

INHERITED BACKLUND TRANSFORMATION (IBT)

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Abstract

In this paper, we present a new theorem (IBT theorem) which helps to define a new inherited property for the ODEs induced from PDE using invariant form method, and we call this property Inherited Backlund. Transformation (IBT).The general work is illustrated by application to the well known nonlinear diffusion equation with convection.

Introduction

In the latter part of the nineteenth century Sophus Lie introduced the notion of continuous group now known as Lie group in order to unify and extend various ad hoc techniques for solving DEs. Asymmetry groups (Infinitesimal generators) of a DEs is Lie group of transformations (Lie algebra) that maps solutions to other solutions of the DEs [2], [3], [7], [11], [12].

Bluman et al. in 1988 [5] presented new classes of symmetries for PDE. For a given PDE one can find useful potential (non-local) symmetries by embedding it (the PDE) in an auxiliary system corresponding to the conserved form of the PDE with auxiliary dependent variable (potential variable).If the auxiliary system induce a scalar PDE then the system will be considered as a type of Backlund transformation.

Backlund Transformations (BT) were devised in 1880 for use the theories of differential geometry and of differential equations(DEs). BT is essentially defined as a pairs of relations involving two dependent variables and their derivative which together imply that each one of the dependent variables satisfies separately a PDE [1], [3], [6]. In general, find BT of DEs is a tedious and not an easy task.

Recently, Bluman and Reid in 1988 [4] presented an algorithm to find BT of special class for ODE.

If the PDE induce on ODE, then some properties for the PDE will be inherited to the induce ODE. In this paper we present a new inherited property for the ODE induce from PDE using invariant form method namely Inherited Backlund Transformations (IBT

theorem) then we apply the theorem to the diffusion equation with convection.We demonstrate the importance of the IBT on two examples. Our main result in this paper (IBT theorem) will be given in section (2). And in section (3) we give it's application to the diffusion equation used by Gandarise et al. [8].

Our Main Result.

Let R denote a scalar PDE, X be infinitesimal generator of R of the commonly occurring case,

$$X = \xi_1(x_1, x_2) \frac{\partial}{\partial x_1} + \xi_2(x_1, x_2) \frac{\partial}{\partial x_2} + \eta^1(x_1, x_2, u) \frac{\partial}{\partial u}$$

..... (1)

Theorem (The IBT Theorem)

Assume that R is written in the conserved form corresponding to the following auxiliary system S

$$\frac{\partial v}{\partial x_1} = f(x_1, x_2) u \text{(2a)}$$

$$\frac{\partial v}{\partial x_2} = g(x_1, x_2, u, u_1, \dots, u_{n-1}) \text{ (2b)}$$

and $R_X\{z, U\}$ is the ODE induced from the PDE R such that $\frac{\partial z}{\partial x_1} \neq 0$. If S admits an

infinitesimal generator X_s such that $X_S = X + \eta^2(x_1, x_2, v) \frac{\partial}{\partial v}$ then S will induce a BT for R_X .

Proof:

Since

$$X = \xi_1(x_1, x_2) \frac{\partial}{\partial x_1} + \xi_2(x_1, x_2) \frac{\partial}{\partial x_2} + \eta^1(x_1, x_2, u) \frac{\partial}{\partial u}$$

and

$$X_S = X + \eta^2(x_1, x_2, v) \frac{\partial}{\partial v}$$

Therefore

$$X = \xi_1(x_1, x_2) \frac{\partial}{\partial x_1} + \xi_2(x_1, x_2) \frac{\partial}{\partial x_2} + \eta^1(x_1, x_2, u) \frac{\partial}{\partial u} + \eta^2(x_1, x_2, v) \frac{\partial}{\partial v}$$

Now, using invariant form method [3],[7] of the system of PDEs S, from X_S we will have the similarity variable $z=z(x_1, x_2)$ with two similarity solutions $U=U(z)$ and $V=V(z)$ such that u is a function of x_1, x_2 and U and v is a function of x_1, x_2 and V .

Consequently eq. (2a) leads to:

$$U = H(z, V, V_1) \dots \dots \dots (3a)$$

It is clear that, $\frac{\partial H}{\partial V_1} \neq 0$ since $\frac{\partial z}{\partial x_1} \neq 0$

while eq. (2b) leads to:

$$K(z, U, U_1, \dots, U_{n-1}, V, V_1) = 0 \dots \dots (3b)$$

Therefore X_S must reduce the system of PDEs S to system of ODEs $S_{X_S} \{z, U, V\}$ given in the form (3a, b).

Now, we claim that the transformation (3a) represent a BT for R_X ;

Eq. (3b) gives:

$$DK(z, U, U_1, \dots, U_{n-1}, V, V_1) = 0 \dots \dots (4)$$

such that D is defined in the form

$$D = \frac{\partial}{\partial z} + U_1 \frac{\partial}{\partial U} + \dots + U_n \frac{\partial}{\partial U_{n-1}} + V_1 \frac{\partial}{\partial V}$$

Eq. (4) represents a conserved form for R_X

since $X_S = X + \eta^2(x_1, x_2, v) \frac{\partial}{\partial v}$

Eq. (3a) gives:

$$\begin{aligned} U_1 &= DU = DH(z, V, V_1) = DH \\ U_2 &= D^2U = D^2H(z, V, V_1) = D^2H \\ &\vdots \\ U_{n-1} &= D^{n-1}U = D^{n-1}H(z, V, V_1) = D^{n-1}H \end{aligned}$$

then the auxiliary ODE related to R_X is given by

$$K(z, H, DH, D^2H, \dots, D^{n-1}H, V, V_1) = 0 \dots \dots (5)$$

Therefore S induces a BT given in eq. (2a) for R_X with auxiliary ODE of nth order ($n \geq 2$) given in eq. (5). □

Application of the (Ibt Theorem).

In this section, we will apply our theorem to two examples, using the results obtained by Gandarias et al [8] in 1998. In their work they found the infinitesimal generators $\{P_i\}_{i=1}^4$ of the nonlinear diffusion equation with convection $R\{x, t, u\}$,

$$u_t = (u^n)_{xx} + \frac{c}{x + \lambda} (u^n)_x \dots \dots \dots (6)$$

where $n \in \mathbb{Q} \setminus \{-1, 0, 1\}$, $c = \frac{3n + 1}{n + 1}$ and $\lambda \in \mathbb{R}$, given by:

$$P_1 = \frac{\partial}{\partial t}; P_2 = (x + \lambda) \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}; P_3 = -t \frac{\partial}{\partial t} + \frac{u}{n-1} \frac{\partial}{\partial u}$$

$$\text{and } P_4 = \frac{(x + \lambda)^{2-c}}{(c-2)(c-1)} \frac{\partial}{\partial x} - \frac{2(x + \lambda)^{1-c}}{(c-1)(n-1)} u \frac{\partial}{\partial u}$$

also, they found that its auxiliary system $S\{x, t, u, v\}$ given by

$$\frac{\partial v}{\partial x} = (x + \lambda) u \dots \dots \dots (7a)$$

$$\frac{\partial v}{\partial t} = (x + \lambda)(u^n)_x + (c-1)(u)^n \dots \dots \dots (7b)$$

and the infinitesimal generators $\{X_i\}_{i=1}^6$ of S are given in the following Table (3.1).

Table (1)

X_i	ξ_1	ξ_2	η^1	η^2	N	C
X_1	0	1	0	0	$\neq 0,1$	$\frac{3n+1}{n+1}$
X_2	0	0	0	1		$\frac{3n+1}{n+1}$
X_3	$(x + \lambda)$	0	$\frac{2u}{n-1}$	$\frac{2nv}{n-1}$	$\neq 0,1$	Arbitrary
X_4	$(x + \lambda)$	2nt	-2u	0		
X_5	$k(x + \lambda)^{\frac{1-n}{1+n}}$	0	$\frac{-2k(x + \lambda)^{\frac{-2n}{n+1}}}{n+1}$	0	$\neq 0, -1, 1$	$\frac{3n+1}{n+1}$
X_6	$2(x + \lambda)v$	0	$-2((x+\lambda)^2u^2+uv)$	v^2	-1	$-\frac{1}{2}$

Infinitesimal Generators of S

Example (3.1)

We will apply our theorem on the auxiliary system S of the diffusion eq. (6) given in section one. In order to satisfy the conditions of the theorem we will define a new infinitesimal generator of S. From table (3.1), we can see that:

$$X_3 + X_4 \equiv (x + \lambda) \frac{\partial}{\partial x} + n t \frac{\partial}{\partial t} + \left(\frac{2-n}{n-1} \right) u \frac{\partial}{\partial u} + \frac{2nv}{n-1} \frac{\partial}{\partial v}$$

Now, if we put $n=2$ and $c = \frac{3n+1}{n+1} = \frac{7}{3}$;

$X_S = X_3 + X_4$
then

$$X_S = (x+\lambda) \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t} + 2v \frac{\partial}{\partial v} = X + 2v \frac{\partial}{\partial v}$$

where

$$X = (x+\lambda) \frac{\partial}{\partial x} + 2t \frac{\partial}{\partial t}$$

Note that $X = P_2$ which is the infinitesimal generator of R given in section one, using invariant form method [10], with $X=P_2$, we get the induced ODEs, as

$$2z UU'' + 2z (U')^2 + \left(\frac{14}{3} U + \frac{z^2}{2} \right) U' = 0 \tag{8}$$

the auxiliary system S of R given in section one at $n=2$ as:

$$\frac{\partial v}{\partial x} = (x+\lambda)u$$

$$\frac{\partial v}{\partial t} = (x+\lambda)(u^2)_x + \frac{4}{3}(u)^2$$

Now, our example above seems satisfy all the conditions of the IBT theorem. Therefore in order to find the IBT of eq. (8) from S, we need only to find S_{X_S} .

By using invariant form method, the characteristic equations are

$$\frac{dx}{x+\lambda} = \frac{dt}{2t} = \frac{du}{0} = \frac{dv}{2v}$$

then the similarity variable $z = z(x, t)$ with two similarity solutions for (u,v) are given by

$$z = \frac{x + \lambda}{\sqrt{t}} \quad \text{and} \quad (u, v) = (U(z), t V(z))$$

substitution (u, v) above in S, we will get S_{X_S} as:

$$V' = z U \tag{9a}$$

$$-\frac{1}{2} z V' + V = 2zUU' + \frac{4}{3}(U)^2 \tag{9b}$$

we can say that eq. (8) has the IBT from eq. (9a) as:

$$U = H(z, V, V') = \frac{V'}{z}$$

and the auxiliary ODE related to eq. (6) from eq. (7b) as:

$$\frac{1}{2} z V' + V - 2z V' \frac{d}{dz} \left(\frac{V'}{z} \right) - \frac{4}{3} \left(\frac{V'}{z} \right)^2 = 0$$

such that

$$K(z, U, U', V, V') = -\frac{1}{2} z V' + V - 2z V' \frac{d}{dz} \left(\frac{V'}{z} \right) - \frac{4}{3} (U)^2$$

Example (3.2):

X_4 is one of the infinitesimal generators of S (see Table (3.1)) and since it's infinitesimal coefficients of x , t and u are independent of v then X_4 must be infinitesimal generator of R [3],[10]. It is clear that X_4 is different from the infinitesimal generators admitted by R , therefore X_4 will define a new ODE induced from R under X_4 denoted by R_{X_4} (using invariant form method [10]) R_{X_4} is

$$z(U)^{n-1} U'' + (n-1)z (U)^{n-2} (U')^2 + (c U^{n-1} + \frac{z^2}{2n^2})U' + \frac{z}{n^2} U = 0$$

where $z = \frac{x + \lambda}{(t)^{\frac{1}{2n}}}$ and $u = (t)^{\frac{1}{n}} U(z)$

we can see that, the conditions of our theorem are satisfied, since the infinitesimal coefficient of v equal to zero in X_4 then X_4 is an infinitesimal generator of R and S in the same time.

Therefore, in order to find the IBT of R_{X_4} from S when $n \neq 0,1$ and c arbitrary, we need only to find the system of ODEs $S_{X_4} \{z,U,V\}$ (using invariant form method [10]), we get S_{X_4} as,

$$V' = z U \dots \dots \dots (10a)$$

$$-\frac{1}{2n} z V' = n z (U)^{n-1} U' + (c-1)(U)^n \dots (10b)$$

Now, we can say that the equation R_{X_4} has IBT from eq. (10 a) as:

$$U = H(z, V, V') = \frac{V'}{z}$$

and the auxiliary ODE related to the ODE R_{X_4} from eq. (10 b) as:

$$nz \left(\frac{V'}{z}\right)^{n-1} \frac{d}{dz} \left(\frac{V'}{z}\right) + (c-1) \left(\frac{V'}{z}\right)^n + \frac{1}{2n} z V' = 0$$

such that

$$K(z,U,U',V,V') = nz(U)^{n-1}U' + (c-1)(U)^{n-1} + \frac{1}{2n} zV$$

Conclusions

In example (3.1) we can see that eq. (8) belongs to (Bluman-Reid) class, while in Example (3.2), we can see that (Bluman-Reid) algorithm fail to obtain a BT for the ODE R_{X_4} .

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