observability if and only if the observability matrix constant matrices is completely state observability if and only if the observability matrix $\left[C^T \vdots C^T A \vdots \cdots \vdots C^T A^{n-1}\right]$ has full rank equal to n(T denoted to conjugate transpose matrix).

Theorem

If all eignvalues of the system $\dot{x} = Ax$, where $A \in R^{n\times n}$ is a constant matrix have negative real parts then the solution of this system is asymptotically stable (exponentially stable).

Lemma

Let A be a symmetric matrix and let $\lambda_{\min}(A)$ *and* $\lambda_{\max}(A)$ be the smallest and largest eignvalue of A, respectively,

then $\lambda_{\min}(A)\|x\|^2 \le x^T Ax \le \lambda_{\max}(A)\|x\|^2$, $\forall x \in R^n$

where $\|x\|^2 = \sum_{i=1}^{n} |x_i|^2$, x_i is the i-the component of x.

Remark

The Euclidean norm of marix can be defined as :

$$\|A\| = (\sum_{i=1}^{n}\sum_{j=1}^{n} |a_{ij}|^2)^{1/2}$$

where $|a_{ij}|$ is the absolute value of the matrix coefficient a_{ij} .

Problem Formulation

Consider the non-linear dynamical control system described as follows:

$$\left.\begin{aligned} &\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x) \\ &y(t) = \sum_{j=1}^{p} c_j x_j + g(x) \quad \textit{such that} \sum_{j=1}^{p} c_j x_j = c^T x \\ &x(0) = x_0 \end{aligned}\right\} \quad ..(2)$$

Where $x(t) \in R^n$ is unmeasurable state vector (see introduction [5]), $u(t)$ is the control input and $y(t) \in R^n$ is the output vector.

Suppose that the matrices A, B and C^T are constant matrices $g(x(t)), f(x(t))$, $i = 1,2,,...r$ are non-linrear vector functions, respectively of dimensions n and p.

Then a dynamical state observer of non-linear dynamical control system (2) is constructed as follows :

$$\hat{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i + \sum_{i=1}^{r} f_i(\hat{x}) + k_e\left[y(t) - c^T\hat{x} - g(\hat{x})\right]$$

..............................(3)

Where the observed state is denoted by $\hat{x}(t)$, and k_e is the observer gain matrix.

Define $e(t) = x(t) - \hat{x}(t)$ ....(4)

$e(t)$ is the dynamical error between the actual state and state observer $\hat{x}(t)$

The dynamical error in state observer (4) of non-linear dynamical control system((2),(3)) has the following dynamic equation:

$$\dot{e}(t) = \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] e(t) + \sum_{i=1}^{r}\left[f_i(x) - f_i(\hat{x})\right] - k_e\left[g(x) - g(\hat{x})\right]$$

..............................(5)

and as the following the linear part of equation (5)

$$\dot{e}(t) = \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] e(t) \text{ ....(6)}$$

$$e(0) = x(0) - \hat{x}(0) \text{ ....(7)}$$

If the dynamic behavior of dynamical error (5) is a symptotically stable, then the dynamical error (5) will tend to zero with an adequate speed as the time tend to infinite then the state x(t) given in (2) will converge to the state observer $\hat{x}(t)$.

Lemma

Consider the linrear dynamical control system

$$\left.\begin{aligned}\dot{x}(t) &= \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) \\ y(t) &= c^T x(t)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8)$$

satisfied the following conditions $\left(\sum_{i=0}^{r} q_i A_i \,,\, c^T\right)$ is observable matrices and $q_\circ = 1, q_i > 0$, $i = 1,2,\ldots,r$ are arbitrary, and let as the dynamical observer by

$$\hat{\dot{x}} = \left(\sum_{i=0}^{r} q_i A_i\right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_e\left[y(t) - c^T \hat{x}\right] \ldots\ldots (9)$$

Then the state observer has the dynamical obsever state exponentially stabilization.

Proof :

Let $e(t) = x(t) - \hat{x}(t)$ from equations (8),(9) we have

$$\dot{e}(t) = \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] e(t) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

with initial condition $e(0) = e_\circ$ has the solution

$$e(t) = e_\circ \exp\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] t$$

Due to the observability condtions of system (8) the matrix $\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]$ is Hurwitz stable matrix

$$\|e(t)\|_1 = \|e_\circ\|_1 \left\| \exp\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] t \right\|_2$$

where $\|\cdot\|_1$ is Eucledian norm and $\|\cdot\|_2$ is stable matrix norm

since $\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]$ is asymptotically stable thus there exist positive numbers α , $\mu_\circ$ such that

$$\|e(t)\|_1 \le \mu_\circ \|e_\circ\|_1 \exp[-\alpha t]$$

$$\Rightarrow \|e(t)\|_1 \le \mu \exp[-\alpha t] \qquad where \;\; \mu = \mu_\circ \|e_\circ\|_1$$

since α , μ are positive numbers

Then $\|e(t)\|_1 \to 0$ *as* $t \to \infty$ and hence $\|e(t)\|_1 \to 0$ (exponentially)

$\Rightarrow x(t) \to \hat{x}(t)$ (exponentially stabilizable)

i.e $x(t) \cong \hat{x}(t)$ as $t \to \infty$.

Theorem

Consider the linear dynamical system (8)

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t)$$

$$y(t) = \sum_{j=1}^{p} c_j x_j$$

$$x(0) = x_0$$

where $x \in R^n, u \in R^m$, $y \in R^p$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{p\times n}$

and the state variables are not available for measurement .

Consider the observer of linear dynamical control system (9)

$$\hat{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_e\left[y(t) - c^T \hat{x}\right]$$

$$\hat{x}(0) = \hat{x}_0$$

satisfied the following conditions

1. $\left[\left(\sum_{i=0}^{r} q_i A_i\right), C^T\right]$ of linear dynamical system (8) is completely state observer
2. The uncertain condition $q_\circ = 1, q_i > 0$; $i = 1,2,\ldots,r$ are arbitrary
3. The observer gain k_e selected such that $\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]$ is asymptotically stable matrix
4. The Riccati equation $\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]^T P + P\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] = -Q$ as a unique positive definite solution P for arbitrary positive definite matrix Q
5. assuming the Lyapunove function $V(x-\hat{x}) = (x-\hat{x})^T P (x-\hat{x})$ Then the dynamical error is asymptotically stable via observer gain Ke

Proof :

Consider the linear dynamical system (8)

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t)$$

$$y(t) = \sum_{j=1}^{p} c_j x_j \qquad suchthat \quad C^T x = \sum_{j=1}^{p} c_j x_j$$

$$x(0) = x_0$$

the state observer of linear dynamical control system (8) is given as follows (9)

$$\hat{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right)\hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_e\left[y(t) - c^T \hat{x}\right]$$

$\hat{x}(0) = \hat{x}_0$

Let $e(t) = x(t) - \hat{x}(t)$

We have dynamical error in state observer (9) of linear dynamical control system (8) is obtained to subtract (8) from (9) as follows

$$\left.\begin{aligned} \dot{e}(t) &= \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] e(t) \\ e(0) &= e_0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

To examine the stability of e(t), we consider the following quadratic Lyapunove function

$V(e(t)) = e^T(t) P e(t)$

thus

$$\dot{V}(e) = \left[\left(\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right) e\right]^T Pe + e^T P\left[\left(\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right) e\right]$$

$$\dot{V}(e) = \left[\left(\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right)^T e^T\right] Pe + e^T P\left[\left(\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right) e\right]$$

$$\dot{V}(e) = e^T\left\{\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]^T P + P\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]\right\} e \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

since $\left(\sum_{i=0}^{r} q_i A_i , c^T\right)$ is observable matrices and hence $\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]$ stabilizable (see Remark (2) [2])

hence the following equation has a unique positive definite solution P such that

$$\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]^T P + P\left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] = -Q \quad \ldots\ldots\ldots (7)$$

Thus from equation (6)

$\dot{V}(e) = -e^T Q e$; Q is positive definite solution and then $\dot{V}(e) < 0$, $\forall e, t$

since

$V(e) = e^T P e > 0$, $(P = P^T > 0)$ *and* $\dot{V}(e) < 0$

and of $V(e) = e^T P e \to 0$

as $\| e \| \to 0$ when *as* $\| e \| \to 0$

$\therefore$ The error dynamic system (10) is asymptotically stable via observer gain k_e

Thus $x(t) \cong \hat{x}(t)$ as $t \to \infty$.

Theorem

Consider the linear dynamical system (2)

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x)$$

$$y(t) = C^T x + g(x) \quad \textit{suchthat} \quad C^T x = \sum_{j=1}^{p} c_j x_j$$

$$x(0) = x_0$$

where $x \in R^n, u \in R^m$, $y \in R^p$, $A \in R^{n \times n}$, $B \in R^{n \times m}$, $C \in R^{p \times n}$,

$f : R^n \to R^n$, $g : R^n \to R^n$

Consider the observer of linear dynamical control system (2) is given in equation (3)

$$\hat{\dot{x}}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) \hat{x}(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + k_e\left[y(t) - c^T \hat{x} - g(\hat{x})\right]$$

$\hat{x}(0) = \hat{x}_0$

satisfied all the conditions (1,2,3,4 and 5) in theorem (2) and two conditions as the following

The non-linearity function

$f_i(x) : R^n \to R^n$; $i = 1, 2, \ldots, r$ and

$g(x) : R^n \to R^n$ are assumed to be Lipschitz condition with Lipschitz constant

L_i ; $i = 1, 2, \ldots, r$ and respectively

i.e , $\sum_{i=1}^{r} \| f_i(x) - f_i(\hat{x}) \| \le \sum_{i=1}^{r} L_i \| x - \hat{x} \|$, $L_i > 0$

and $\| g(x) - g(\hat{x}) \| \le L \| x - \hat{x} \|$, $L > 0$, $\hat{x} \in R^n$

If $\left[\left(\sum_{i=1}^{r} L_i\right) - L\right] \le \dfrac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$

Then the dynamical error (5)

$$\dot{e}(t) = \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] e(t) + \sum_{i=1}^{r}\left[f_i(x) - f_i(\hat{x})\right] - k_e\left[g(x) - g(\hat{x})\right]$$

is asymptotically stable via observer gain parameter k_e .

Proof:

The linear dynamical system (2)

$$\dot{x}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x)$$

$$y(t) = C^T x + g(x)$$

$$x(0) = x_0$$

The state observer of non-linear dynamical control system (2) is given as follows (3)

$$\hat{\dot{x}}(t) = \left(\sum_{i=0}^{r} q_i A_i\right) x(t) + \sum_{i=0}^{r} q_i B_i u_i(t) + \sum_{i=1}^{r} f_i(x) + k_e\left[y(t) - (x) - C^T \hat{x} - g(\hat{x})\right]$$

$x(0) = x_0$

The dynamical error instate observer (3) of non-linear linear dynamical control system is obtained to subtract (2) from (3) as follows:

$$\dot{e}(t) = \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right] e + \sum_{i=1}^{r}\left[f_i(x) - f_i(\hat{x})\right] - k_e\left[g(x) - g(\hat{x})\right]$$

Let $e(t) = x(t) - \hat{x}(t)$

Now, we consider the following quadratic Lyapunove function

For examining the stability of $e(t)$

Let $V(e) = e^T Pe$ $V(0) = 0$, $V(e) > 0$

And $\dot{V}(e) = \dot{e}^T Pe + e^T P\dot{e}$

$$\therefore \dot{V}(e) = \left\{ \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] e(t) + \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] - k_e[g(x) - g(\hat{x})] \right\}^T Pe$$

$$+ e^T P \left\{ \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] e(t) + \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] - k_e[g(x) - g(\hat{x})] \right\}$$

$$\Rightarrow \dot{V}(e) = \left\{ \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right]^T e^T(t) + \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})]^T - k_e^T [g(x) - g(\hat{x})]^T \right\} Pe$$

$$+ e^T P \left\{ \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] e(t) + \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] - k_e[g(x) - g(\hat{x})] \right\}$$

$$\dot{V}(e) = e^T \left\{ \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right]^T P + P \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] \right\} +$$

$$\left\{ \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})]^T - k_e^T [g(x) - g(\hat{x})]^T \right\} Pe +$$

$$e^T P \left\{ \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] - k_e[g(x) - g(\hat{x})] \right\}$$

by using theorem (2) we have

$$\dot{V}(e) = -e^T Qe + \left\{ \left(\sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] \right)^T - k_e^T [g(x) - g(\hat{x})]^T \right\} Pe +$$

$$e^T P \left\{ \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] - k_e[g(x) - g(\hat{x})] \right\}$$

$$\Rightarrow \dot{V}(e) = -e^T Qe + \left[\sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] \right]^T Pe + e^T P \left[\sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] \right] -$$

$$\left\{ [k_e(g(x) - g(\hat{x}))]^T Pe + [k_e(g(x) - g(\hat{x}))] e^T P \right\}$$

but

$$\left[\sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] \right]^T Pe + e^T P \left[\sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] \right] \le 2\|P\|\|e\| \left\| \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] \right\|$$

$$\le 2\|P\|\|e\| \sum_{i=1}^{r} \|f_i(x) - f_i(\hat{x})\|$$

$$\le 2\|P\|\|e\| \left(\sum_{i=1}^{r} L_i \right) \|x - \hat{x}\|$$

$$\le 2\|P\|\|e\|^2 \left(\sum_{i=1}^{r} L_i \right)$$

since $\sum_{i=1}^{r} \|f_i(x) - f_i(\hat{x})\|$ is satisfied Lipschitze condition and

$$[k_e(g(x) - g(\hat{x}))]^T Pe + e^T P[k_e(g(x) - g(\hat{x}))] \le 2\|P\|\|e\|\|k_e(g(x) - g(\hat{x}))\|$$

$$\le 2\|P\|\|e\| L \|x - \hat{x}\|$$

$$\le 2L\|P\|\|e\|^2$$

$$\Rightarrow \dot{V}(e) \le -e^T Qe + 2\|P\|\|e\|^2 \left(\sum_{i=1}^{r} L_i \right) - 2L\|P\|\|e\|^2$$

$$\le -\lambda_{\min}(Q)\|e\|^2 + 2\left[\left(\sum_{i=1}^{r} L_i \right) - L \right] \lambda_{\max}(P)\|e\|^2$$

$$\le \left(-\lambda_{\min}(Q) + 2\left[\left(\sum_{i=1}^{r} L_i \right) - L \right] \lambda_{\max}(P) \right) \|e\|^2$$

on using Lemma(1) and condition (2) , we get $\Rightarrow \dot{V}(e) < 0$

Since P is unique positive definite solution and it is clear that $V(e) > 0$, $V(0) = 0$ and by $\Rightarrow \dot{V}(e) < 0$ we have conclude that error dynamic system (5) is asymptotically stable via observer gain parameter

$t \to \infty$ as $x(t) \cong \hat{x}(t)$ Thus

Algorithem

Step(1). Consider the linear and non-linear dynamical system (8)-(2) with respectively.

Step(2). Check the pair $\left(\sum_{i=0}^{r} q_i A_i, C^T \right)$ is observable

Step(3). Check the following Lipschitz conditions

$$\sum_{i=1}^{r} \|f_i(x) - f_i(\hat{x})\| \le \sum_{i=1}^{r} L_i \|x - \hat{x}\|$$

$\|g(x) - g(\hat{x})\| \le L\|x - \hat{x}\|$ for $t \in R$, $x \in R^n$

And design the observer dynamic by (9)-(3)

Step(4). Select k_e that makes $\left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right]$ asymptotically stable by using pole placement (see[3])

Step(5). Let the dynamic error $e(t) = x(t) - \hat{x}(t)$ and $e'(0) = e_0$

$$\dot{e}(t) = \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] e + \sum_{i=1}^{r} [f_i(x) - f_i(\hat{x})] - k_e[g(x) - g(\hat{x})]$$

$e(0) = e_0$

Step(6). Set $V(t) = V(e(t)) = e^T(t)Pe(t)$ k_e where P is the unique positive definite solution of

$$\left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] P + P \left[\left(\sum_{i=0}^{r} q_i A_i \right) - k_e c^T \right] = -Q$$

for arbitrary positive definite matrix Q

Step(7). Check $\left[\left(\sum_{i=1}^{r} L_i\right) - k_e L\right] \le \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$, where $\sum_{i=0}^{r} L_i$, L and k_e are found in step(3), step(4) with respectively and $\lambda_{\min}(Q)$ denotes the smallest eigenvalue of Q, $\lambda_{\max}(P)$ denotes the largest eigenvalue of P.

Illustration (1)

Consider the linear dynamical control observation error description

$$\dot{e}(t) = \left[\left(\sum_{i=1}^{2} q_i A_i\right) - k_e c^T\right] e(t)$$

where $q_\circ = 1$, $q_1, q_2 > 0$; $for\ n = 2$ gives the matrices

$$A_\circ = \begin{bmatrix} -3 & 0 \\ -2 & -1 \end{bmatrix}_{2\times 2} \quad , \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix}_{2\times 2} \quad , \quad A_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}_{2\times 2}$$

$$c^T = \begin{bmatrix} 1 & 1 \end{bmatrix}_{1\times 2}$$

where $q_\circ = 1$, $q_1 = 1$, $q_2 = 2$; *where*

$$\sum_{i=0}^{2} q_i A_i = q_\circ A_\circ + q_1 A_1 + q_2 A_2 = A_\circ + A_1 + 2A_2 = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$

Let us examine state observability of the matrix

$\left(\sum_{i=0}^{2} q_i A_i , c^T\right)$, notice that

$$rank\left[(c^T)^T : A^T (c^T)^T\right] = rank\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = 2$$

$\Rightarrow$ the vectors $(c^T)^T, A^T(c^T)^T$ are independent and the rank of the matrix $\left[(c^T)^T : A^T(c^T)^T\right]$ is two

$\Rightarrow \left(\sum_{i=0}^{2} q_i A_i , c^T\right)$ is observable .

Then, the system is completely state observable

$\Rightarrow \left[\left(\sum_{i=0}^{r} q_i A_i\right) - k_e c^T\right]$ is asymptotically stable (see problem formulation)

Now, by using the formula

$\left[\left(\sum_{i=0}^{2} q_i A_i\right) - k_e c^T\right]_{2\times 2} = -E_{2\times 2}$ such that $E_{2\times 2}$ is arbitrary diagonal matrix.

$$\left[\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} k_{e1} \\ k_{e2} \end{pmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}\right] = -\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} -1-k_{e1} & 1-k_{e1} \\ 1-k_{e2} & -2-k_{e2} \end{bmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$$

$$\left.\begin{matrix} -1-k_{e1} = -2 \\ 1-k_{e1} = 0 \\ 1-k_{e2} = 0 \\ -2-k_{e2} = -3 \end{matrix}\right\} \Rightarrow k_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \left(\left(\sum_{i=0}^{2} q_i A_i\right) - k_e c^T\right) = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}$$

$\Rightarrow \left[\left(\sum_{i=0}^{2} q_i A_i\right) - k_e c^T\right]$ is diagonal matrix

$\Rightarrow \left[\left(\sum_{i=0}^{2} q_i A_i\right) - k_e c^T\right]$ is stable matrix

we have , $\dot{e}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} e(t)$

Now, find the positive defined solution $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ of the following algebraic equation

$$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = -Q \ ; \ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2P_{11} & -2P_{12} \\ -3P_{21} & -3P_{22} \end{bmatrix} + \begin{bmatrix} -2P_{11} & -3P_{12} \\ -2P_{21} & -3P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and thus we have

$$P_{11} = \frac{1}{4} \quad , \quad P_{12} = 0 \quad , \quad P_{21} = 0 \quad , \quad P_{22} = \frac{1}{6}$$

set the Lyapunove function $V(e) = e^T P e$, where $P = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$

$$V(e) = \begin{bmatrix} e_1 & e_2 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$V(e) = \begin{bmatrix} \frac{1}{4} e_1 & \frac{1}{6} e_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$V(e) = \frac{1}{4} e_1^2 \quad \frac{1}{6} e_2^2 \quad \Rightarrow V(e) = 0.25 e_1^2 + 0.1666666 e_2^2$$

$\Rightarrow V(e)$ positive defined function and have the system (10) is stabilized by using Lyapunove $V(e)$ and observer gain k_e

In example (1) we apply the proceeding main theorem (2) and we find Linear Dynamical control observation error (5) is asymptotically stabilizable .

Now, we illustrate the proceeding main theorem (3), we consider the non-linear dynamical control observation error.

Illustration (2)

where $g(x)$, $f_i(x)$; $i = 1,2$ are non linear vector functions and

$A_1 \in R^2 \quad , \quad c^T \in R^2 \quad ; \; i = 0,1,2$

and

$$\left[\left(\sum_{i=0}^{2} q_i A_i\right) - k_e c^T\right] = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \text{ is stable matrix}$$

$$f_1(x) = \begin{bmatrix} 0.03x\sin x \\ 0.05x \end{bmatrix} \quad , \quad f_2(x) = \begin{bmatrix} -0.2\cos x \\ -0.02\cos x \end{bmatrix}$$

and $g(x) = \frac{\|x\|}{5}$ are defined on the region $D = \{(t,x) \in R : |t| \le 1 \; , \; |x| \le 0.01\}$

$$\|f_1(x) - f_1(\hat{x})\| = \|0.03x\sin x - 0.03\hat{x}\sin\hat{x}\| + \|0.05x - 0.05\hat{x}\|$$

$$\le \left\|\frac{\partial(0.03x\sin x)}{\partial x}\right\| \|x - \hat{x}\| + 0.05\|x - \hat{x}\|$$

$$\le \left(0.03\|x\|\|\cos x\| + 0.03\|\sin x\|\right)\|x - \hat{x}\| + 0.05\|x - \hat{x}\|$$

$$\le 0.0803\|x - \hat{x}\|$$

$\Rightarrow$ Lipschitz constant $L_1 = 0.0803$

$$\|f_2(x) - f_2(\hat{x})\| = \|-0.2\cos x + 0.2\cos\hat{x}\| + \|-0.02\cos x + 0.02\cos\hat{x}\|$$

$$\le \left\|\frac{\partial(-0.2\cos x)}{\partial x}\right\| \|x - \hat{x}\| + 0.02\|x - \hat{x}\|$$

$$\le 0.2\|x - \hat{x}\| + 0.02\|x - \hat{x}\|$$

$$\le 0.22\|x - \hat{x}\|$$

$\Rightarrow$ Lipschitz condition $L_2 = 0.223$

and $g(x)\frac{\|x\|}{5} \quad , \quad k_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow k_e g(x) = \begin{bmatrix} \frac{\|x\|}{5} \\ \frac{\|x\|}{5} \end{bmatrix}$

$$\Rightarrow \|k_e(g(x) - g(\hat{x}))\| = \left\|\frac{\|x\|}{5} - \frac{\|\hat{x}\|}{5}\right\| + \left\|\frac{\|x\|}{5} - \frac{\|\hat{x}\|}{5}\right\|$$

$$= \frac{2}{5}\|\|x\| - \|\hat{x}\|\|$$

$$= \frac{2}{5}\|x - \hat{x}\|$$

$$\Rightarrow \quad L = \frac{2}{5} = 0.4$$

finally, we can discussed the condition $\left[\left(\sum_{i=1}^{2} L_i\right) - L\right] < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$ as the following

since the matrix $P = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.1666666 \end{bmatrix}$ been found by solution

$$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}^T P + P\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = -Q \; , \; Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

we have $\lambda_{\max}(P) = 0.25$ *and* $\lambda_{\min}(Q) = 1$

since $L_1 = 0.0803 \;\; , \;\; L_2 = 0.22 \;\; , \;\; L = 0.4$

$$\left[\left(\sum_{i=1}^{2} L_i\right) - L\right] = -0.0997 < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} = 2$$

the condition $\left[\left(\sum_{i=1}^{r} L_i\right) - L\right] < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$ is satisfied the system (4) is asymptotically stable in the large using the Lyapunove function

$$V(e) = 0.25e_1^{\,2} + 0.1666666e_2^{\,2}$$

$$\Rightarrow V(x - \hat{x}) = 0.25e_1^{\,2} + 0.1666666e_2^{\,2}$$

Conclusion

- Sufficient conditions were given for the design observers for a class of non-linear system
- This system is characterized by non-linear functions which are Lipschitz in nature.

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الخلاصة

ان الهدف الرئيسي من هذا البحث هو تصميم مخمن دينامي كامل الرتبة غير خطي لتخمين فضاء الحالة من خلال نظام سيطرة دينامي مدخل–مخرج (input-output) غير خطي كما وتم تطوير اسلوبية دالة ليابانوف التربيعية للاستقرارية وتم عرض العديد من الشروط الكافية لوجودية النظام المخمن الدينامي غير الخطي, قدمت كذلك امثلة توضيحية لتحديد دراسة صحة الاسلوبية المقدمة.