Journal of Al-Nahrain University Vol.11(1), April, 2008, pp.105-119 Science

ON FINDING THE COEFFICIENTS c_{d-3} OF THE EHRHART POLYNOMIALS OF A POLYHEDRON IN $\Re^d$

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Abstract

Computing the volume and integral points of a polyhedron in $\Re^d$ is a very important subject in different areas of mathematics, such as: number theory, toric Hilbert functions, Kostant's partition function in representation theory, Ehrhart polynomial in combinatorics, cryptography, integer programming, statistical contingency and mass spectroscope analysis.

Therefore a method for finding the coefficients of this polynomial are to be listed. A program in visual basic language is made for finding the general differentiation of the function that used for finding the coefficients of the Ehrhart polynomial which illustrated by a flow chart in Fig.(1).

Introduction

The Ehrhart polynomial of a convex lattice polytope counts the number of integral points in an integral dilate of the polytope(Every bounded polyhedron is said to be a polytope). Ehrhart proved that, the function which counts the number of lattice points that lie inside the dilated polytope tP(Let $P \subset \Re^d$ be an integral polytope, for positive integer t, let tP={tX : X∈P} denote the dilated polytope P) is a polynomial in t and it is denoted by L(P,t), which is the cardinal of $(tP \cap Z^d)$ where Z^d is the integer lattice in $\Re^d$. From the definitions of the Ehrhart polynomial, the leading coefficient is the volume of the polytope and the constant term is one; these are termed as the trivial coefficient of the Ehrhart polynomial, the other coefficients are nontrivial, [1].

In this work we present a method for computing the coefficients of the Ehrhart polynomial that depends on the concepts of Dedekind sum and residue theorem in complex analysis. General formula that counts the derivatives in the introduced method is given. To the best of our knowledge this method seems to be new.

Formulation of this method

Before we discuss the method the following theorm where needed.

Theorem (1), [2]:

Let $P \subset \Re^d$ be a lattice d-polytope, with the Ehrhart polynomial $L(P,t) = \sum_{i=0}^{d} c_i t^i$. Then c_d is the volume of P, while the constant term is one, which is equal to the Euler characteristic of P.

The other coefficients of L(P,t) are not easily accessible. In fact, a method of computing these coefficients was unknown until quite recently, [1], [3] and [4].

Counting integral points using Dedekind sums

In this section we describe the relation between the Dedekind sum and the Ehrhart polynomial of a polytope and discussed a theorem that counts the number of integral points in a polytope.

Recall that the Dedekind sum of two relatively prime positive integers a and b, denoted by S(a,b), is defined as

$$S(a,b) = \sum_{i=1}^{b} ((\frac{i}{b}))((\frac{ai}{b}))$$

where

$$((x)) = \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \notin Z \\ 0 & \text{if } x \in Z \end{cases}$$

$\lfloor x \rfloor$ is the greatest integer $\le x$ and Z is the set of integer numbers.

Remark (1):

The discrete Fourier expansions can be used to rewrite the Dedekind sum in terms of the Dedekind cotangent sum, that is, for two relatively prime positive integers a and b:

$$S(a,b)=\frac{1}{4b}\sum_{k=1}^{b-1}\cot\left(\frac{\pi ka}{b}\right)\cot\left(\frac{\pi k}{b}\right)$$

where S(a,b) is the Dedekind sum of a and b, [5, p. 72].

Counting integral points using the residue theorem

This section is concerned with a method given in [6] to count the integral points of a given polytope by means of the residue theorem.

Theorem (2), [6]:

Let P be a polytope defined as

$$P=\left\{(x_1,\ldots,x_d)\in\Re^d:\sum_{k=1}^{d}\frac{x_k}{a_k}\le 1 \text{ and } x_k>0\right\}$$
.... (1)

with vertices $(0,0,\ldots,0)$, $(a_1,0,\ldots,0)$, $(0,a_2,0,\ldots,0),\ldots,(0,\ldots,0,a_d)$, where $a_1,\ldots,a_d$ are positive integers, $A=a_1a_2\ldots a_d$, $A_k=a_1a_2..\hat{a}_k..a_d$ (where $\hat{a}_k$ means the factor a_k is omitted), and $A_{j,k}$ denotes $a_1..\hat{a}_j\ldots\hat{a}_k\ldots a_d$.

$f_{-t}(z)$ and Ω are defined as

$$f_{-t}(z)=\frac{z^{-tA}-1}{(1-z^{A_1})(1-z^{A_2})\ldots(1-z^{A_d})(1-z)z},$$

and

$$\Omega=\{z\in C\setminus\{1\}: z^{\frac{A}{a_ka_j}}=1, 1\le k<j\le d\}, \text{ then}$$

$$L(P,t)=1-\operatorname{Res}(f_{-t}(z),z=1)-\sum_{\lambda\in\Omega}\operatorname{Res}(f_{-t}(z),z=\lambda)$$

The Ehrhart coefficients

In this section, some details for deriving formula of Ehrhart coefficients are given. For each coefficient of the Ehrhart polynomial

$$L(P,t)=c_dt^d+c_{d-1}t^{d-1}+\ldots+c_0.$$

a formula for finding these coefficients can be derived with a small modification of $f_t(z)$. Consider the function,

$$g_k(z)=\frac{(z^{-tA}-1)^k}{(1-z^{A_1})(1-z^{A_2})\ldots(1-z^{A_d})(1-z)z}$$

$$g_k(z)=\frac{\sum_{j=0}^{k}\binom{k}{j}z^{-tA(k-j)}(-1)^j}{(1-z^{A_1})(1-z^{A_2})\ldots(1-z^{A_d})(1-z)z}$$

If $-\sum_{j=0}^{k}\binom{k}{j}(-1)^j=0$ is inserted in the numerator of the above equation, we get

$$g_k(z)=\frac{\sum_{j=0}^{k}\binom{k}{j}z^{-tA(k-j)}(-1)^j-\sum_{j=0}^{k}\binom{k}{j}(-1)^j}{(1-z^{A_1})(1-z^{A_2})\ldots(1-z^{A_d})(1-z)z}$$

$$g_k(z)=\sum_{j=0}^{k-1}\frac{\binom{k}{j}(-1)^j(z^{-tA(k-j)}-1)}{(1-z^{A_1})(1-z^{A_2})\ldots(1-z^{A_d})(1-z)z}$$

$$=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^jf_{-t(k-j)}(z)$$

Recall that,

$L(P,t)=\operatorname{Res}(f_{-t}(z),\ z=0)+1$, using this relation, we obtain,

$$\operatorname{Res}(g_k(z),z=0)=\operatorname{Res}\left(\sum_{j=0}^{k-1}\binom{k}{j}(-1)^jf_{-t(k-j)}(z),z=0\right)$$

$$=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j\operatorname{Res}(f_{-t(k-j)}(z),z=0)$$

$$=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j(L(P,(k-j)t)-1)$$

$$=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j(L(P,(k-j)t)-\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j$$

$$g_k(z)=\sum_{j=0}^{k-1}\binom{k}{j}(-1)^j(L(P,(k-j)t)+(-1)^k.$$

The following lemma is needed to derive the formula of the coefficients of the Ehrhart polynomial.

Lemma (1), [6]

Suppose that $L(P,t)=c_dt^d+c_{d-1}t^{d-1}+\ldots+c_0$, then for $1\le k\le d$

$\operatorname{Res}(g_k(z),z=0)=k!\sum_{m=k}^{d}S_2(m,k)c_mt^m$, where $S_2(m,k)$ denotes the Stirling number (number of partition of an m-set into k-blocks) of the second kind of m and k and $c_0=1$.

Theorem (3), [6] :

Let P be a lattice d-polytope given by expression (1), with the Ehrhart polynomial $L(P,t)=c_d t^d + c_{d-1}t^{d-1}+...+c_0$, then for $1 \le k \le d$

$$\sum_{m=k}^{d} S_2(m,k)c_m t^m = \frac{-1}{k!}(\mathrm{Res}(g_k(z), z=1) + \sum_{\lambda\in\Omega_k}\mathrm{Res}(g_k(z), z=\lambda))$$

where $\Omega_k = \{z \in C \setminus \{1\} : z^{\frac{A}{a_{j_1}\cdots a_{j_{k+1}}}} = 1, 1 \le j_1 < j_2 < ... < j_{k+1} \le d\}$

Corollary (1), [6]:

For m > 0, c_m is the coefficient of t^m in

$$\frac{-1}{m!}(\mathrm{Res}(g_m(z), z=1) + \sum_{\lambda\in\Omega_m}\mathrm{Res}(g_m(z), z=\lambda))$$

Theorem (4) [6]:

Let $P \subset \Re^d$ be a lattice d-polytope, with vertices $(0,0,...,0), (a_1,0,...,0),...,(0,0,...,a_d)$ where $a_1, a_2 ..., a_d$ are pairwise relatively prime integers. The first nontrivial Ehrhart coefficients c_{d-2}, $d \ge 3$ is given by,

$$c_{d-2} = \frac{1}{(d-2)!}(C_d - S(A_1,a_1) - ... - S(A_d,a_d))$$

where $S(a,b)$ denotes the Dedekind sum and

$$C_d = \frac{1}{4}(d + A_{1,2} + ... + A_{d-1,d}) + \frac{1}{12}(\frac{1}{A} + \frac{A_1}{a_1} + ... + \frac{A_d}{a_d})$$

Computing c_{d-3} of The Ehrhart Polynomial using V.B.P:

As seen before, the leading coefficient of the Ehrhart polynomial represents the volume of the polytope, the second coefficient represents half of the surface area of the polytope and the constant term is one, while the other coefficients are unknown.

In this section we find the non trivial coefficients c_{d-3} for the d–polytope with d≥4, where P is represented by a list of vertices $(0,0,...,0), (a_1,0,...,0), (0, a_2, 0,...,0)$ $,...,(0,0,...,0,a_d)$, such that $a_1,...,a_d$ are pairwise relatively prime positive integers.

Theorem (5):

Let P denote the polytope in $\Re^d (d \ge 4)$ with vertices $(0,0,...,0), (a_1,0,...,0),...,(0,0,...,a_d)$ where $a_1,...,a_d$ are pairwise relatively prime positive integers. Then c_{d-3} is given by

$$c_{d-3} = \frac{-1}{(d-3)!}\left[D\begin{pmatrix}\frac{d}{4} - \frac{1}{4}\left(\frac{1}{a_1} + ... + \frac{1}{a_d}\right) \\ -S(A_1,a_1) - ... - S(A_d,a_d)\end{pmatrix} - C\right]$$

Where S(a,b) is the Dedekind sum of a and b,

$$D = \frac{1}{B}\left(1 - \frac{1}{2!}(B_2 + B_3 + ... + B_d)\right),$$

$$C = \frac{(A)^{d-3}}{(A_1)\cdots(A_d)} \cdot \frac{\phi^{(3)}(0)}{3!},$$

$$\phi(z) = \frac{\left(1 - \frac{(tBz)}{2!} + \frac{(tBz)^2}{3!} + ...\right)^{d-3}}{\left(1 + \frac{(A_1 z)}{2!} + \frac{(A_1 z)^2}{3!} + ...\right)\cdots\left(1 + \frac{(A_d z)}{2!} + \frac{(A_d z)^2}{3!} + ...\right)\cdot\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + ...\right)}$$

$A = a_1 a_2 .. a_d$, $A_k = a_1 a_2 .. \hat{a}_k .. a_d$, ($\hat{a}_k$ means the factor a_k is omitted), B =$a_2 a_3 \dots a_3$ and $B_k = a_2 a_3 ... \hat{a}_k ... a_d$.

Proof:

By corollary (1), if we define $g_{d-3}(z)$ as

$$g_{d-3}(z) = \frac{(z^{-tA} - 1)^{d-3}}{(1 - z^{A_1})(1 - z^{A_2})\cdots(1 - z^{A_d})(1 - z)}$$

Where $A = a_1 a_2 \cdots a_d$, $A_k = a_1 a_2 \cdots \hat{a}_k \cdots a_d$ and $\hat{a}_k$ means that the factor a_k is omitted, then the poles of the function $g_{d-3}(z)$ are at z= 0, 1 and the roots of unity.

We find the residues of the function $g_{d-3}(z)$ at these poles.

Since $a_1,...,a_d$ are pairwise relatively prime therefore $g_{d-3}(Z)$ has simple poles at $a_1,...,a_d$ *-th* roots of unity. Let $\lambda^{a_1} = 1 \ne \lambda$

and since, $A = a_1 \cdots a_d$, $A_1 = a_2 a_3 \cdots a_d, \ldots$, $A_d = a_1 a_2 \cdots a_{d-1}$, therefore

$$g_{d-3}(z) = \frac{(z^{-t(a_1 \cdots a_d)} - 1)^{d-3}}{(1 - z^{a_2 \cdots a_d})(1 - z^{a_1 a_3 \cdots a_d}) \cdots (1 - z^{a_1 a_2 \cdots a_{d-1}})(1 - z)z}$$

Now at $z = \lambda$,
$1 - \lambda^{a_2 \cdots a_d} \neq 0$ and $1 - \lambda \neq 0$.
Therefore

A change of variables $Z = \omega^{1/a_1} = \exp\left(\frac{1}{a_1}\log\omega\right)$ is made, where a suitable branch of logarithm such that $\exp\left(\frac{1}{a_1}\log(1)\right) = \lambda$, thus

$$\mathrm{Res}(g_{d-3}(z), z = \lambda) = \frac{1}{(1 - \lambda^{A_1})(1 - \lambda)\lambda} \frac{\lambda}{a_1}$$

$$\mathrm{Res}\left(\frac{(\omega^{-tB} - 1)^{d-3}}{(1 - \omega^{B_2}) \ldots (1 - \omega^{B_d})}, \omega = 1\right)$$

Wheren $B = a_2, a_3, \ldots, a_d, B_k = a_2 a_3 \ldots \hat{a}_k \ldots a_d$.
Since $\mathrm{Res}(f(z), z = 1) = \mathrm{Res}(e^Z f(e^Z), z = 0)$, then

$$\mathrm{Res}\left(\frac{(z^{-tB} - 1)^{d-3}}{(1 - z^{B_2}) \ldots (1 - z^{B_d})}, z = 1\right)$$

$$= \mathrm{Res}\left(\frac{e^Z (e^{-tBZ} - 1)^{d-3}}{(1 - e^{B_2 Z}) \ldots (1 - e^{B_d Z})}, z = 0\right)$$

Let $\alpha = tB$, then

$$\mathrm{Res}\left(\frac{e^z (e^{-tBz} - 1)^{d-3}}{(1 - e^{B_2 z}) \ldots (1 - e^{B_d z})}, z = 0\right)$$

$$= \mathrm{Res}\left(\frac{e^z (e^{-\alpha z} - 1)^{d-3}}{(1 - e^{B_2 z}) \ldots (1 - e^{B_d z})}, z = 0\right).$$

By writing the Maclaurin series for exponential function one can get,

$$\mathrm{Res}\left(\frac{e^z \left(1 - \alpha z + \frac{(\alpha z)^2}{2!} - \frac{(\alpha z)^3}{3!} + \ldots + (-1)\right)^{d-3}}{\left(1 - 1 - B_2 z - \frac{(B_2 z)^2}{2!} - \ldots\right) \cdots \left(1 - 1 - B_d z - \frac{(B_d z)^2}{2!} - \ldots\right)}, z = 0\right)$$

After simple computations the above residue can be written as,

$$\mathrm{Res}\left(\frac{(-\alpha)^{d-3} e^z}{(-B_2) \cdots (-B_d) z^{-d+3+d-1}} \left[\frac{\left(1 - \frac{(\alpha z)}{2!} + \frac{(\alpha z)^2}{3!} - \ldots\right)^{d-3}}{\left(1 + \frac{(B_2 z)}{2!} + \frac{(B_2 z)^2}{3!} \ldots\right) \cdots \left(1 + \frac{(B_d z)}{2!} + \frac{(B_d z)^2}{3!} + \ldots\right)}\right], z = 0\right)$$

Let

$$I = \left(1 - \frac{\alpha}{2!} z + \frac{\alpha^2}{3!} z^2 - \frac{\alpha^3}{4!} z^3 + \ldots\right)^{d-3}$$

$$J_1 = \left(1 + \frac{A_1}{2!} z + \frac{A_1^2}{3!} z^2 + \frac{A_1^3}{4!} z^3 + \ldots\right)^{-1}$$

$$J_2 = \left(1 + \frac{B_2}{2!} z + \frac{B_2^2}{3!} z^2 + \frac{B_2^3}{4!} z^3 + \ldots\right)^{-1}$$

$$J_3 = \left(1 + \frac{B_3}{2!} z + \frac{B_3^2}{3!} z^2 + \frac{B_3^3}{4!} z^3 + \ldots\right)^{-1}$$

$$\vdots$$

$$J_d = \left(1 + \frac{B_d}{2!} z + \frac{B_d^2}{3!} z^2 + \frac{B_d^3}{4!} z^3 + \ldots\right)^{-1}$$

Then

$$\mathrm{Res}\left(\frac{e^Z (e^{-\alpha Z} - 1)^{d-3}}{(1 - e^{B_2 Z}) \ldots (1 - e^{B_d Z})}, z = 0\right)$$

$$= \mathrm{Res}\left[\frac{(-\alpha)^{d-3} e^Z}{(-B_2) \cdots (-B_d) z^2} (IJ_2 \cdots J_d), z = 0\right]$$

For the function $\frac{(-\alpha)^{d-3} e^z}{(-B_2) \cdots (-B_d) z^2} (IJ_2 \cdots J_d)$ we have a pole of order two at zero.

Let $\phi(z) = e^Z IJ_2 J_3 \cdots J_d$,

and $\gamma = \frac{(\alpha)^{d-3}}{(-B_2) \ldots (-B_d)}$

After simple computations on γ, we get $\gamma = \frac{t^{d-3}}{B}$. By the formula for finding the residues, if we consider

$f(z) = \frac{\gamma\phi(z)}{z^2}$, then

$\mathrm{Res}(f(z), z = 0) = \frac{\phi'(0)\gamma}{1!}$, where

$$\phi'(z) = \phi(z) + e^z I' J_2 J_3 \cdots J_d + \ldots + e^Z IJ_2 J_3 \cdots J_d'$$

Let

$K_1 = e^z I' J_2 J_3 \cdots J_d$, $K_2 = e^z IJ_2' J_3 \cdots J_d, \ldots$,

$$K_d = e^z I J_2 J_3 \cdots J'_d$$

therefore

$$\phi'(z) = \phi(z) + K_1 + K_2 + \ldots + K_d$$

at z = 0, we compute $\phi'(0)$, after simple computations we get

$$\phi'(0) = 1 - \frac{1}{2!}(B_2 + B_3 + \ldots + B_d) - \frac{\alpha(d-3)}{2!}$$

therefore,

$$\operatorname{Res}(f(z), z = 0) = \frac{t^{d-3}}{B}\left(1 - \frac{1}{2!}(B_2 + B_3 + \ldots + B_d) - \frac{\alpha(d-3)}{2!}\right).$$

Let $D = \frac{1}{B}\left(1 - \frac{1}{2!}(B_2 + B_3 + \ldots + B_d)\right)$

Therefore,

$$\operatorname{Res}(g_{d-3}(z), z = \lambda) = \frac{D}{a_1(1-\lambda^{A_1})(1-\lambda)} t^{d-3}.$$

all the a_1-*th* root of unity ≠ 1 are added up to get

$$\sum_{\lambda^{a_1}=1\neq\lambda} \operatorname{Res}(g_{d-3}(z), z = \lambda) = \frac{Dt^{d-3}}{a_1} \sum_{\lambda^{a_1}=1\neq\lambda} \frac{1}{(1-\lambda^{A_1})(1-\lambda)}.$$

Let ξ be a primitive $a_1 - th$ roots of unity, therefore

$$\frac{Dt^{d-3}}{a_1} \sum_{\lambda^{a_1}=1\neq\lambda} \frac{1}{(1-\lambda^{A_1})(1-\lambda)}$$

$$= \frac{Dt^{d-3}}{a_1} \sum_{k=1}^{a_1-1} \frac{1}{(1-\xi^{kA_1})(1-\xi^k)}$$

then

$$\frac{1}{a_1} \sum_{k=1}^{a_1-1} \frac{1}{(1-\xi^{kA_1})(1-\xi^k)}$$

$$= \frac{1}{a_1} \sum_{k=1}^{a_1-1} \frac{\xi^{kA_1} - \xi^{kA_1} + 1 + 1}{2(1-\xi^{kA_1})} \cdot \frac{\xi^k - \xi^k + 1 + 1}{2(1-\xi^k)}$$

$$= \frac{1}{4a_1} \sum_{k=1}^{a_1-1} \left(1 + \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right) \cdot \left(1 + \frac{1+\xi^k}{1-\xi^k}\right)$$

$$= \frac{1}{4a_1} \sum_{k=1}^{a_1-1} \left(1 + \frac{1+\xi^k}{1-\xi^k} + \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}} + \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}} \cdot \frac{1+\xi^k}{1-\xi^k}\right)$$

$$= \frac{1}{4a_1}\left[\sum_{k=1}^{a_1-1} 1 + \sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k} + \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right) + \sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k} \cdot \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\right]$$

$$= \frac{1}{4a_1}(a_1 - 1) + \frac{1}{4a_1}\left[\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k} + \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right) + \sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k} \cdot \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\right]$$

Now, since $\xi = 1^{\frac{1}{a_1}}$, then by using the formula for finding the roots in the complex plane, $r_k = e^{\frac{2k\pi}{a_1}i}$, $k = 0, 1, \ldots, a_1 - 1$. We obtain

$$\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k} + \frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right) = \sum_{k=1}^{a_1-1}\left[\frac{1+e^{(\frac{2k\pi}{a_1})i}}{1-e^{(\frac{2k\pi}{a_1})i}} + \frac{1+e^{(\frac{2k\pi A_1}{a_1})i}}{1-e^{(\frac{2k\pi A_1}{a_1})i}}\right].$$

and

$$\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\right) \cdot \left(\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right) = \sum_{k=1}^{a_1-1}\left[\left(\frac{1+e^{(\frac{2k\pi}{a_1})i}}{1-e^{(\frac{2k\pi}{a_1})i}}\right) \cdot \left(\frac{1+e^{(\frac{2k\pi A_1}{a_1})i}}{1-e^{(\frac{2k\pi A_1}{a_1})i}}\right)\right]$$

But $\cot(z) = \frac{\cos(z)}{\sin(z)} = -i\left(\frac{1+e^{2iz}}{1-e^{2iz}}\right)$

hence

$$\sum_{k=1}^{a_1-1}\left[\frac{1+e^{(\frac{2k\pi}{a_1})i}}{1-e^{(\frac{2k\pi}{a_1})i}} + \frac{1+e^{(\frac{2k\pi A_1}{a_1})i}}{1-e^{(\frac{2k\pi A_1}{a_1})i}}\right] = \sum_{k=1}^{a_1-1} \frac{-1}{i}\left(\cot\frac{\pi k}{a_1} + \cot\frac{\pi k A_1}{a_1}\right)$$

and

$$\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\right) \cdot \left(\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)$$

$$= \sum_{k=1}^{a_1-1} -\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi k A_1}{a_1}\right)$$
$$= -\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi k A_1}{a_1}\right)$$

therefore

$$\frac{1}{4a_1}(a_1-1)+\frac{1}{4a_1}\left[\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}+\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)+\sum_{k=1}^{a_1-1}\left(\frac{1+\xi^k}{1-\xi^k}\cdot\frac{1+\xi^{kA_1}}{1-\xi^{kA_1}}\right)\right]$$
$$=\frac{1}{4a_1}(a_1-1)+\frac{i}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}+\cot\frac{\pi k A_1}{a_1}\right)$$
$$-\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi k A_1}{a_1}\right)$$

The imaginary terms disappear, and then the above equation can be written as

$$\frac{1}{4}-\frac{1}{4a_1}-\frac{1}{4a_1}\sum_{k=1}^{a_1-1}\left(\cot\frac{\pi k}{a_1}\cdot\cot\frac{\pi k A_1}{a_1}\right)$$
$$=\frac{1}{4}-\frac{1}{4a_1}-\frac{4a_1}{4a_1}S(A_1,a_1)$$

where $S(A_1,a_1)$ is the Dedekind sum of A_1 and a_1. Hence

$$\sum_{\lambda^{a_1}=1\neq\lambda}\operatorname{Res}(g_{d-3}(z), z=\lambda)$$
$$= Dt^{d-3}\left(\frac{1}{4}-\frac{1}{4a_1}-S(A_1,a_1)\right).$$

Similar expressions are obtained for the residues at the other roots of unity.

Now we find the residue at $g_{d-3}(z)$ at z = 1, we have

$\operatorname{Res}(g_{d-3}(z), z=1)=\operatorname{Res}(e^{Z}g_{d-3}(e^{z}), z=0)$ then

$$\operatorname{Res}(g_{d-3}(z), z=1)=$$
$$\operatorname{Res}\left(\frac{e^{Z}(e^{-tAZ}-1)^{d-3}}{(1-e^{A_1Z})(1-e^{A_2Z})\ldots(1-e^{A_dZ})(1-e^{Z})e^{Z}}, z=0\right).$$

By writing the Maclaurin series for exponential function we get,

$$\operatorname{Res}\left(\frac{\left(1-\alpha z+\frac{(\alpha z)^2}{2!}-\frac{(\alpha z)^3}{3!}+\ldots+(-1)\right)^{d-3}}{\left(1-1-A_2z-\frac{(A_2z)^2}{2!}-\ldots\right)\cdots\left(1-1-A_dz-\frac{(A_dz)^2}{2!}-\ldots\right)\cdot\left(1-1-z-\frac{z^2}{2!}-\ldots\right)}, z=0\right)$$

where $\alpha = tA$, then the above residue becomes

$$\operatorname{Res}\left(\frac{(\alpha)^{d-3}}{(A_1)\cdots(A_d)z^4}\left[\frac{\left(1-\frac{(\alpha z)}{2!}+\frac{(\alpha z)^2}{3!}-\ldots\right)^{d-3}}{\left(1+\frac{(A_1z)}{2!}+\frac{(A_1z)^2}{3!}\ldots\right)\cdots\left(1+\frac{(A_dz)}{2!}+\frac{(A_dz)^2}{3!}+\ldots\right)\left(1+\frac{z}{2!}+\frac{z^2}{3!}+\ldots\right)}\right], z=0\right)$$

the function for which we want to find the residue has a pole of order four at zero.

Let

$$\phi(z)=\frac{\left(1-\frac{(\alpha z)}{2!}+\frac{(\alpha z)^2}{3!}+\ldots\right)^{d-3}}{\left(1+\frac{(A_1z)}{2!}+\frac{(A_1z)^2}{3!}+\ldots\right)\cdots\left(1+\frac{(A_dz)}{2!}+\frac{(A_dz)^2}{3!}\ldots\right)\cdot\left(1+\frac{z}{2!}+\frac{z^2}{3!}\ldots\right)}$$

$$\gamma=\frac{\alpha^{d-3}}{A_1\ldots A_d}\ \text{and}\ f(z)=\frac{\gamma\phi(z)}{z^4}$$

By the formula for finding the residue, we get

$$\operatorname{Res}(f(z), z=0)=\frac{\phi^{(3)}(0)\gamma}{3!}.$$

Let

$$I=\left(1-\frac{\alpha}{2!}z+\frac{\alpha^2}{3!}z^2-\frac{\alpha^3}{4!}z^3+\ldots\right)^{d-3}$$
$$h=\left(1+\frac{1}{2!}z+\frac{1}{3!}z^2+\frac{1}{4!}z^3+\ldots\right)^{-1}$$
$$J_1=\left(1+\frac{A_1}{2!}z+\frac{A_1^2}{3!}z^2+\frac{A_1^3}{4!}z^3+\ldots\right)^{-1}$$
$$J_2=\left(1+\frac{A_2}{2!}z+\frac{A_2^2}{3!}z^2+\frac{A_2^3}{4!}z^3+\ldots\right)^{-1},\ldots,$$
and
$$J_d=\left(1+\frac{A_d}{2!}z+\frac{A_d^2}{3!}z^2+\frac{A_d^3}{4!}z^3+\ldots\right)^{-1}.$$

Then $\phi(z)=IJ_1J_2\cdots J_dh$

and

$$\phi'(z)=I'J_1J_2\cdots J_dh+IJ_1'J_2\cdots J_dh$$
$$+\ldots+IJ_1J_2\cdots J_d'h+IJ_1J_2\cdots J_dh'$$

let

$$K_1 = I'J_1J_2\cdots J_d h$$
$$K_2 = IJ_1'J_2\cdots J_d h$$
$$\vdots$$
$$K_{d+1} = IJ_1J_2\cdots J_d' h \text{ and}$$
$$K_{d+2} = IJ_1J_2\cdots J_d h'$$

Hence

$$\phi'(z) = K_1 + K_2 + \ldots + K_{d+1} + K_{d+2}$$

and

$$\phi'''(z) = K_1'' + K_2'' + \ldots + K_{d+1}'' + K_{d+2}''$$

Now,

$$I = \left(1 - \frac{tA}{2!}z + \frac{(tA)^2}{3!}z^2 - \frac{(tA)^3}{4!}z^3 + \ldots\right)^{d-3}$$

therefore

$$I'(z) = (d-3)\left[\left(1 - \frac{tA}{2!}z + \frac{(tA)^2}{3!}z^2 - \frac{(tA)^3}{4!}z^3 + \ldots\right)^{d-4} \cdot \left(-\frac{tA}{2!} + \frac{2(tA)^2}{3!}z + \ldots\right)\right]$$

Differentiating I' to get I'' and I''', then put z = 0 in the obtained expression to get

$$I(O) = 1$$
$$I'(0) = (d-3)\left(\frac{-tA}{2!}\right)$$
$$I''(0) = (d-3)\left[(d-4)(-\frac{tA}{2!})^2 + \frac{2(tA)^2}{3!}\right]$$
$$I'''(0) = (d-3)\left[(d-5)(d-6)\left(-\frac{tA}{2!}\right)^3 + (d-4)(-tA)\left(\frac{2(tA)^2}{3!} + (d-4)\right)\left(\frac{-tA}{2}\right)\left(\frac{2(tA)^2}{3!} + \left(\frac{-3!(tA)^3}{4!}\right)\right)\right]$$

For

$$J_1 = \left(1 + \frac{A_1}{2!}z + \frac{A_1^2}{3!}z^2 + \frac{A_1^3}{4!}z^3 + \ldots\right)^{-1}$$
$$J_1'(z) = -\left(1 + \frac{A_1}{2!}z + \frac{A_1^2}{3!}z^2 + \frac{A_1^3}{4!}z^3 + \ldots\right)^{-2}\left(\frac{A_1}{2!} + \frac{2A_1^2}{3!}z + \frac{3A_1^3}{4!}z^2 + \ldots\right)$$

Differentiate J' to get J'' and J''', then put z = 0 in the obtained expressions to get

$J_1(0) = 1$, $J_1'(0) = -\frac{A_1}{2!}$, $J_1''(0) = \frac{A_1^2}{3!}$ and $J_1'''(0) = 0$.

In a similar way, we get the other differentiation of $J_2, J_3, \ldots, J_d$ and h, then

$$\mathrm{Res}\left(\frac{(tA)^{d-3}}{(A_1)\cdots(A_d)z^4}\phi(z), z=0\right) = \frac{(A)^{d-3}}{(A_1)\cdots(A_d)}\frac{t^{d-3}}{3!}\phi^{(3)}(0)\cdot$$

Let $C = \frac{(A)^{d-3}}{(A_1)\cdots(A_d)}\cdot\frac{\phi^{(3)}(0)}{3!}$

So by corollary (1) we get for d≥4, c_{d-3}, which is the coefficient of t^{d-3} of

$$\frac{-1}{(d-3)!}(\mathrm{Res}(g_{d-3}(z), z=1) + \sum_{\lambda\in\Omega_{d-3}}\mathrm{Res}(g_{d-3}(z), z=\lambda))$$

So

$$c_{d-3} = \frac{-D}{(d-3)!}\left[\left(\frac{d}{4} - \frac{1}{4}\left(\frac{1}{a_1} + \ldots + \frac{1}{a_d}\right) - S(A_1, a_1) - \ldots - S(A_d, a_d)\right) - C\right]$$

General formula for the differentiation of $I, J_1, J_2, \ldots, J_d, h$

In this section, we get a general form for the differentiation of the terms $I, J_1, J_2, \ldots, J_d$ and h that appears throughout the process of finding the coefficients of the Ehrhart polynomial, we begin by considering

$$I^{[j]} = e^z I^{(j)} J_2 J_3 \cdots J_d h \qquad j = 1, 2, \ldots$$

where $I^{[j]}$ means that only I in the expression $e^z IJ_2J_3\cdots J_d h$ is differentiated j times.

Let $E_1 = 1 + \frac{J_2'}{J_2} + \ldots + \frac{J_d'}{J_d}$, then

$$I'' = I^{[2]} + E_1 I',$$
$$I''' = I^{[3]} + E_1(I^{[2]} + I'') + E_1'I',$$
$$I^{(4)} = I^{[4]} + E_1(2I^{[3]} + I''') + E_1^2 I^{[2]} + E_1'(I^{[2]} + 2I'') + E_1 I'$$
$$I^{(5)} = I^{[5]} + E_1(3I^{[4]} + I^{(4)}) + 3E_1^2 I^{[3]} + 3E_1'(I^{[3]} + I''') + 3E_1E_1'I^{[2]} + E_1''(I^{[2]} + 3I'') + E_1'''I' + E_1^3 I^{[2]},$$

$$I^{(6)} = I^{[6]} + E_1(4I^{[5]} + I^{(5)}) + 6E_1^2 I^{[4]} + 4E_1^3 I^{[3]} + E_1^4 I^{[2]} + E_1'(6I^{[4]} + 4I^{(4)}) + 3E_1'^2 I^{[2]} + 12E_1E_1'I^{[3]} + 6E_1^2E_1'I^{[2]} + 4E_1E_1''I^{[2]} + E_1''(4I^{[3]} + 6I''') + E_1'''(I^{[2]} + 4I'') + E_1^{(4)}I' + E_1^4 I^{[2]},$$

$$I^{(7)} = I^{[7]} + E_1(5I^{[6]} + I^{(6)}) + 10E_1^2 I^{[5]} + 10E_1^3 I^{[4]} + 5E_1^4 I^{[3]} + E_1^5 I^{[2]} + E_1'(10I^{[5]} + I^{(5)}) + E_1''(10I^{[4]} + 10I^{(4)}) + E_1'''(5I^{[3]} + 10I''') + E_1^{(4)}(I^{[2]} + 5I'') + E_1^{(5)}I' + 30E_1E_1'I^{[4]} + 30E_1^2E_1'I^{[3]} + 10E_1^3E_1'I^{[2]} + 20E_1E_1''I^{[3]} + 10E_1^2E_1''I^{[2]} + 5E_1E_1'''I^{[2]} + 15E_1'^2 I^{[3]} + 15E_1E_1'^2 I^{[2]} + 10E_1'E_1''I^{[2]},$$

$$I^{(8)} = I^{[8]} + E_1(6I^{[7]} + I^{(7)}) + 5E_1^2 I^{[6]} + 20E_1^3 I^{[5]} + 15E_1^5 I^{[3]} + E_1^6 I^{[3]} + E_1'(15I^{[6]} + 6I^{(6)}) + E_1''(20I^{[5]} + 15I^{(5)}) + E_1'''(15I^{[4]} + 20I^{(4)}) + E_1^{(4)}(6I^{[3]} + 15I''') + E_1^{(5)}(I^{[2]} + 6I'') + E_1^{(6)}I' + 60E_1E_1'I^{[5]} + 90E_1^2E_1'I^{[4]} + 60E_1^3E_1'I^{[3]} + 15E_1^4E_1'I^{[2]} + 60E_1E_1''I^{[4]} + 60E_1^2E_1''I^{[3]} + 20E_1^3E_1''I^{[2]} + 30E_1E_1'''I^{[3]} + 15E_1^2E_1'''I^{[2]} + 6E_1E_1^{(4)}I^{[2]} + 45E_1'^2 I^{[4]} + E_1''^2 I^{[2]} + 90E_1E_1'^2 I^{[3]} + 45E_1^2E_1'^2 I^{[2]} + 60E_1'E_1''I^{[3]} + 15E_1'E_1'''I^{[2]} + 60E_1E_1'E_1''I^{[2]} + 15(E_1')^3 I^{[2]},$$

In order to differentiate $J_2, J_3, \ldots, J_d$ we need to find a general formula for these differentiations so we work on these elements and find a general formula. To illustrate this, consider for example,

$$J_2 = \left(1 + \frac{w_2}{2!}z + \frac{w_2^2}{3!}z^2 + \frac{w_2^3}{4!}z^3 + \ldots\right)^{-1} = \frac{w_2 z}{e^{w_2 Z} - 1}$$

The $e^{w_2 Z}J_2 - J_2 = w_2 z$. By assuming the implicit differentiation for both sides of the above equation, we get

$$\frac{d}{dz}(e^{w_2 Z}J_2) - \frac{d}{dz}(J_2) = w_2$$

and the second derivative of the above equation is

$$\frac{d^2}{dz^2}(e^{w_2 Z}J_2) - \frac{d^2}{dz^2}(J_2) = 0$$

when we differentiate $e^{w_2 Z}J_2$ d-times we get a shape like a binomial formula

$$(a+b)^d = a^d + da^{d-1}b + \frac{d(d-1)}{2!}a^{d-2}b^2 + \ldots + b^d = \sum_{i=1}^{d}\binom{d}{i}b^i a^{d-i}.$$

Therefore,

$$e^{w_2 Z}(J_2 + w_2)^m - J_2^{(m)} = 0$$

where $J_2^{(m)}$ is the m-th derivative of J_2, since w_2 is constant therefore w_2^m means w_2 raised to the power m. For example,

let $h = J_2 e^{w_2 Z}$,

then

$$h' = J_2' e^{w_2 Z} + w_2 e^{w_2 Z} J_2 = e^{w_2 Z}(J_2' + w_2 J_2),$$

$$h'' = J_2'' e^{w_2 Z} + w_2 J_2' e^{w_2 Z} + w_2^2 e^{B_2 z} J_2 + w_2 J_2' e^{B_2 z} = e^{w_2 Z}(J_2'' + 2w_2 J_2' + w_2^2 J_2),$$

$$h''' = e^{w_2 Z}(J_2''' + 3w_2 J_2'' + 3w_2^2 J_2' + w_2^3 J_2).$$

and so on. Therefore

$$J_2' = e^Z I J_2' \cdots J_d = J_2^{[1]},$$

$$J_2'' = J_2^{[2]} + E_2 J_2'$$

where $E_2 = 1 + \frac{I'}{I} + \ldots + \frac{J_d'}{J_d}$.

$$J_2''' = J_2^{[3]} + 2E_2 J_2^{[2]} + E_2^2 J_2' + E_2' J'$$

$$J_2^{(4)} = J_2^{[4]} + 3E_2 J_2^{[3]} + 3E_2^2 J_2^{[2]} + E_2^3 J_2' + 3E_2' J_2^{[2]} + E_2'' J_2' + 3E_2 E_2' J_2'$$

and similarly for highest derivative.

By arranging them together we obtain

$$J_2' = e^Z I J_2' \cdots J_d$$

$$J_2'' = J_2^{[2]} + E_2 J_2',$$

$$J_2''' = J_2^{[3]} + E_2(J_2^{[2]} + J_2'') + E_2' J',$$

$$J_2^{(4)} = J_2^{[4]} + E_2(2J_2^{[3]} + J_2''') + E_2^2 J_2^{[2]} + E_2'(J_2^{[2]} + 2J_2'') + E_2'' J_2',$$

$$
\begin{aligned}
J_2^{(5)} &= J_2^{[5]} + E_2(3J_2^{[4]} + J_2^{(4)}) + E_2^2(3J_2^{[3]}) \\
&\quad + E_2^3 J_2^{[2]} + E_2'(3J_2^{[3]} + 3J_2''') + E_2''(J_2^{[2]} + 3J_2'') \\
&\quad + E_2''' J_2' + 3E_2 E_2' J_2^{[2]},
\end{aligned}
$$

$$
\begin{aligned}
J_2^{(6)} &= J_2^{[6]} + E_2(4J_2^{[5]} + J_2^{(5)}) + 6E_2^2 J + 4E_2^3 J_2^{[3]} \\
&\quad + E_2^4 J_2^{[2]} + E_2'(6J_2^{[4]} + 4J_2^{(4)}) + E_2''(4J_2^{[3]} \\
&\quad + 6J_2^{(3)}) + E_2'''(J_2^{[2]} + 4J_2'' + \ E_2^{(4)} J_2' \\
&\quad + 12E_2 E_2' J_2^{[3]} + 6E_2' E_2^2 J_2^{[2]} + 4E_2 E_2'' J_2^{[2]} \\
&\quad + 3(E_2')^2 J_2^{[2]},
\end{aligned}
$$

$$
\begin{aligned}
J_2^{(7)} &= J_2^{[7]} + E_2(5J_2^{[6]} + J_2^{(6)}) + 10E_2^2 J_2^{[5]} + 10E_2^3 J_2^{[4]} \\
&\quad + 5E_2^4 J_2^{[3]} + E_2^5 J_2^{[2]} + E_2'(10J_2^{[5]} + 5J_2^{(5)}) + E_2''(10J_2^{[4]} \\
&\quad + 10J_2^{(4)}) + E_2'''(5J_2^{[3]} + 10J_2''') + E_2^{(4)}(J_2^{[2]} + 5J_2'') \\
&\quad + E_2^{(5)} J_2' + 30E_2 E_2' J_2^{[4]} + 30E_2' E_2^2 J_2^{[3]} + 10E_2' E_2^3 J_2^{[2]} \\
&\quad + 20E_2 E_2'' J_2^{[3]} + 10E_2^2 E_2'' J_2^{[2]} + 5E_2 E_2''' J_2^{[2]} \\
&\quad + 15(E_2')^2 J_2^{[3]} + 15(E_2')^2 E_2 J_2^{[3]} + 10E_2' E_2'' J_2^{[2]},
\end{aligned}
$$

$$
\begin{aligned}
J_2^{(8)} &= J_2^{[8]} + E_2(6J_2^{[7]} + J_2^{(7)}) + 15E_2^2 J_2^{[6]} + 20E_2^3 J_2^{[5]} \\
&\quad + 15E_2^4 J_2^{[4]} + 6E_2^5 J_2^{[3]} + E_2^6 J_2^{[2]} + E_2'(15J_2^{[6]} \\
&\quad + 6J_2^{(6)}) + E_2''(20J_2^{[5]} + 15J_2^{(5)}) + E_2'''(15J_2^{[4]} \\
&\quad + 20J_2^{(4)}) + E_2^{(4)}(6J_2^{[3]} + 13J_2^{(3)}) + E_2^{(5)}(J_2^{[2]} \\
&\quad + 6J_2'') + E_2^{(6)} J_2' + 60E_2 E_2' J_2^{[5]} + 90E_2' E_2^2 J_2^{[4]} \\
&\quad + 60E_2^3 E_2'' J_2^{[2]} + 15E_2^4 E_2' J_2^{[2]} + \ 60E_2 E_2'' J_2^{[4]} \\
&\quad + 60E_2^2 E_2'' J_2^{[3]} + 20E_2^3 E_2'' J_2^{[2]} + 30E_2 E_2''' J_2^{[3]} \\
&\quad + 15E_2^2 E_2''' J_2^{[2]} + 6E_2 E_2^{(4)} J_2^{[2]} + 45(E_2')^2 J_2^{[4]} \\
&\quad + 15(E_2')^3 J_2^{[2]} + 90E_2(E_2')^2 J_2^{[3]} + 45E_2^2(E_2')^2 \\
&\quad J_2^{[2]} + 60E_2' E_2'' J_2^{[3]} + 15E_2' E_2''' J_2^{[2]} + 60E_2 E_2' \\
&\quad E_2'' J_2^{[2]} + 10(E_2'')^2 J_2^{[2]},
\end{aligned}
$$

$$
\begin{aligned}
J_2^{(9)} &= J_2^{[9]} + E_2(7J_2^{[8]} + J_2^{(8)}) + 21E_2^2 J_2^{[7]} + 35E_2^3 J_2^{[6]} \\
&\quad + 35E_2^4 J_2^{[5]} + 21E_2^5 J_2^{[4]} + 7E_2^6 J_2^{[3]} + E_2^7 J_2^{[2]} \\
&\quad + E_2'\left(21J_2^{[7]} + 7J_2^{(7)}\right) + E_2''\left(35J_2^{[6]} + 21J_2^{(6)}\right) \\
&\quad + E_2'''\left(35J_2^{[5]} + 35J_2^{(5)}\right) + E_2^{(4)}\left(21J_2^{[4]} + 35J_2^{(4)}\right) \\
&\quad + E_2^{(5)}\left(7J_2^{[3]} + 21J_2^{(3)}\right) + E_2^{(6)}\left(J_2^{[2]} + 7J_2''\right) + E_2^{(7)} J_2' \\
&\quad + 105E_2 E_2' J_2^{[6]} + 210E_2' E_2^2 J_2^{[5]} + 210E_2^3 E_2' J_2^{[4]} \\
&\quad + 105E_2^4 E_2' J_2^{[3]} + 21E_2^5 E_2' J_2^{[2]} + 140E_2 E_2'' J_2^{[5]} \\
&\quad + 210E_2^2 E_2'' J_2^{[4]} + 140E_2^3 E_2'' J_2^{[3]} + \ 35E_2^4 E_2'' J_2^{[2]} \\
&\quad + 105E_2 E_2''' J_2^{[4]} + 105E_2^2 E_2''' J_2^{[3]} + 35E_2^3 E_2''' J_2^{[2]} \\
&\quad + 42E_2 E_2^{(4)} J_2^{[3]} + \ 21E_2^2 E_2^{(4)} J_2^{[2]} + 7E_2 E_2^{(5)} J_2^{[2]} \\
&\quad + 105(E_2')^2 J_2^{[5]} + 315E_2(E_2')^2 J_2^{[4]} + 315E_2^2 \\
&\quad (E_2')^2 J_2^{[3]} + 105E_2^3(E_2')^2 J_2^{[2]} + 105(E_2')^3 J_2^{[3]} \\
&\quad + 105E_2(E_2')^3 J_2^{[2]} + 210E_2' E_2'' J_2^{[4]} + 105E_2' E''' \\
&\quad {}_2 J_2^{[3]} + 21E_2' E_2^{(4)} J_2^{[2]} + 70(E_2'')^2 J_2^{[3]} + 10E_2 \\
&\quad (E_2'')^2 J_2^{[2]} + 35E_2'' E_2''' J_2^{[2]} + 420E_2 E_2' E_2'' J_2^{[3]} \\
&\quad + 210E_2^2 E_2' E_2'' J_2^{[2]} + 105E_2 E' E_2''' J_2^{[2]}.
\end{aligned}
$$

By similar procedure we get the derivatives of $J_3, J_4, \ldots, J_d$ and h that are used in the definition of $\phi(z)$ in the preceding sections. When we arrange the obtained results we get a triangle like a Polya triangle [8, p.20] where the contents of the triangle are the coefficients of $E_2{}^2, E_2^3, \ldots$in the expression $J_2^{(4)}, J_2^{(5)}, \ldots$

	E_2^2														
$J_2^{(4)}$	1	E_2^3	$J_2^{[2]}$												
$J_2^{(5)}$	3	1	E_2^4	$J_2^{[3]}$	$J_2^{[2]}$										
$J_2^{(6)}$	6	4	1	E_2^5	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$								
$J_2^{(7)}$	10	10	5	1	E_2^6	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$						
$J_2^{(8)}$	15	20	15	6	1	E_2^7	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$				
$J_2^{(9)}$	21	35	35	21	7	1	E_2^8	$J_2^{[7]}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$		
$J_2^{(10)}$	28	56	70	56	28	8	1	E_2^9	$J_2^{[8]}$	$J_2^{[7]}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$

Also, the first terms of the coefficients of $E_2', E_2'', \ldots$ in the expression of $J_2^{(4)}, J_2^{(5)}, \ldots$ are

	E'_2												
$J_2^{(4)}$	1	E''_2	$J_2^{[2]}$										
$J_2^{(5)}$	3	1	E'''_2	$J_2^{[3]}$	$J_2^{[2]}$								
$J_2^{(6)}$	6	4	1	$E_2^{(4)}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$						
$J_2^{(7)}$	10	10	5	1	$E_2^{(5)}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$				
$J_2^{(8)}$	15	20	15	6	1	$E_2^{(6)}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$		
$J_2^{(9)}$	21	35	35	21	7	1	$E_2^{(7)}$	$J_2^{[7]}$	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$

The second terms of the coefficients of $E'_2, E''_2,\ldots$ in the expression of $J_2^{(4)}, J_2^{(5)},\ldots$ are

	E'_2	E''_2													
$J_2^{(4)}$	1	1	E'''_2	J''_2	J'_2										
$J_2^{(5)}$	3	3	1	$E_2^{(4)}$	J'''_2	J''_2	J'_2								
$J_2^{(6)}$	4	6	4	1	$E_2^{(5)}$	$J_2^{(4)}$	J'''_2	J''_2	J'_2						
$J_2^{(7)}$	5	10	10	5	1	$E_2^{(6)}$	$J_2^{(5)}$	$J_2^{(4)}$	J'''_2	J''_2	J'_2				
$J_2^{(8)}$	6	15	20	15	6	1	$E_2^{(7)}$	$J_2^{(6)}$	$J_2^{(5)}$	$J_2^{(4)}$	J'''_2	J''_2	J'_2		
$J_2^{(9)}$	7	21	35	35	21	7	1	$E_2^{(8)}$	$J_2^{(7)}$	$J_2^{(6)}$	$J_2^{(5)}$	$J_2^{(4)}$	J'''_2	J''_2	J'_2

The coefficients of $E_2E'_2,\ E_2^2E'_2,\ldots$ in the expression of $J''_2, J'''_2,\ldots$ are arranged as follows.

J''_2	0										
J'''_2	0										
$J_2^{(4)}$	0										
	$E_2E'_2$										
$J_2^{(5)}$	3	$E_2^2E'_2$	$J_2^{[2]}$								
$J_2^{(6)}$	12	6	$E_2^3E'_2$	$J_2^{[3]}$	$J_2^{[2]}$						
$J_2^{(7)}$	30	30	10	$E_2^4E'_2$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$				
$J_2^{(8)}$	60	90	60	15	$E_2^5E'_2$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$		
$J_2^{(9)}$	105	210	210	105	21	E_2^6E'	$J_2^{[6]}$	$J_2^{[5]}$	$J_2^{[4]}$	$J_2^{[3]}$	$J_2^{[2]}$

The diagonal of the above results is the second column of the preceding Polya triangle, and the first column for the above results is obtained as follows:

By multiplying the diagonal by 4,5,6… we get the line under the diagonal, which are:

(3)(4)=12,
(6)(5)=30,
(10)(6)=60,
(15)(7)=105,

The general formula of the differentiation is given by

$$J_2^{(m)} = J_2^{[m]} + E_2\left((m-2)J_2^{[m-1]} + J_2^{(m-1)}\right) + W$$

where 1<m≤8 and W can be obtained from the given tables as follow

when m=3 then

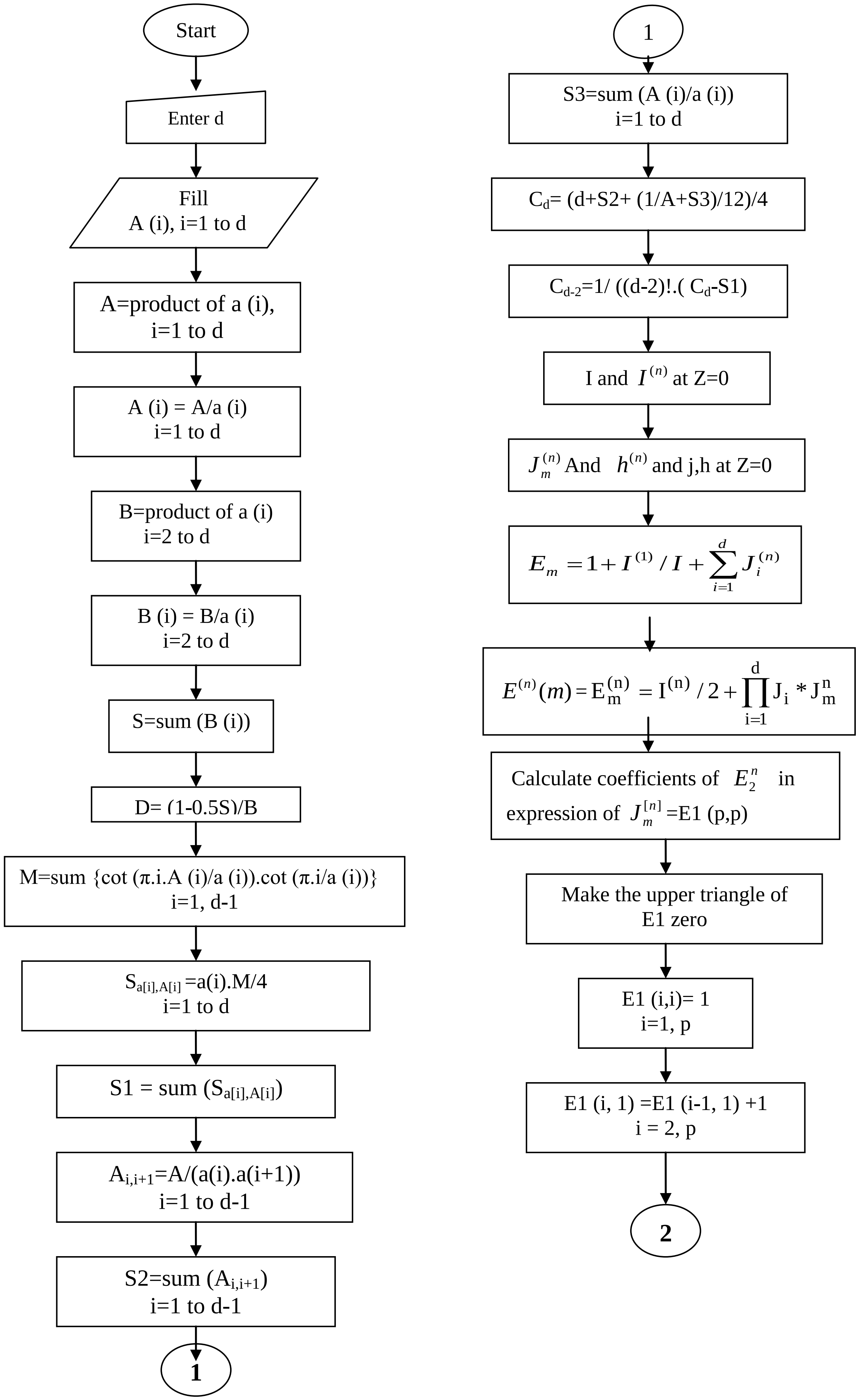
$$J_2^{(3)} = J_2^{[3]} + E_2\left(J_2^{[2]} + J_2^{(2)}\right)$$

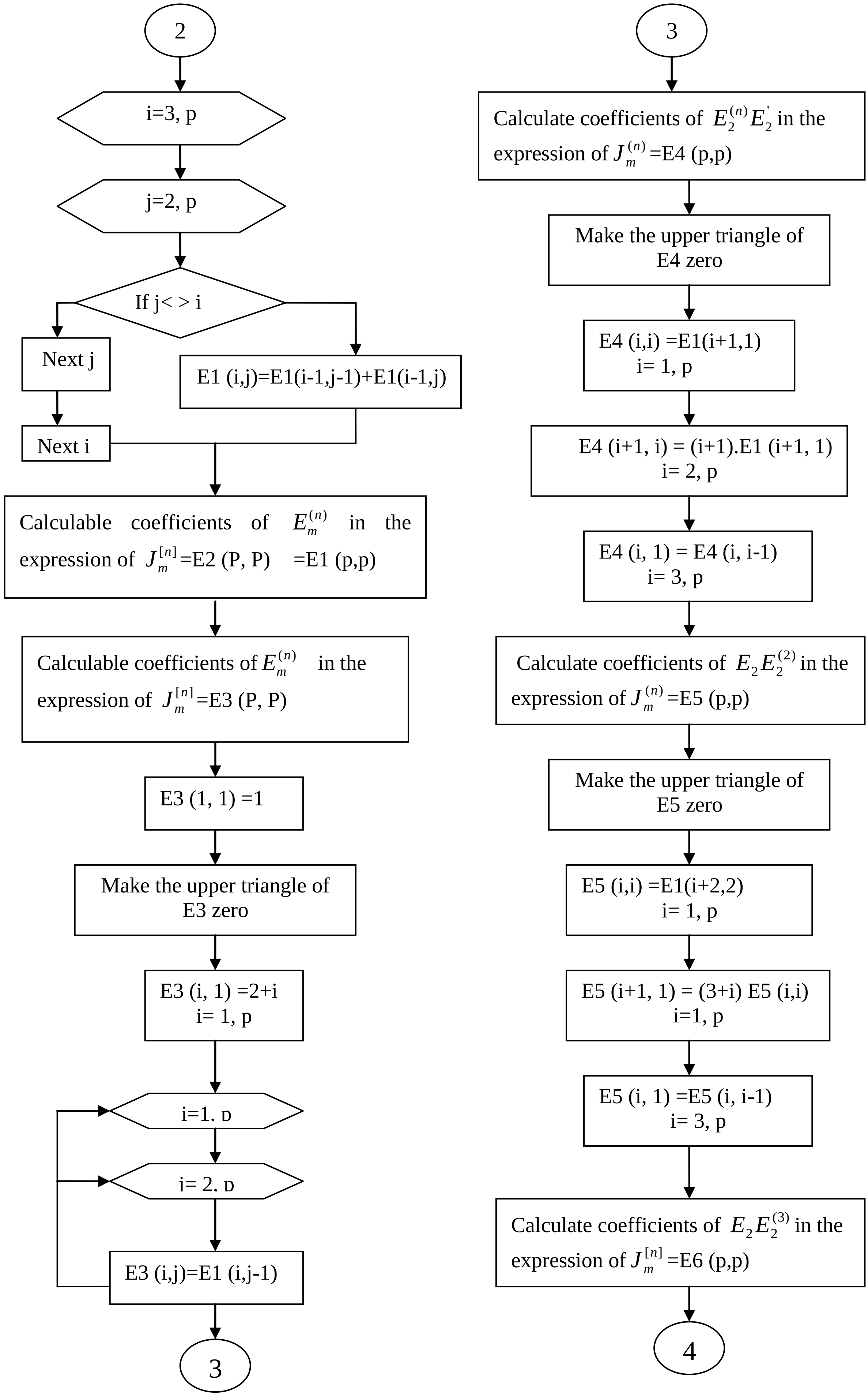
when m=4 then

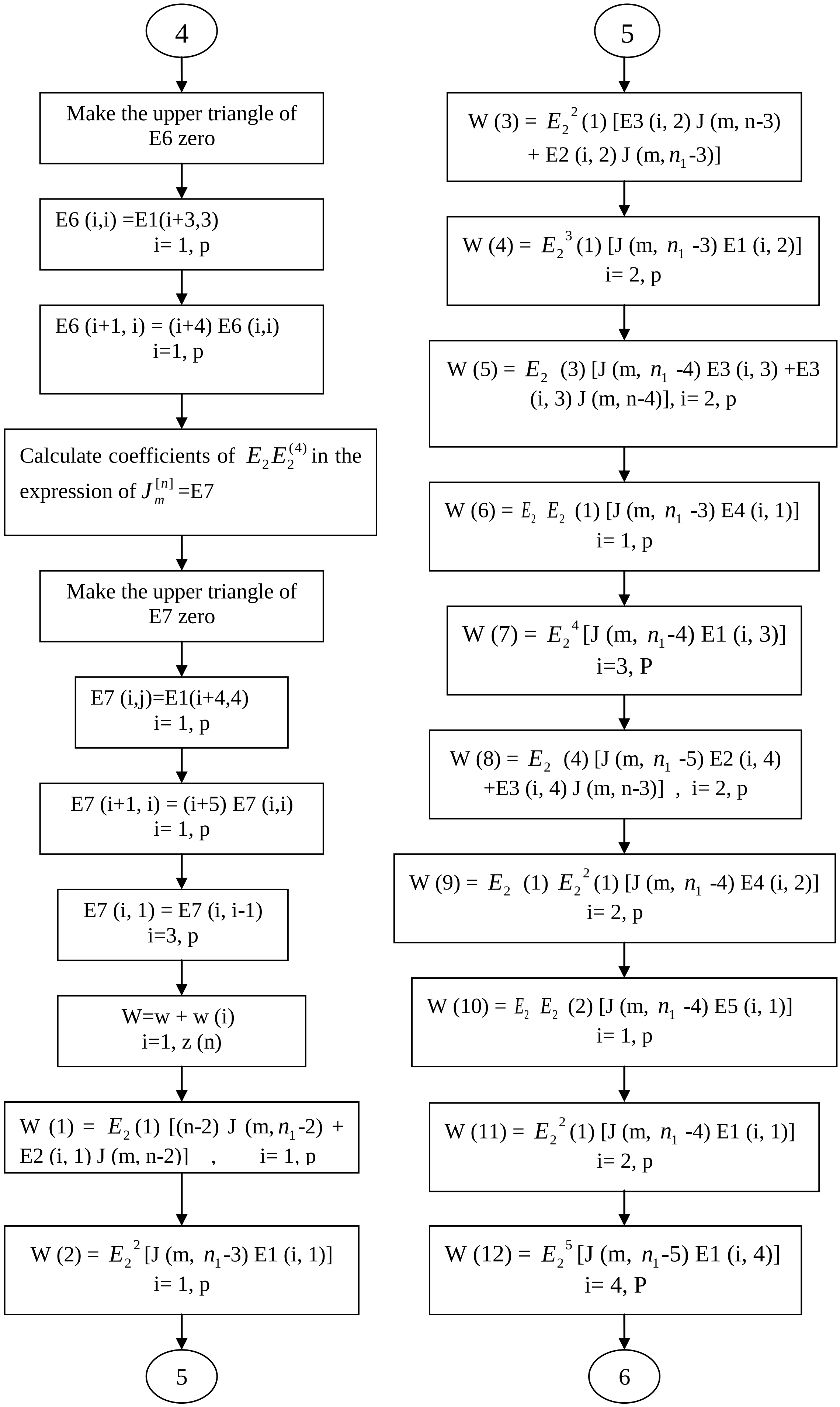
$$J_2^{(4)} = J_2^{[4]} + E_2\left(2J_2^{[3]} + J_2^{(3)}\right) + W$$

from the tables, W can be found as follows

$$W = E_2^2 J_2^{[2]} + E'_2\left(J_2^{[2]} + 2J_2^{(2)}\right) + E''_2 J'_2 .$$


Start
Enter d
Fill
A (i), i=1 to d
A=product of a (i),
i=1 to d
A (i) = A/a (i)
i=1 to d
B=product of a (i)
i=2 to d
B (i) = B/a (i)
i=2 to d
S=sum (B (i))
D= (1-0.5S)/B
M=sum {cot (π.i.A (i)/a (i)).cot (π.i/a (i))}
i=1, d-1
$S_{a[i],A[i]}$ =a(i).M/4
i=1 to d
S1 = sum ($S_{a[i],A[i]}$)
$A_{i,i+1}$=A/(a(i).a(i+1))
i=1 to d-1
S2=sum ($A_{i,i+1}$)
i=1 to d-1
1
1
S3=sum (A (i)/a (i))
i=1 to d
C_d= (d+S2+ (1/A+S3)/12)/4
C_{d-2}=1/ ((d-2)!.(C_d-S1)
I and $I^{(n)}$ at Z=0
$J_m^{(n)}$ And $h^{(n)}$ and j,h at Z=0
$E_m = 1 + I^{(1)} / I + \sum_{i=1}^{d} J_i^{(n)}$
$E^{(n)}(m) = E_m^{(n)} = I^{(n)} / 2 + \prod_{i=1}^{d} J_i * J_m^n$
Calculate coefficients of E_2^n in
expression of $J_m^{[n]}$ =E1 (p,p)
Make the upper triangle of
E1 zero
E1 (i,i)= 1
i=1, p
E1 (i, 1) =E1 (i-1, 1) +1
i = 2, p
2


2
i=3, p
j=2, p
If j< > i
Next j
E1 (i,j)=E1(i-1,j-1)+E1(i-1,j)
Next i
Calculable coefficients of $E_m^{(n)}$ in the expression of $J_m^{[n]}$=E2 (P, P) =E1 (p,p)
Calculable coefficients of $E_m^{(n)}$ in the expression of $J_m^{[n]}$=E3 (P, P)
E3 (1, 1) =1
Make the upper triangle of E3 zero
E3 (i, 1) =2+i
i= 1, p
i=1, p
i= 2, p
E3 (i,j)=E1 (i,j-1)
3
3
Calculate coefficients of $E_2^{(n)}E_2^{'}$ in the expression of $J_m^{(n)}$=E4 (p,p)
Make the upper triangle of E4 zero
E4 (i,i) =E1(i+1,1)
i= 1, p
E4 (i+1, i) = (i+1).E1 (i+1, 1)
i= 2, p
E4 (i, 1) = E4 (i, i-1)
i= 3, p
Calculate coefficients of $E_2E_2^{(2)}$ in the expression of $J_m^{(n)}$=E5 (p,p)
Make the upper triangle of E5 zero
E5 (i,i) =E1(i+2,2)
i= 1, p
E5 (i+1, 1) = (3+i) E5 (i,i)
i=1, p
E5 (i, 1) =E5 (i, i-1)
i= 3, p
Calculate coefficients of $E_2E_2^{(3)}$ in the expression of $J_m^{[n]}$=E6 (p,p)
4


4
Make the upper triangle of E6 zero
E6 (i,i) =E1(i+3,3) i= 1, p
E6 (i+1, i) = (i+4) E6 (i,i) i=1, p
Calculate coefficients of $E_2 E_2^{(4)}$ in the expression of $J_m^{[n]}$ =E7
Make the upper triangle of E7 zero
E7 (i,j)=E1(i+4,4) i= 1, p
E7 (i+1, i) = (i+5) E7 (i,i) i= 1, p
E7 (i, 1) = E7 (i, i-1) i=3, p
W=w + w (i) i=1, z (n)
W (1) = E_2(1) [(n-2) J (m, n_1-2) + E2 (i, 1) J (m, n-2)] , i= 1, p
W (2) = E_2^2[J (m, n_1-3) E1 (i, 1)] i= 1, p
5
5
W (3) = E_2^2(1) [E3 (i, 2) J (m, n-3) + E2 (i, 2) J (m, n_1-3)]
W (4) = E_2^3(1) [J (m, n_1 -3) E1 (i, 2)] i= 2, p
W (5) = E_2 (3) [J (m, n_1 -4) E3 (i, 3) +E3 (i, 3) J (m, n-4)], i= 2, p
W (6) = E_2 E_2 (1) [J (m, n_1 -3) E4 (i, 1)] i= 1, p
W (7) = E_2^4[J (m, n_1-4) E1 (i, 3)] i=3, P
W (8) = E_2 (4) [J (m, n_1 -5) E2 (i, 4) +E3 (i, 4) J (m, n-3)] , i= 2, p
W (9) = E_2 (1) E_2^2(1) [J (m, n_1 -4) E4 (i, 2)] i= 2, p
W (10) = E_2 E_2 (2) [J (m, n_1 -4) E5 (i, 1)] i= 1, p
W (11) = E_2^2(1) [J (m, n_1 -4) E1 (i, 1)] i= 2, p
W (12) = E_2^5[J (m, n_1-5) E1 (i, 4)] i= 4, P
6

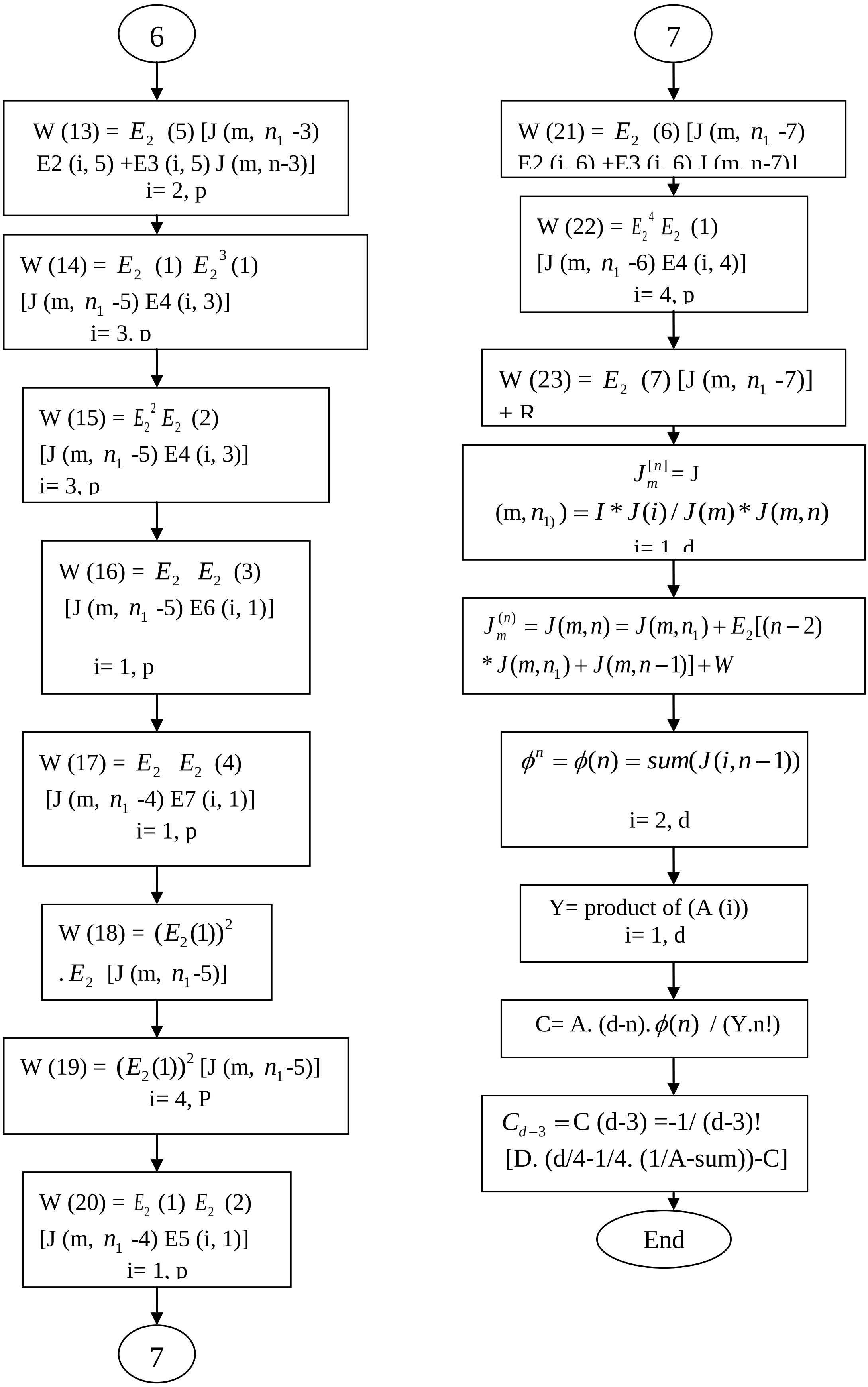


Fig. (1) : Program Flowchart for finding the coefficient of Ehrhart polynomial.

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الخلاصة

حساب حجم متعدد الأضلاع وكذلك حساب عدد النقاط التي احداثياتها أعداد صحيحة في المجال R^d هو موضوع مهم جدا في فروع الرياضيات المختلفة مثل نظرية الاعداد، نظرية التمثيل، متعدد حدود ايرهارت في التوافيقية، التجفير و النظام الاحصائي.

تم حساب متعدد حدود ايرهارت باستخدام بعض الطرق. ا حدى هذه الطرق طورت واستنتجنا مبرهنة لحساب معاملات متعددة الحدود ايرهارت.

كذلك كتب برنامج بلغة فيجوال بيسك لحساب المشتقات للدالة التي أستخدمت لحساب معاملات متعددة الحدود ايرهارت.