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INVARIANTE AND REDUCTION OF THE CLASS $y'' = F(x)y^n$, $(n \neq 0)$, UNDER STRETCHING GROUPS

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Abstract

The aim of this papar, using stretching groups which leaves the form of the class differential equations invariant and reduce to first-order differential equations in the (p,q)-plane.
Whenever satisfied the following conditions

- n=1 the function F(x) = $x^{\beta-1}$.
- n=2,3,... the function F(x)=-1, $\beta < 0$.

with remaind two theorem for the purpose.

Introduction

In the relativistic spherically symmetric perfect fluids, differential equations of the form

$$\frac{d^2y}{dx^2} = F(x)\, y^n \ , \quad n \neq 0 \ \ldots\ldots\ldots\ldots \ (1)$$

arise on two occasions. The first (with n=2) is that of a fluid in shear-free motion, the second with ($n=-\frac{1}{3}$) corresponds to a special class of shearing perfect fluids. In both cases the function F(x) is arbitrary: it is specified only when the equation of state is fixed, and vice versa [1].

Solutions of the differential equations (1) rare and they have been found only for rather restricted classes of functions F(x) [1].

In [2], when the differential equations are invariant under a stretching group, the group can be used to determine the asymptotic form of certain solutions that are important in application.

In [4], every solution y(x) for any differential equation is mapped into a solution $\overset{*}{y}(\overset{*}{x})$ under the effect of group of transformation, if there exist an invariance transformation.

In this paper, the differential equations (1) are invariant under a stretching group by calculating the form

F(x)= $x^{\beta-1}$, $\beta < 0$ when n=1

and

F(x)= $-1, \beta < 0$ when n=2,3,....

With finding way of reduce differential equation (1) to first order differential equation in the (p,q)- plane.

The following definitions and theorems are needed later on.

Definition (1) ***(Group of Transformations)***

Atransformation form n-dimensional Euclidean space E_n of points $\overset{*}{x} = (\overset{*}{x}_1, \overset{*}{x}_2, ..., \overset{*}{x}_n)$ into point x = (x_1,x_2, ..., x_n) be defined by the relation $x_i=F_i(\overset{*}{x},\beta)$, i=1,2,...,n such that F_i are continuous in x and β is a continuous parameter, therefore the transformation is continuous. Such that $T_\beta \overset{*}{x} = x = F(\overset{*}{x},\beta)$ and $T_\delta T_\beta \overset{*}{x} = F\{F(\overset{*}{x},\beta),\delta\}$

$$= F\{\overset{*}{x}, \phi(\beta,\delta)\} = T_\phi \overset{*}{x}$$

provided that the transformations form a group through the parameterβ.

Definition (2)[2] ***(Stretching group)***

The stretching group in two variables x and y is defined by

$$\overset{*}{x} = \lambda x$$

$$\overset{*}{y} = \lambda^\beta y \qquad ;0 < \lambda < \infty$$

where β is a constant.

Theorem (1)[3] ***[Lie's reduction theorem]***

If the function p(x,y) is a group invariant and the function q(x,y,u) =$q(x,y,\dot{y})$ is a first differential invariant. The second-order differential equation $w(x,y,\dot{y},\ddot{y})=0$ will be reduced to a first-order differential equation in p and q.

Remark (1) [3]

The curve in the (p,q)- plan can be determined by studying the direction field of the associated differential equation $\frac{dq}{dp}=f(p,q)$. (The direction field is obtained by drawing a short line segment having the slope $\frac{dq}{dp}$ given by the differential equation at each point (p,q) of the (p,q)- plane).

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Theorem (2)[3]:-

If $X(x,y;\lambda)=\lambda x$, is a stretching. Then $\xi = x$ and the differential equation $\xi F_x + \varsigma F_y = \underline{x} = F$ may be satisfied by choosing F= x.

1- Stretching Groups of the Second order Ordinary Differential Equations [2],[3].

Consider the second-order ordinary differential equation:

$$f(x,y,\dot{y},\ddot{y})=0 \quad \text{.....................................} (2)$$

Also, consider the stretching group in two variables x and y :

$$\left.\begin{aligned} &\overset{*}{x}=\lambda x, && \overset{*}{\dot{y}}=\lambda^{\beta-1}\dot{y} \\ &\overset{*}{y}=\lambda^{\beta}y, && \overset{*}{\ddot{y}}=\lambda^{\beta-2}\ddot{y} \end{aligned}\right\} \text{................} (3)$$

where $0<\lambda<\infty$, β being the group parameter(constant). Then

$$\left.\begin{aligned} &\xi = x, && \varsigma_1=(\beta-1)\dot{y} \\ &\varsigma=\beta y, && \varsigma_2=(\beta-2)\ddot{y} \end{aligned}\right\} \text{.....................} (4)$$

If the differential equation (2) satisfies the following condition

$$\xi f_x + \varsigma f_y + \varsigma_1 f_{\dot{y}} + \varsigma_2 f_{\ddot{y}} = 0$$

Then the differential equation (2) is said to by invariant.

Now, by using Lie's reduction theorem for the characteristic equations

$$\frac{dx}{x}=\frac{dy}{\beta y}=\frac{d\dot{y}}{(\beta-2)\dot{y}}=\frac{d\ddot{y}}{(\beta-2)\ddot{y}} \text{............} (5)$$

we have; p(x,y)= $\frac{y}{x^{\beta}}$ .... (6)

$$q(x,y,\dot{y})=\frac{\dot{y}}{x^{\beta-1}} \text{....................................} (7)$$

such that the function p(x,y) is a group invariant, the function $q(x,y,\dot{y})$ is a first differential equations and the equation G(p,q,c)=0 represents a one-parameter family of curves in the (p,q)-plane.

2- Invariante and Reduction of the Class $y''(x)$= F(x) $y^n(x)$, $(n \neq 0)$ for every n (n=1,2,3,...)

Given a differential equation

$$\frac{d^2y}{dx^2}=F(x)y^n \quad (n\neq 0) \text{ ....} (8)$$

and take a mapping $\{x,y,F(x)\} \longrightarrow \{\overset{*}{x},\overset{*}{y},F(\overset{*}{x})\}$ that maps (8) into $\frac{d^2\overset{*}{y}}{d\overset{*}{x}^2}=F(\overset{*}{x})\,\overset{*}{y}^n$ by using stretching group.

Therefore there are two cases for dealing with this invariance condition which we shall discuss now in details.

2.1 For n=1 [i.e., for the case $y''(x)$=F(x)y(x)].

From equations (6)-(7) one can have.

$$\frac{dp}{dx} = x^{-\beta}\dot{y} - \beta x^{-\beta+1} y$$

$$\frac{dq}{dx} = -(\beta-1)x^{-\beta}\dot{y} + x^{-\beta-1}\ddot{y}$$

Then,

$$x\frac{dp}{dx} = x^{1-\beta}\dot{y} - \beta x^{-\beta} y \text{ .... (9)}$$

$$x\frac{dq}{dx} = -(\beta-1)x^{1-\beta}\dot{y} + x^{-\beta-2}\ddot{y} \text{ .... (10)}$$

Now, subsitituing $\ddot{y}$=F(x)y in the last term of equation(10) such that

$\ddot{y}$=F(x)y$\Rightarrow x^{-\beta-2}\ddot{y} = F(x)x^{-\beta-2}y$

we choose $F(x)\,x^{-\beta-2}y = P(x,y)$

since P(x,y) is a group invariant

$\Rightarrow$ $F(x)\;x^{-\beta-2}y = P = x^{-\beta}y$

$$\Rightarrow F(x) = \frac{x^{-\beta}}{x^{-\beta-2}} = x^{-2}$$

Now, if we see to the first differential invariant $q(x,y,\dot{y}) = \dfrac{\dot{y}}{x^{\beta-1}}$ and F(x)= x^{-2}

$\beta - 1 = -2 \Rightarrow \quad \beta = -1$

Hence the differential equation y''=F(x)y is invariant under stretching group if is satisfi the following conditions

- $\beta = \dfrac{-2}{1+n}$, $\beta < 0 \xrightarrow{n=1} \beta = -1$......(11)
- $F(x) = x^{\beta-1}$, $\beta < 0 \xrightarrow{n=1} F(x) = x^{-2}$ .... (12)

Now, reducing the differential equation $y''(x) = x^{-2}y(x)$ under a stretching group

Here we have replaced $\ddot{y}$ by its value in terms of x,y and $\dot{y}$ obtained from $y''(x) = x^{-2}y(x)$

$$\frac{dp}{dx} = x\dot{y} + y \Rightarrow x\frac{dp}{dx} = x^2\dot{y} + xy = q + p$$
.... (13)

$$\frac{dq}{dx} = 2x\dot{y} + x^2\ddot{y} \Rightarrow x\frac{dq}{dx} = 2x^2\dot{y} + x^3\ddot{y} = 2q + p$$
.... (14)

Dividing equation(14) by equation (13) we obtain

$$\frac{dq}{dp} = \frac{2q+p}{q+p} \text{ .... (15)}$$

Which is a first-order differential equation in the (p,q)-plan.

Theorem (1)

If the linear differential equation $y''(x) = x^{\beta-1}y(x)$, $\beta < 0$ under stretching group satisfied the condition

$$\beta = \frac{-2}{1+n} \quad ,(n=1)$$

Then the linear differential equation $y''(x) = x^{\beta-1}y(x)$ $\beta < 0$ in (p,q)-plan) is invariant and may be reduced to a first-order.

2.2 For the second-order differential equation $y''(x) = F(x)y^n(x)$ $n \neq 0$, (n=2,3,...)

From equations (6)-(7) one can have

$$\frac{dp}{dx} = \frac{\dot{y}}{x^{\beta}} - \frac{\beta y}{x^{\beta+1}} \text{ .... (16)}$$

$$\frac{dq}{dx} = \frac{\ddot{y}}{x^{\beta-1}} - \frac{(\beta-1)\dot{y}}{x^{\beta}} \text{ .... (17)}$$

Subsitituting P(x,y), $q(x,y,\dot{y})$ and $\ddot{y}$=F(x)y^n in equations (16)-(17) to get

$$\frac{dp}{dx} = \frac{q}{x} - \frac{\beta p}{x} = \frac{q-\beta p}{x} \text{ .... (18)}$$

$$\frac{dq}{dx} = \frac{F(x)y^n}{x^{\beta-1}} - \frac{(\beta-1)\dot{y}}{x^{\beta}}$$

if we take F(x)=-1 and $\beta = \dfrac{1/2}{1-n}$, n=2,3,4,...

$$\Rightarrow \frac{dq}{dx} = \frac{-\left(\frac{y}{x^{\beta}}\right)^{\frac{\beta-2}{\beta}} - (\beta-1)\frac{y}{x^{\beta-1}}}{x}$$

$$= \frac{-P^{\frac{\beta-2}{\beta}} - (\beta-1)q}{x} \quad \text{......................} (19)$$

This emplies, the function $q(x, y, \dot{y})$ is an invariant of the one-extended group and separable, is called a first differential invariant. Now, reduce the differential equation $\mathbf{y}'' = -y^{n}$, n=2,3,... under a stretching group. Dividing equation (19) by equation (16), we obtained

$$\frac{dq}{dp} = \frac{-p^{\frac{\beta-2}{\beta}} - (\beta-1)q}{q-\beta p}$$

$$= \frac{-p^{\frac{\beta-2}{\beta}} - (\beta-1)q + 2q - 2q}{q-\beta p}$$

$$\Rightarrow \frac{dq}{dp} = \frac{-p^{\left(\frac{\beta-2}{\beta}\right)} - (\beta+1)q - 2q}{q-\beta p} \quad \text{...........} (20)$$

which is a first-order differential equation in the (p,q)-plan.

If substitue (n=2,3,4,...) in $\beta = \frac{1/2}{1-n}$ we have $(\beta = -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, ...)$. Then equation (20) can be integrated directly.

Theorem(2)

If the differential equation $\mathbf{y}'' = -y^{n}$, $n \neq 0$ $(n = 2,3,4,...)$ under stretching group satisfies the condition

$$\beta = \frac{1/2}{1-n}, n = 2,3,4,...,$$

Then the differential equation $\mathbf{y}'' = -y^{n}$, $n \neq 0$ $(n = 2,3,4,...)$ is invariant and reduce to the first-order differential equation in (p,q)-plane.

Conclusions

- An invariance leaves the form of the differential equation, $y'' = F(x)y^{n}, (n \neq 0)$ invariance under the effects of the function F(x).
- The condition $F(x) = x^{-2}$ when (n=1) and F(x) =–1 where (n=2,3,...) is unchanged under the stretching group because x unchang.
- Reduction of the differential equation $y'' = F(x)y^{n}, (n \neq 0)$ depend on two function p(x,y) is invariant group $q(x, y, \dot{y})$ is a first-order differential invariant in the (p,q)-plane.

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الخلاصة

الهدف من هذا البحث هواستخدام زمر المد

(Sretching groups) التي تجعل المعادلات التفاضلية

الاعتيادي ذات الشكل $y'' = F(x)y^{n}, (n \neq 0)$ لا متغايره

ويمكن أختزالها الى معادلات تفاضلية ذات الرتبة الاولى في

المستوي –(p,q) متى ما حققت الشروط التالية:

- أذا كانت n=1 فأن الداله $F(x) = x^{\beta-1}$ بحيث $\beta < 0$.
- أذا كانت n=2,3,... فأن الداله $F(x) = -1$ بحيث $\beta < 0$.

مع ذكر مبرهنتين لهذا الغرض.