DERIVATION OF TRIDIAGONAL THREE-STAGES IMPLICIT RUNGE-KUTTA METHOD

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Abstract

In this paper, the main objective is to introduce a new modified approach for deriving of implicit Runge-Kutta methods by construction a tridiagonal implicit matrices of unknown coefficients.

Keywords: Runge-Kutta methods, Implicit Runge-Kutta methods.

Introduction

The idea of extending Euler method by allowing for a multiplicity of evolutions of a function within each step was originally proposed by Runge (1895). Further contributions were made by Heun (1900) and by Kutta (1901). The latter completely characterized the set of Runge-Kutta method of order 4 and proposed the first methods of order 5. Special methods for second-order differential equations were proposed by Nyström (1925) who also contributed to the development of methods for first-order equations.

Since the advent of digital computers, fresh interest had been focused on Runge-Kutta methods, and a large number of research workers have contributed to recent extensions to the theory and the development of particular methods. Although, early studies were devoted entirely to explicit Runge-Kutta methods, interest has now extended to implicit methods, which are now recognized as appropriate for stiff differential equations [5].

The general form of an r-stages Runge-Kutta methods is given by:

$$y_{n+1} = y_n + h\sum_{i=1}^{r} c_i k_i$$

where $k_i = f\left(x_n + ha_i, y_n + h\sum_{j=1}^{r} b_{ij} k_j\right)$,h is step length , $a_i = \sum_{j=1}^{r} b_{ij}$, and c_i, a_i and b_{ij}, for all i, j = 1, 2, …, r; are constants to be determined.

For convenience, we design the process by an array of constants, as follows [6]:

b_{11}	b_{12}	…	b_{1j}	a_1
b_{21}	b_{22}	…	b_{2j}	a_2
⋮	⋮	⋱	⋮	⋮
b_{i1}	b_{i2}	…	b_{ij}	a_i
c_1	c_2	…	c_j	

It is easy to classify Runge-Kutta methods, in three categories:

1. If $b_{ij} = 0$, $\forall$ i < j, then the method is called semi-explicit.
2. If $b_{ij} = 0$, $\forall$ i ≤ j, then the method is called explicit.
3. Otherwise it is called implicit.

The next results could be found in [2, 3, 4] either as statement or proof.

Theorem

Consider the system:

$y'(x) = f(y)$, $y = y_0$ at $x = x_0$.................(1)

then $\phi = \frac{1}{\gamma}$, where ϕ is a constant depends on γ, where $\gamma = \frac{i\beta}{\alpha}$, $\forall$ i = 1, 2, …, r and $\alpha, \beta \neq 0$ are numerical coefficients independent of the form of f(y).

Theorem

Consider the system:

$y'(x) = f(y),\ y = y_0$ at $x = x_0$

If $\phi = \frac{1}{\gamma}$, $r \leq \xi$, then $\sum_{j=1}^{r} c_j a_j^{k-1} = \frac{1}{k}$, for $k \leq \xi$, $\xi \geq 1$, where k = 1, 2, …, r, and r is the number of stages of Runge-Kutta method and ξ is the order of the considered method.

Remark

It is will known that the following formula $\sum_{j=1}^{r} b_{ij} a_j^{k-1} = \frac{a_j^k}{k}$ could be obtained to evaluate a_j's for i = 1, 2, …, r and $k \leq \xi$.

Derivation of Tridiagonals Three-Stages Implicit Runge-Kutta Method:

In this section, a new modification is made in order to derive a new formula of triadiagonals implicit Runge-Kutta method with the property that the elements of each diagonal are equal, for simplicity, the parameters related by this method are presented in the following design of process:

Design of process (1)

ω	σ	0	a_1
δ	ω	σ	a_2
0	δ	ω	a_3
c_1	c_2	c_3	

One can find the values of a_1, a_2 and a_3 by solving the third degree Legendre polynomial, the obtained results are:

$a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2} + \frac{\sqrt{15}}{10}$ and $a_3 = \frac{1}{2} - \frac{\sqrt{15}}{10}$

Similarly, using theorem (2), we can find c_1, c_2 and c_3, where:

$$\sum_{j=1}^{3} c_j a_j^{k-1} = \frac{1}{k}, \text{ for } k = 1, 2, 3$$

hence for k = 1, 2 and 3, we have:

$$c_1 + c_2 + c_3 = 1 \quad \text{.......(2)}$$

$$c_1a_1 + c_2a_2 + c_3a_3 = \frac{1}{2} \quad \text{.......(3)}$$

$$c_1 a_1^2 + c_2 a_2^2 + c_3 a_3^2 = \frac{1}{3} \quad \text{.......(4)}$$

Solving the above system for c_1, c_2 and c_3, we have:

$c_1 = 4/9$ and $c_2 = c_3 = 5/18$

Finally to find, ω, δ and σ use is made as given in remark (1) in which the consistent equations for integer k, are:

$$b_{11} a_1^{k-1} + b_{12} a_2^{k-1} + b_{13} a_3^{k-1} = \frac{a_1^k}{k}, \text{ for } i = 1$$

$$b_{21} a_1^{k-1} + b_{22} a_2^{k-1} + b_{23} a_3^{k-1} = \frac{a_2^k}{k}, \text{ for } i = 2$$

$$b_{31} a_1^{k-1} + b_{32} a_2^{k-1} + b_{33} a_3^{k-1} = \frac{a_3^k}{k}, \text{ for } i = 3$$

Hence for k = 1, we have:

$$\left.\begin{array}{l} b_{11} + b_{12} + b_{13} = a_1 \\ b_{21} + b_{22} + b_{23} = a_2 \\ b_{31} + b_{32} + b_{33} = a_3 \end{array}\right\} \text{.......(5)}$$

For k = 2, we have:

$$\left.\begin{array}{l} b_{11}a_1 + b_{12}a_2 + b_{13}a_3 = \frac{a_1^2}{2} \\ b_{21}a_1 + b_{22}a_2 + b_{23}a_3 = \frac{a_2^2}{2} \\ b_{31}a_1 + b_{32}a_2 + b_{33}a_3 = \frac{a_3^2}{2} \end{array}\right\} \text{.......(6)}$$

and for k = 3, we have:

$$\left.\begin{aligned} b_{11}a_1^2 + b_{12}a_2^2 + b_{13}a_3^2 &= \frac{a_1^3}{3} \\ b_{21}a_1^2 + b_{22}a_2^2 + b_{23}a_3^2 &= \frac{a_2^3}{3} \\ b_{31}a_1^2 + b_{32}a_2^2 + b_{33}a_3^2 &= \frac{a_3^3}{3} \end{aligned}\right\} \quad \text{..................(7)}$$

Since $b_{11} = b_{22} = b_{33} = \omega$, $b_{12} = b_{23} = \sigma$, $b_{21} = b_{32} = \delta$ and $b_{13} = b_{31} = 0$

From equations (5), we have:

$$\omega + \sigma + 0 = \frac{1}{2} \quad \text{......................................(8)}$$

$$\delta + \omega + \sigma = \frac{1}{2} + \frac{\sqrt{15}}{10} \quad \text{..........................(9)}$$

$$0 + \delta + \omega = \frac{1}{2} - \frac{\sqrt{15}}{10} \quad \text{.........................(10)}$$

Solving equations (8), (9) and (10) for ω, δ and σ, we have:

$$\omega = \frac{1}{2} - \frac{\sqrt{15}}{5}, \ \sigma = \frac{\sqrt{15}}{5} \text{ and } \delta = \frac{\sqrt{15}}{10}$$

Accordingly the design of process(1) become

$\frac{1}{2}-\frac{\sqrt{15}}{5}$	$\frac{\sqrt{15}}{5}$	0	$\frac{1}{2}$
$\frac{\sqrt{15}}{10}$	$\frac{1}{2}-\frac{\sqrt{15}}{5}$	$\frac{\sqrt{15}}{5}$	$\frac{1}{2}+\frac{\sqrt{15}}{10}$
0	$\frac{\sqrt{15}}{10}$	$\frac{1}{2}-\frac{\sqrt{15}}{5}$	$\frac{1}{2}-\frac{\sqrt{15}}{10}$
$\frac{4}{9}$	$\frac{5}{18}$	$\frac{5}{18}$	

Stability of Tridiagonal Three-Stages Implicit Runge-Kutta Method:

To obtain intervals of stability of 3-stages Runge-Kutta method, we consider the test problem $y' = \lambda y$, where $\text{Re}(\lambda) < 0$. Recall the tridiagonals three steps implicit Runge-Kutta method:

$$y_{n+1} = y_n + h(c_1k_1 + c_2k_2 + c_3k_3) \quad \text{...........(11)}$$

where:

$$k_1 = \lambda(y_n + hb_{11}k_1 + hb_{12}k_2 + hb_{13}k_3)$$

$$k_2 = \lambda(y_n + hb_{21}k_1 + hb_{22}k_2 + hb_{23}k_3)$$

$$k_3 = \lambda(y_n + hb_{31}k_1 + hb_{32}k_2 + hb_{33}k_3)$$

Now, let $\hbar = \lambda h$, we have:

$$k_1 = \lambda y_n + \hbar\left(\frac{1}{2} - \frac{\sqrt{15}}{5}\right)k_1 + \hbar\frac{\sqrt{15}}{5}k_2, \quad \hbar = \lambda h \quad \text{...................................(12)}$$

$$k_2 = \lambda y_n + \hbar\frac{\sqrt{15}}{10}k_1 + \hbar\left(\frac{1}{2} - \frac{\sqrt{15}}{5}\right)k_2 + \hbar\frac{\sqrt{15}}{5}k_3 \quad \text{...............................(13)}$$

$$k_3 = \lambda y_n + \hbar\frac{\sqrt{15}}{10}k_2 + \hbar\left(\frac{1}{2} - \frac{\sqrt{15}}{5}\right)k_3 \quad \text{...............................(14)}$$

Substituting equations (12) and (14) in equation (13), we get:

$$k_2 = \lambda y_n + \hbar\frac{\sqrt{15}}{10}\left[\frac{\lambda y_n + \hbar\frac{\sqrt{15}}{5}k_2}{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar}\right] + \hbar\left(\frac{1}{2} - \frac{\sqrt{15}}{5}\right)k_2 + \hbar\frac{\sqrt{15}}{5}\left[\frac{\lambda y_n + \hbar\frac{\sqrt{15}}{10}k_2}{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar}\right]$$

Then after some simplifications, we have:

$$k_2 = \lambda y_n\left[\frac{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{2}\right)\hbar}{\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2 - \frac{3}{5}\hbar^2}\right],$$

provided that $\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2 - \frac{3}{5}\hbar^2 \neq 0$

Substituting k_2 in equation (12), yields:

$$k_1=\frac{\lambda y_n+\lambda y_n\frac{\sqrt{15}}{5}\hbar\left[\dfrac{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{2}\right)\hbar}{\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2-\frac{3}{5}\hbar^2}\right]}{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar}$$

Then after some simplification, we get:

$$k_1=\lambda y_n\left[\frac{1+\left(\frac{3\sqrt{15}}{5}-1\right)\hbar+\left(\frac{7}{4}-\frac{3\sqrt{15}}{10}\right)\hbar^2}{\left[1-\left(1-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]}\right]$$

, provided that

$$\left[1-\left(1-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]\neq 0$$

substituting k_2 in equation (14)

$$k_3=\frac{\lambda y_n+\lambda y_n\frac{\sqrt{15}}{10}\hbar\left[\dfrac{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{2}\right)\hbar}{\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2-\frac{3}{5}\hbar^2}\right]}{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar}$$

Hence, after some simplifications:

$$k_3=\lambda y_n\left[\frac{1+\left(\frac{\sqrt{15}}{2}-1\right)\hbar+\left(1-\frac{\sqrt{15}}{4}\right)\hbar^2}{\left[1-\left(1-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]}\right],$$

provided that

$$\left[1-\left(1-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]\neq 0$$

Therefore equation (11),become:

$$y_{n+1}=y_n+ h y_n\left[\frac{4}{9}\frac{1+\left(\frac{3\sqrt{15}}{5}-1\right)\hbar+\left(\frac{7}{4}-\frac{3\sqrt{15}}{10}\right)\hbar^2}{\left[1-\left(1-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]} + \frac{5}{18}\frac{1-\left(\frac{1}{2}-\frac{\sqrt{15}}{2}\right)\hbar}{\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2-\frac{3}{5}\hbar^2} + \frac{5}{18}\frac{1+\left(\frac{\sqrt{15}}{2}-1\right)\hbar+\left(1-\frac{\sqrt{15}}{4}\right)\hbar^2}{\left[1-\left(1-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]}\right]$$

Hence, as a result, we have:

$$y_{n+1}=y_n\left[1+\frac{13\hbar+\left(\frac{73\sqrt{15}}{10}-13\right)\hbar^2+\left(19-\frac{49\sqrt{15}}{20}\right)\hbar^3}{18\left[1-\left(\frac{1}{2}-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]} + \frac{5\hbar+\left(\frac{5}{2}-\frac{5\sqrt{15}}{2}\right)\hbar^2}{18\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2-\frac{54}{5}\hbar^2}\right], \text{ where } y_n=r^n$$

Hence the corresponding root is given by:

$$r=1+\frac{13\hbar+\left(\frac{73\sqrt{15}}{10}-13\right)\hbar^2+\left(19-\frac{49\sqrt{15}}{20}\right)\hbar^3}{18\left[1-\left(\frac{1}{2}-\frac{2\sqrt{15}}{5}\right)\hbar+\left(\frac{1}{4}-\frac{\sqrt{15}}{5}\right)\hbar^2\right]\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]} + \frac{5\hbar+\left(\frac{5}{2}-\frac{5\sqrt{15}}{2}\right)\hbar^2}{18\left[1-\left(\frac{1}{2}-\frac{\sqrt{15}}{5}\right)\hbar\right]^2-\frac{54}{5}\hbar^2}$$

using computer facilities, one can find the values of $\hbar$, by solving the inequality $|r| < 1$, and the following interval of absolute stability is obtained, which is $\hbar \in (-9.5,-1) \cup (-0.637, 0)$.

Example:

Consider the first order differential equation:

$$y' = -y + x + 1$$

with initial condition $y(0) = 1$.

In order to give a comparison and describe the precision of the previously derived methods of Runge-Kutta, we can easily find the exact solution, which is:

$$y(x) = e^{-x} + x$$

Therefore using the 2-stage explicit, 2-stage semi-explicit, 2-stage implicit and triadiagonal implicit Runge-Kutta methods, we get the results presented in tables (4.1)) with step lengths $h = 0.1$.

From the results, it is easily noticed that the triadiagonal Runge-Kutta method has less accuracy than the other methods, but still it has more simplified form than the other methods and more simple than the other methods in applications.

Table (1)
Numerical results of example with step length h = 0.1.

x_i	*Exact*	*Explicit*		*Semi-explicit*		*Two stages implicit*		*Triadiagonal method*	
		Numerical solution	*Error*	*Numerical solution*	*Error*	*Numerical solution*	*Error*	*Numerical solution*	*Error*
0.0	1.00000000	1.00000000	0.00000000	1.00000000	0.00000000	1.00000000	0.00000000	1.00000000	0.00000000
0.1	1.00483741	1.00500000	0.00016258	1.00482757	0.00000984	1.00483743	0.00000001	1.00466161	0.00017580
0.2	1.01873075	1.01902500	0.00029424	1.01871293	0.00001781	1.01873077	0.00000002	1.01841263	0.00031811
0.3	1.04081822	1.04121762	0.00039940	1.04079403	0.00002418	1.04081825	0.00000003	1.04038650	0.00043171
0.4	1.07032004	1.07080195	0.00048190	1.07029087	0.00002917	1.07032008	0.00000003	1.06979924	0.00052079
0.5	1.10653065	1.10707576	0.00054510	1.10649766	0.00003299	1.10653070	0.00000004	1.10594167	0.00058898
0.6	1.14881163	1.14940356	0.00059193	1.14877580	0.00003582	1.14881168	0.00000004	1.14817217	0.00063946
0.7	1.19658530	1.19721022	0.00062492	1.19654748	0.00003782	1.19658535	0.00000004	1.19591032	0.00067498
0.8	1.24932896	1.24997525	0.00064629	1.24928985	0.00003911	1.24932901	0.00000005	1.24863103	0.00069793
0.9	1.30656965	1.30722760	0.00065794	1.30652984	0.00003981	1.30656971	0.00000005	1.30585927	0.00071038
1.0	1.36787944	1.36854098	0.00066154	1.36783941	0.00004002	1.36787949	0.00000005	1.36716530	0.00071413

Concluding remarks

1. One can see from error estimation of the results that (2-stage implicit) is the more accurate. Also three stages implicit tri-diagonal gives reasonable agreement exact solution.
2. The improved tridiagonal method is so easy to drive which are indeed implicit method and therefore to drive improved method with five diagonal and proving its stability.
3. Using Runge-Kutta method for solving delay differential equations.

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الخلاصة

في هدا البحث، الهدف الرئيسي هو أستحداث أسلوب جديد لأشتقاق طرائق رانك كوتا الضمنية ودلك عن طريق أستخدام مصفوفات ثلاثية الاقطار والتي تعطي معاملات الطريقة.