# TENSOR PRODUCT OF CONTINUOUS OPERATOR

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## Abstract

In this paper we prove that some properties of tensor product and we show that if $A_1, A_2$ are θ-adjoint then $A_1 \otimes I + I \otimes A_2$ is normal .Also we prove that a continuous operator is invariant under tensor product.

## Introduction

Let $H$ be an infinite dimensional separable complex Hilbert space with inner product $\langle , \rangle$ and let $B(H)$ be the algebra of all bounded linear operators on $H$, given $A_1, A_2 \in B(H)$, the tensor product $A_1 \otimes A_2$ on the Hilbert space $H \otimes H$ has been considered variously by many of authors (see [2],[3],[5],[6],[7]) .

When $A_1 \otimes A_2$ is defined as follows

$$\langle A_1 \otimes A_2 \, (x_1 \otimes y_1), (x_2 \otimes y_2) \rangle = \langle A_1 x_1, x_2 \rangle \langle A_2 y_1, y_2 \rangle$$

The operation of taking tensor product $A_1 \otimes A_2$ preserves many properties of $A_1$ and $A_2 \in B(H)$ but by no means all of them , Thus, whereas the binormal property is invariant under tensor product ,the *-paranormal property is not [9] .a gain ,whereas $A_1 \otimes A_2$ is posinormal if and only if $A_1$ and $A_2 \in B(H)$ are [9] and is similarly for U-operator ,pseudo normal ,subnormal and normaloid operators [3],[9],[10] .it was shown in [9]that paranormal is not invariant under tensor product .

In this section we prove some properties of tensor product .

***Proposition***

If $A \geq B \geq 0$ and $C \geq D \geq 0$ then $A \otimes C \geq B \otimes D \geq 0$

***Proof:***

Sinc e $A \geq B \geq 0$ then

$\langle Ax, x \rangle \geq \langle Bx, x \rangle \ \forall x \in H$

And Since $C \geq D \geq 0$ then $\langle Cx_1, x_1 \rangle \geq \langle Dx_1, x_1 \rangle$ $\forall x_1 \in H$

$$\langle Ax, x \rangle \langle Cx_1, x_1 \rangle \geq \langle Bx, x \rangle \langle Dx_1, x_1 \rangle$$

$$\langle A \otimes C \, (x \otimes x_1), (x \otimes x_1) \rangle \geq \langle B \otimes D \, (x \otimes x_1), (x \otimes x_1) \rangle$$

$$\langle A \otimes C - B \otimes D \, (x \otimes x_1), (x \otimes x_1) \rangle \geq 0$$

$\forall x \otimes x_1 \in H \otimes H$

then $A \otimes C \geq B \otimes D \geq 0$ .

**Proposition**

If $A \geq C \geq B \geq 0$ then $C \otimes A^2 \otimes B \geq C \otimes B^2 \otimes B$

***proof***

Since $A \geq C \geq B \geq 0$ then $\langle Ax, x \rangle \geq \langle Cx, x \rangle \geq \langle Bx, x \rangle \geq 0 \ \forall x \in H$

And $\langle A^2 x, x \rangle \geq \langle B^2 x, x \rangle \geq 0$

$$\langle Cx, x \rangle \langle A^2 x, x \rangle \langle Bx, x \rangle \geq \langle Cx, x \rangle \langle B^2 x, x \rangle \langle Bx, x \rangle \geq 0$$

$$\langle C \otimes A^2 \otimes B \, (x \otimes x \otimes x), (x \otimes x \otimes x) \rangle \geq \langle C \otimes B^2 \otimes B \, (x \otimes x \otimes x), (x \otimes x \otimes x) \rangle \geq 0$$

$$\langle C \otimes A^2 \otimes B - C \otimes B^2 \otimes B \text{-} \, (x \otimes x \otimes x), (x \otimes x \otimes x) \rangle \geq 0$$

then $C \otimes A^2 \otimes B \geq C \otimes B^2 \otimes B$

***Proposition***

If $A \geq B \geq 0$ then

$$\left( B^{\frac{r}{2}} \otimes A^p \otimes B^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq \left( B^{\frac{r}{2}} \otimes B^p \otimes B^{\frac{r}{2}} \right)^{\frac{1}{q}} \quad \text{and}$$

$$\left( A^{\frac{r}{2}} \otimes A^p \otimes A^{\frac{r}{2}} \right)^{\frac{1}{q}} \geq \left( A^{\frac{r}{2}} \otimes B^p \otimes A^{\frac{r}{2}} \right)^{\frac{1}{q}} \quad \text{for}$$

each $r \geq 0, q \geq 1, p \geq 0$

***Proof:***

Since $A \geq B \geq 0$ then $A^p \geq B^p$ hence $\langle A^p x_1, x_1 \rangle \geq \langle B^p x_1, x_1 \rangle$

$$\left\langle B^{\frac{r}{2}}x,x\right\rangle\left\langle A^{p}x_1,x_1\right\rangle\left\langle B^{\frac{r}{2}}x_2,x_2\right\rangle\geq$$
$$\left\langle B^{\frac{r}{2}}x,x\right\rangle\left\langle B^{p}x_1,x_1\right\rangle\left\langle B^{\frac{r}{2}}x_2,x_2\right\rangle\geq 0$$
$$\left\langle B^{\frac{r}{2}}\otimes A^{P}\otimes B^{\frac{r}{2}}\ (x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\ \right\rangle\geq$$
$$\left\langle B^{\frac{r}{2}}\otimes B^{P}\otimes B^{\frac{r}{2}}\ (x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\ \right\rangle\geq 0$$
$$\left\langle B^{\frac{r}{2}}\otimes A^{P}\otimes B^{\frac{r}{2}}-B^{\frac{r}{2}}\otimes B^{p}\otimes B^{\frac{r}{2}}\right.$$
$$\left.(x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\ \right\rangle\geq 0$$

hence $\left(B^{\frac{r}{2}}\otimes A^{p}\otimes B^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq\left(B^{\frac{r}{2}}\otimes B^{p}\otimes B^{\frac{r}{2}}\right)^{\frac{1}{q}}$

Similarly $\left(A^{\frac{r}{2}}\otimes A^{p}\otimes A^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq\left(A^{\frac{r}{2}}\otimes B^{p}\otimes A^{\frac{r}{2}}\right)^{\frac{1}{q}}$

***Proposition***

If $A\geq C\geq B\geq 0$ then for each $r\geq 0,q\geq 1,p\geq 0$

$$\left(C^{\frac{r}{2}}\otimes A^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq\left(C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq$$
$$\left(C^{\frac{r}{2}}\otimes B^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq 0$$

***Proof:***

$\Rightarrow$) since $A\geq C\geq B\geq 0$ then $\langle Ax,x\rangle\geq\langle Cx,x\rangle\geq\langle Bx,x\rangle\geq 0$ and
$\langle A^{p}x,x\rangle\geq\langle C^{p}x,x\rangle\geq\langle B^{P}x,x\rangle\geq 0 \qquad \forall x\in H$

$$\left\langle C^{\frac{r}{2}}x_1,x_1\right\rangle\left\langle A^{p}x,x\right\rangle\left\langle C^{\frac{r}{2}}x_2,x_2\right\rangle\geq$$
$$\left\langle C^{\frac{r}{2}}x_1,x_1\right\rangle\left\langle C^{p}x,x\right\rangle\left\langle C^{\frac{r}{2}}x_2,x_2\right\rangle\geq$$
$$\left\langle C^{\frac{r}{2}}x_1,x_1\right\rangle\left\langle B^{p}x,x\right\rangle\left\langle C^{\frac{r}{2}}x_2,x_2\right\rangle\geq 0$$
$$\langle\ C^{\frac{r}{2}}\otimes A^{p}\otimes C^{\frac{r}{2}}\ (x_1\otimes x\otimes x_2),(x_1\otimes x\otimes x_2)\ \rangle\geq$$
$$\langle\ C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}\ (x_1\otimes x\otimes x_2),(x_1\otimes x\otimes x_2)\ \rangle\geq$$
$$\langle\ C^{\frac{r}{2}}\otimes B^{p}\otimes C^{\frac{r}{2}}\ (x_1\otimes x\otimes x_2),(x_1\otimes x\otimes x_2)\ \rangle\geq 0$$

then $\langle\ C^{\frac{r}{2}}\otimes A^{p}\otimes C^{\frac{r}{2}}-C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}$
$(x_1\otimes x\otimes x_2),(x_1\otimes x\otimes x_2)\ \rangle\geq 0$

hence $\left(C^{\frac{r}{2}}\otimes A^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq\left(C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq 0$

and

$\langle\ C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}-C^{\frac{r}{2}}\otimes B^{p}\otimes C^{\frac{r}{2}}$
$(x_1\otimes x\otimes x_2),(x_1\otimes x\otimes x_2)\ \rangle\geq 0$

$\left(C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq\left(C^{\frac{r}{2}}\otimes B^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq 0$ then

$$\left(C^{\frac{r}{2}}\otimes A^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq\left(C^{\frac{r}{2}}\otimes C^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq$$
$$\left(C^{\frac{r}{2}}\otimes B^{p}\otimes C^{\frac{r}{2}}\right)^{\frac{1}{q}}\geq 0$$

The following example to show that how to find the Log – operator on a Hilbert space

**Example**

Let H be a Hilbert space and let $A,B$ be positive operator matrices such that

$B=\begin{pmatrix}e^2 & 0\\ 0 & e\end{pmatrix}$ and $A=V\begin{pmatrix}e^4 & 0\\ 0 & e^{3/2}\end{pmatrix}V^*$ where

$V=\frac{1}{\sqrt{5}}\begin{pmatrix}\sqrt{3} & \sqrt{2}\\ \sqrt{2} & -\sqrt{2}\end{pmatrix}$ ( unitary) ,then we have

$\mathrm{Log}(B)=\begin{pmatrix}2 & 0\\ 0 & 1\end{pmatrix}$ and $Log(A)=\begin{pmatrix}2 & \sqrt{3/2}\\ \sqrt{3/2} & 5/2\end{pmatrix}$

and $\mathrm{Log}(A)-\mathrm{Log}(B)\geq 0$ .

## *Theore*

Let $A,B,C$ be a positive operator if $Log(A)\geq Log(C)\geq Log(B)\geq 0$ then

$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(A^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq$$
$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(C^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq$$
$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(B^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq 0$$

***Proof:***

Since $\mathrm{Log}(A)\geq \mathrm{Log}(C)\geq \mathrm{Log}(B)$ then

$$\underbrace{\mathrm{Log}(A)+...+\mathrm{Log}(A)}_{\text{Ptime}}\geq \underbrace{\mathrm{Log}(C)+....+\mathrm{Log}(C)}_{\text{Ptime}}\geq \underbrace{\mathrm{Log}(B)+...+\mathrm{Log}(B)}_{\text{Ptime}}$$

$$p\mathrm{Log}(A)\geq p\mathrm{Log}(C)\geq p\mathrm{Log}(B)$$
$$\mathrm{Log}(A^{p})\geq \mathrm{Log}(C^{p})\geq \mathrm{Log}(B^{p})$$
$$\langle \mathrm{Log}(A^{p})x_1,x_1\rangle \geq \langle \mathrm{Log}(C^{p})x_1,x_1\rangle \geq \langle \mathrm{Log}(B^{p})x_1,x_1\rangle \quad \forall x_1\in H$$

$$\left\langle Log\left(C^{\frac{r}{2}}\right)x,x\right\rangle\left\langle Log\left(A^{p}\right)x_1,x_1\right\rangle\left\langle Log\left(C^{\frac{r}{2}}\right)x_2,x_2\right\rangle\geq$$
$$\left\langle LogC^{\frac{r}{2}}x,x\right\rangle\left\langle Log\left(C^{p}\right)x_1,x_1\right\rangle\left\langle Log\left(C^{\frac{r}{2}}\right)x_2,x_2\right\rangle\geq$$
$$\left\langle Log\left(C^{\frac{r}{2}}\right)x,x\right\rangle\left\langle Log\left(B^{p}\right)x_1,x_1\right\rangle\left\langle Log\left(C^{\frac{r}{2}}\right)x_2,x_2\right\rangle\geq 0$$

$$\langle\, \mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(A^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)(x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\,\rangle\geq$$
$$\langle\, \mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(C^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)(x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\,\rangle\geq$$
$$\langle\, \mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(B^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)(x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\,\rangle\geq 0$$

then

$$\langle\, \mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(A^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right) - \mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(C^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)$$
$$(x\otimes x_1\otimes x_2),(x\otimes x_1\otimes x_2)\,\rangle\geq 0$$
$$\forall x\otimes x_1\otimes x_2\in H\otimes H\otimes H$$

Then

$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(A^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq$$
$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(C^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)$$

Similarly

$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(C^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq$$
$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(B^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)$$

Hence

$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(A^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq$$
$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(C^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)\geq$$
$$\mathrm{Log}\left(C^{\frac{r}{2}}\right)\otimes \mathrm{Log}\left(B^{p}\right)\otimes \mathrm{Log}\left(C^{\frac{r}{2}}\right)$$

recall that an operator $A\in B(H)$ is **θ-**adjoin if $A^{*}=e^{i\theta}A$, $\theta\in R$ .[4]

***Theorem***

If $A_1\otimes I+I\otimes A_2$ is θ-adjoin then $A_1,A_2$ are θ-adjoin .

***Proof:***

Since $A_1\otimes I+I\otimes A_2$ is θ-adjoin then

$$A_1^{*}\otimes I+I\otimes A_2^{*}=e^{i\theta}\left(A_1\otimes I+I\otimes A_2\right)$$
$$A_1^{*}\otimes I+I\otimes A_2^{*}-e^{i\theta}A_1\otimes I-I\otimes e^{i\theta}A_2^{*}=0$$

$A_1^{*},I,e^{i\theta}A_1$ are linear independent then $I=A_1^{*}=e^{i\theta}A_1=0$ see [2] hence $I=0$ contradiction .then $A_1^{*},I,e^{i\theta}A_1$ are linear dependent hence $A_1^{*}=re^{i\theta}A_1$ , $\left|\frac{1}{r}\right|=e^{i\theta}$ and $A_2^{*},I,e^{i\theta}A_2$ are linear independent then $I=A_2^{*}=e^{i\theta}A_2=0$ hence $I=0$ contradiction .

then $A_2^*, I, e^{i\theta}A_2$ are linear dependent then $A_2^* = re^{i\theta}A_2$ , $|r| = e^{i\theta}$ then $r = 1$,hence $\theta = 0$

$A_1 = A_1^*$ , $A_1$ is θ-adjoin (since every self – adjoint is θ-adjoin where $\theta = 0$ ).

and $A_2^* = e^{i\theta}A_2$ , $A_2$ is θ-adjoin.

***Theorem***

If $A_1, A_2$ are θ-adjoint then $A_1 \otimes I + I \otimes A_2$ is normal .

***Proof:***

Since $A_1, A_2$ are θ-adjoin then $A_1^* = e^{i\theta_1}A_1$ and $A_2^* = e^{i\theta_2}A_2$ , $\theta_1, \theta_2 \in R$

$A_1A_1^* \otimes I + I \otimes A_2A_2^* - A_1^*A_1 \otimes I - I \otimes A_2^*A_2 =$

$e^{-i\theta_1}A_1^*e^{i\theta_1}A_1 \otimes I + I \otimes e^{-i\theta_2}A_2^*e^{i\theta_2}A_2 -$

$A_1^*A_1 \otimes I - I \otimes A_2^*A_2 = 0$

hence $A_1 \otimes I + I \otimes A_2$ is normal

In this section we may assume that $A \notin B(H)$

If $A : H \to H$ let $x \in H$ , $A$ is continuous at $x$ incase $x_n \to x$ impels $Ax_n \to Ax$ [1]

***Proposition***

If $A_i, i = 1,..., n$ are operator and $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ on Hilbert space $H_1 \otimes H_2 \otimes \cdots H_n$, $A_1 \otimes A_2 \otimes \cdots \otimes A_n$ is continuous operator if each $A_i, i = 1,..., n$ is continuous operator .

***Proof:***

By induction it is enough to show that $A_1 \otimes A_2$ is continuous if $A_1$ and $A_2$ are

If $\|x_n \otimes y_m - x \otimes y\| = 0$ we prove that

$\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y\| = 0$

since $A_1$ is continuous if $x_n \to x$ then $A_1x_n \to A_1x$

and $A_2$ is continuous if $y_m \to y$ then $A_2y_m \to A_2y$

$\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y\| =$

$\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y -$

$A_1x_n \otimes A_2y + A_1x_n \otimes A_2y\| =$

$\|A_1x_n \otimes (A_2y_m - A_2y) + (A_1x_n - A_1x) \otimes A_2y\| = 0$

then $A_1 \otimes A_2$ is continuous .

***Proposition***

If $A_1 \otimes A_2 \otimes \cdots \otimes A_n \neq 0$ is continuous operator and $A_1$ is continuous then $A_j, j = 2, \cdots, n$ are continuous operators .

***Proof:***

By induction ,it is suffices to show that $A_1 \otimes A_2$ then $A_1 \otimes A_2 \otimes \cdots \otimes A_n \neq 0$

Let $A_1 \otimes A_2$ is continuous.

if $x_n \otimes y_m \to x \otimes y$ then

$\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y\| = 0$

since $\left\| \|A_1x_n \otimes A_2y_m\| - \|A_1x \otimes A_2y\| \right\| \leq$

$\|A_1x_n \otimes A_2y_m - A_1x \otimes A_2y\| = 0$

then $\|A_1x_n \otimes A_2y_m\| - \|A_1x \otimes A_2y\| = 0$

$\|A_1x_n\|\|A_2y_m\| - \|A_1x\|\|A_2y\| = 0$

$\|A_1x_n\|\|A_2y_m\| - \|A_1x\|\|A_2y_m\| +$

$\|A_1x\|\|A_2y_m\| - \|A_1x\|\|A_2y\| = 0$

$(\|A_1x_n\| - \|A_1x\|)\|A_2y_m\| -$

$\|A_1x\|(\|A_2y_n\| - \|A_2y\|) = 0$

since $A_1$ is continuous if $x_n \to x$ then $A_1x_n \to A_1x$

then $\|A_1x\|(\|A_2y_n\| - \|A_2y\|) = 0$ hence $A_2$ is continuous.

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**الخلاصة**

نبرهن في هذا البحث بعض خواص الجداء التنسوري ونوضح ان المؤثر المستمر يحافظ على خواصه تحت تأثير الجداء التنسوري.