Journal of Al-Nahrain University Vol.10(2), December, 2007, pp.118-123 Science

G-SPLINE INTERPOLATION FOR APPROXIMATING THE SOLUTION OF FRACTIONAL DIFFERENTIAL EQUATIONS USING LINEAR MULTI-STEP METHODS

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Abstract

In this paper, we consider fractional differential equations of the form:

$$y^{(q)}(x) = F(x, y),\ x \in [a, b] \qquad \text{............(1)}$$
$$y(a) = \xi$$

where n < q < n + 1 and n is a positive integer number.

The aim of this paper is to approximate the solution of fractional differential equations using linear multi-step methods with the cooperation of G-spline interpolation.

Keywords: G-spline interpolation, Fractional calculus, Linear multi-step methods.

Introduction:

The subject of fractional calculus has a long history whose infancy dates back to the beginning of classical calculus and it is an area having interesting applications in real life problems. This type of calculus has its origin in the generalizations of the differential and integral calculus, [Oldham, 1998].

The traditional integral and derivative are, to say the least, a stable for the technology professional, essential as a means of understanding and working with natural and artificial systems, [Loverro, 2004].

Fractional calculus is a field of mathematics that grows of the traditional definitions of the calculus of the integral and derivative operators in which the same way fractional exponents is an outgrowth of exponents with integer value. According to our primary school teachers exponents provide a short notation for what is essentially a repeated multiplication of a numerical value. While any one can verify that $x^3 = x \times x \times x$, how might one describe the physical meaning of $x^{3.4}$, or moreover the transcendental exponent x^{π}. One can not conceive what it might like to multiply a number or quantity by itself 3.4-times or π-times, and yet these expressions have a definite value for any value x, verifiable by infinite series expansion, or more practically by calculator, [Loverro, 2004].

Now, in the same way, consider the integral and derivative. Although they are indeed concepts of a higher complexity by nature, it is still fairly easy to physically represent their meaning.

The analytical solution of fractional differential equations, in general, has many difficulties, therefore numerical methods may be suitable for approximating the solution.

Once mastered, the idea of completing numerous of these operations, integrations or differentiations follows naturally. Given the satisfaction of a very few restrictions (e.g., function continuity) completing n integrations can become as methodical as multiplication. Fractional calculus follows quite naturally from our traditional definitions. And just as fractional exponents, such as the square root may find their way into innumerable equations and applications, it will become apparent that integrations of order 1/2 and beyond ca find practical use in many modern problems.

Nearly 60 years ago, I. J. Schoenberg [Schoenberg, 1946] introduced the subject of "Spline Functions". Since then splines, have proved to be enormously important in various branches of mathematics, such as approximation theory, numerical analysis, numerical treatment of ordinary, integral, partial differential equations, and statistics, etc.

There are several types of splines [deBoor, 1978], [Powel, 1981] and [Stephen, 2002]. The most important of these types of splines which is necessary to the work of this paper is the so called G-spline interpolation.

Ahlberg and Nilson [Ahlberg, 1966] and Schoenberg [Schoenberg, 1968] have treated the best approximation of linear functionals using G-spline interpolation formula. Therefore, we shall use their techniques to solve problem (1) which gives the best approximation in the sense of Sard [Sard, 1963].

G-Spline Interpolation Formula:

G-spline interpolation was first present-ed by Schoenberg [Schoenberg, 1968] as a tool used to specify the interpolatory conditions:

$f^{(j)}(x_i) = y_i^{(j)}$, for $(i, j) \in e$

which is so called the Hermite-Birkhoff problem (and abbreviated by HB-problem), where e is a certain set of ordered pairs

Schoenberg in 1968, extended the idea of Hermite for splines to specify that the orders of the derivatives specified may vary from knot to knot. Again, Schoenberg has defined G-spline as a smooth piecewise poly-nomials, where the smoothness is governed by the incidence matrix E, which will be defined later in this section and then he proved that G-splines, satisfies what we call the "minimum norm property", which is used for the optimality of the G-spline functions, which is given mathematically by the inequality:

$$\int_I \left[f^{(m)}(x)\right]^2 dx > \int_I \left[S^{(m)}(x)\right]^2 dx$$

where the function S is called a G-spline function and it is a polynomial of degree $2m - 1$ over the interval I.

If the only polynomial that solves the homogeneous HB-interpolation problem is identically the zero polynomial, then the problem is said to be m-poised, [Mohammed, 2006].

The consideration of the HB problem to be m-poised problem will play an important role for the uniqueness of the HB-problem.

The HB-Problem, [Schoenberg, 1968], [Ahlberg, 1966]:

It is convenient in this subsection to discuss the HB-problem, before; we give the tractable formal definition of the natural G-spline interpolation. Let us consider the knots points:

$x_1 < x_2 < \ldots < x_k$

to be distinct and real and let α be the maximum of the orders of the derivatives to be specified at the knots.

Define an incidence matrix E, by:

$E = [a_{ij}]$, $i = 1, 2, \ldots, k$; $j = 0, 1, \ldots, \alpha$

where:

$$a_{ij} = \begin{cases} 1, & \text{if } (i, j) \in e \\ 0, & \text{if } (i, j) \notin e \end{cases}$$

Here $e = \{(i, j): i = 1, 2, \ldots, k; j = 0, 1, \ldots, \alpha\}$ has been chosen in such a way that i takes the values 1, 2, …, k; one or more times, while $j \in \{0, 1, \ldots, \alpha\}$ and $j = \alpha$ is attained in at least one element (i, j) of e, assume also that each row of the incidence matrix E and last column of E should contain some element equals 1.

Let $y_i^{(j)}$ be prescribed real numbers for each $(i, j) \in e$. The HB-problem is to find $f(x) \in C^\alpha$, which satisfies the interpolatory condition:

$f^{(j)}(x_i) = y_i^{(j)}$, for $(i, j) \in e$......................(2)

The matrix E will likewise describes the set of equations (2) if we define the set e by:

$e = \{(i, j) \mid a_{ij} = 1\}$

then the integer $n = \sum_{i,j} a_{ij}$, really is the number of interpolatory conditions required to constitute the system (2).

<u>Definition</u>

Let m be a natural number, then the HB-problem (2) is said to be m-poised provided that if:

$p(x) \in \Pi_{m-1}$

$p^{(j)}(x_i) = 0$ if $(i, j) \in e$

then:

$p(x) = 0$.

(Π_{m-1} is the class of polynomials of degree m – 1 or less).

At this point, the G-spline interpolant of order m to f can be given in terms of the fundamental G-spline L_{ij}, by:

$$S_m(x) = \sum_{(i,j)\in e} L_{ij}(x) y_i^{(j)} \text{ ....(3)}$$

where:

$$L_{ij}^{(s)}(x_r) = \begin{cases} 0, & \text{if } (r,s) \neq (i,j) \\ 1, & \text{if } (r,s) = (i,j) \end{cases}$$

The definition of G-spline is facilitated by defining a matrix E* which is obtained from the incidence matrix E by adding m – α – 1 columns of zeros to the matrix E. Let $E^* = [a_{ij}^*]$, where (i = 1, 2, …, k; j = 0, 1, …, m – 1), and:

$$a_{ij}^* = \begin{cases} a_{ij}, & \text{if } j \le \alpha \\ 0, & \text{if } j = \alpha+1, \alpha+2, \ldots, m-1 \end{cases}$$

If j = α + 1, then E* = E.

Definition

A function S(x) is called natural G-spline for the knots $x_1, x_2, \ldots, x_k$ and the matrix E* of order m provided that it satisfies the following conditions:

1. $S(x) \in \Pi_{2m-1}$ in (x_i, x_{i+1}), i = 1, 2, …, k – 1.
2. $S(x) \in \Pi_{m-1}$ in $(-\infty, x_1)$ and in (x_k, ∞).
3. $S(x) \in C^{m-1}(-\infty, \infty)$.
4. If $a_{ij}^* = 0$, then $S^{(2m-j-1)}(x)$ is continuous at $x = x_i$.

Let $\mathcal{S}(E^*; x_1, x_2, \ldots, x_k)$ denotes the class of all G-spline of order m.

Approximation of Linear Functionals with the Sense of G-Spline Formula, [Schoenberg, 1968]:

Let I = [a,b] be a finite interval contain-ing the knots points $x_1, x_2, \ldots, x_k$ and let us consider a linear functional:

$$\mathcal{L}f : C^{\alpha}[a, b] \longrightarrow \mathbb{R}$$

of the form:

$$\mathcal{L}f = \sum_{j=0}^{\alpha} \int_a^b a_j(x) f^{(j)}(x)\,dx + \sum_{j=0}^{\alpha} \sum_{i=1}^{n_j} b_{ji} f^{(j)}(x_{ji}) \text{ ....(4)}$$

where the $a_j(x)$ are piecewise continuous functions in I, $x_{ji} \in I$ and b_{ji} are real constants, we can approximate the functional (4) using the formula:

$$\mathcal{L}f = \sum_{(i,j)\in e} \beta_{ij} f^{(j)}(x_i) + Rf \text{ ....(5)}$$

Therefore, in order to find the approximation $\mathcal{L}f$ given by (5), which is best in some sense, we propose to determine the reals β_{ij}.

I. J. Schoenberg [Schoenberg, 1968] states two procedures to determine β_{ij}. One of them is the so called Sard procedure, which can be summarized by the following theorem:

Theorem , [Schoenberg, 1968]:

If α < m < n and the HB-problem (2) is m-poised, then Sard's best approximation (5) to $\mathcal{L}F$ of order m is obtained by operating with $\mathcal{L}$ on both sides of the G-spline interpolation formula (3) of order m.

In other words, the coefficients β_{ij} are given by:

$\beta_{ij} = \mathcal{L}L_{ij}(x)$

where $L_{ij}(x)$ are the fundamental functions of (3).

G-Spline Interpolation Techniques for Approximating the Solution of Fractional Differential Equations:

The definition of the fractional derivatives and some well known results of fractional calculus tell us that we interpret fractional differential equations such as, [Oldham, 1998]:

$$D^q y(x) = F(x, y(x)),\ y(a) = \xi \text{ ....(6)}$$

where n < q < n + 1 and $D^q := \frac{d^q}{dx}$. Hence, upon carrying D^{1-q} to the both sides of (6), yields:

$$D^{1-q}D^q y(x) = D^{1-q}F(x, y(x)),\ y(a) = \xi \text{.....(7)}$$

with n < q < n + 1, $n \in \mathbb{N}$.

Equation (7) can be simplified using formulas and definitions of the fractional derivatives, to get:

$y'(x) = g(x, y), y(a) = \xi, x \in [a, b]$ ....(8)

The exact solution of (8) evaluated at $x_k = a + kh$, as:

$$Y(x_k) - Y(x_\ell) = \int_{x_\ell}^{x_k} g(x, y)\, dx, 0 \le \ell \le k \quad .....(9)$$

and then replacing g by its G-spline Interpolant.

An m-th order linear multistep formula of the general type can be given by, [Chi, 1972]:

$$y_{n+k} - y_{n+\ell} = \sum_{j=0}^{p} \sum_{i=0}^{k} \beta_{ij} h^{j+1} g^{(j)}(x_{n+i}, y_{n+i})$$

...........(10)

where y_j is an approximation to $Y(x_j)$ and $[x_n, x_{n+k}] \subset [a, b]$.

Now, we pick k and α along with the m-poised HB-problem corresponding to the n values:

$\{ \varphi_i^{(j)} = \varphi^{(j)}(i), (i, j) \in e\}$

where $\varphi(s) \equiv g(x_n + sh, Y(x_n + sh))$, for $0 \le s \le k$.

Then as we mention previously in section two, the G-spline interpolant to φ can be given in terms of the fundamental G-splines $L_{ij}(x)$, by:

$$S_m(s) = \sum_{(i,j) \in e} L_{ij}(s) \varphi_i^{(j)}$$

Referring again to our HB-problem, there is a unique G-spline S_m in $\mathcal{S}(E^*; x_1, x_2, \ldots, x_k)$, such that $S_m^{(j)}(i) = \varphi_i^{(j)}$, [Schoenberg, 1968].

In order to determine the coefficients β_{ij} in (10), we replace g in (9) by its G-spline interpolant, make a change of variables, integrate and compare the results with (10).

As a consequence of the uniqueness of the G-spline interpolant, and the sense of theorem (1) it follows that:

$$\beta_{ij} = \int_{\ell}^{k} L_{ij}(s)\, ds \quad ....(11)$$

Finally, it is the time to summarize the above results as follows:

Equation (10) can have approximate solution using linear multistep methods in terms of G-spline interpolation, as follows:

$$y(x_{n+k}) - y(x_{n+\ell}) = \sum_{(i,j) \in e} h^{j+1} \beta_{ij} g^{(j)}(x_{n+i}, y_{n+i})$$

where β_{ij} are presented in equation (11).

Illustrative Examples:

Next, we give two examples as an illustration to the above discussed approach for solving fractional differential equations. The obtained results are compared with the exact solution which are available in these two examples.

Example (1):

Consider the fractional differential equation:

$$y^{(1/2)}(x) = -y(x) + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)}$$

$y(0) = 0$

where the exact solution is given by $Y(x) = x^2$.

Consider it is required that a three-step method be constructed in such a way that it is exact for $Y \in \Pi_4$.

To construct such a method via G-splines, an HB-problem must be first chosen. The choice:

$\Delta = \{0, 1, 2\}$

are taken to be the knote points and let:

$e = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0)\}$

We shall seek for $S_4(s) \in \mathcal{S}_4(E^*, \Delta)$, with:

$$E = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and for which:

$S_4^{(j)}(i) = \varphi^{(j)}(i), (i,j) \in e$

Integrating $S_4(s)$ over [1, 2] yields the closed formula:

$$y_{n+2} = y_{n+1} + [h\{\beta_{00} g(x_n, y_n) + \beta_{10} g(x_{n+1}, y_{n+1}) + \beta_{20} g(x_{n+2}, y_{n+2})\} + h^2\{\beta_{01} g'(x_n, y_n) + \beta_{11} g'(x_{n+1}, y_{n+1})\}] \quad ....(12)$$

where:

$$\beta_{00} = \int_1^2 L_{00}(s)\, ds = \frac{503}{3072},$$

$$\beta_{10} = \int_1^2 L_{10}(s)\, ds = \frac{1748}{3072},$$

$$\beta_{20} = \int_1^2 L_{20}(s)\, ds = \frac{821}{3072},$$

$$\beta_{01} = \int_1^2 L_{01}(s)\, ds = \frac{150}{3072},$$

$$\beta_{11} = \int_1^2 L_{11}(s)\, ds = \frac{1068}{3072}.$$

and

$$g(x, y) = D^{1/2}\left[-y + x^2 + \frac{2x^{3/2}}{\Gamma(5/2)}\right]$$

$$= y(x) - x^2 + 2x$$

$$g'(x, y) = y - x^2 + 2$$

where the fundamental G-spline functions are defined by:

$$L_{00}(s) = [128 - 494s^2 + 411s^3 + 25s_+^7 - 70s_+^6 - 20(s-1)_+^7 - 140(s-1)_+^6 - 5(s-2)_+^7]/128.$$

$$L_{01}(s) = [64s - 150s^2 + 95s^3 + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/64.$$

$$L_{10}(s) = [118s^2 - 95s^3 - 5s_+^7 + 14s_+^6 + 4(s-1)_+^7 + 28(s-1)_+^6 + (s-2)_+^7]/32.$$

$$L_{11}(s) = [-54s^2 + 63s + 5s_+^7 - 14s_+^6 - 4(s-1)_+^7 - 28(s-1)_+^6 - (s-2)_+^7]/32.$$

$$L_{20}(s) = [22s^2 - 31s^3 - 5s_+^7 + 14s_+^6 + 4(s-1)_+^7 + 28(s-1)_+^6 + (s-2)_+^7]/128.$$

where:

$$(s-a)_+^n = \begin{cases} (s-a)^n, & \text{if } s>a \\ 0, & \text{if } s \le a \end{cases}$$

Figure (1) illustrate the approximated results obtained by using eq.(12) and its comparison with the exact solution $Y(x) = x^2$.

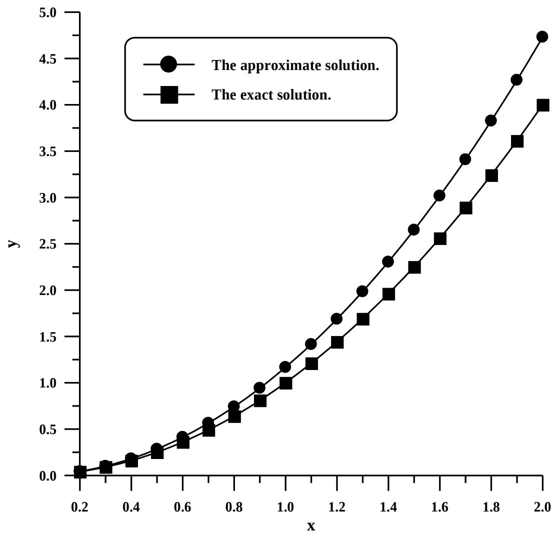

Fig. (1): Results of example (1).

Example (2):

Consider the previous fractional differential equation given in example (1), with the same initial condition, but we take the fundamental G-splines for the HB-problem with the HB-set:

$$e = \{(0, 0), (1, 0), (2, 0)\}$$

and knots 0, 1, 2. Integrating from 1 to 2 with h = 0.1, then we get a closed formula of the two step implicit method as:

$$y_{n+2} = y_{n+1} + h[\beta_{00}g(x_n, y_n) + \beta_{10}(x_{n+1}, y_{n+1}) + \beta_{20}g(x_{n+2}, y_{n+2})]$$

where:

$$\beta_{00} = \int_1^2 L_{00}(s)\, ds = -0.063,$$

$$\beta_{10} = \int_1^2 L_{10}(s)\, ds = 0.625,$$

$$\beta_{20} = \int_1^2 L_{20}(s)\, ds = 0.438.$$

and the fundamental G-spline functions are given by:

$$L_{00}(s) = 1 - 5/4s + 1/4\,(s-0)_+^3 - 1/2\,(s-1)_+^3 + 1/4\,(s-2)_+^3 .$$

$$L_{10}(s) = 3/2s - 1/2\,(s-0)_+^3 + (s-1)_+^3 - 1/2\,(s-2)_+^3 .$$

$$L_{20}(s) = -1/4s + 1/4\,(s-0)_+^3 - 1/2\,(s-1)_+^3 + 1/4\,(s-2)_+^3 .$$

Figure (2) illustrate the approximated results obtained by using eq.(12) and its comparison with the exact solution $Y(x) = x^2$.

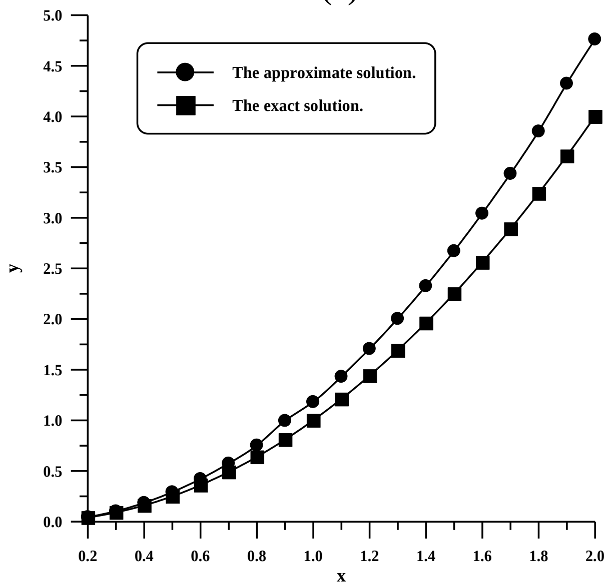

Fig. (2): Results of example (2).

Conclusions

1. One can notice that the error increases when x_i approaches to 2, because of the accumulation error. An approach to avoid this problem is to decrease the step size h.
2. In this paper, we had used the one step method to calculate the predictor value of y_{n+k} in order to evaluate the corrector value of y_{n+k} in equation (10).

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الخلاصة

في هذا البحث، قمنا بدراسة المعادلات التفاضلية الكسرية بالشكل:

$y^{(q)}(x) = F(x, y),\ x \in [a, b],\ y(a) = \xi$

حيث أن $n < q < n + 1$، و n عدد صحيح موجب.

بصورة عامة، هنالك العديد من الصعوبات في إيجاد الحل التحليلي للمعادلات التفاضلية الكسرية، ولذلك فأن الطرق العددية تكون مناسبة في بعض الاحيان لتقريب هذه الحلول.

الهدف من هذا البحث هو لتقريب حلول المعادلات التفاضلية الكسرية باستخدام طرائق متعددة الخطوات (وبمساعدة دوال التقريب-G).