On the Solvability of the Operator Equation $AT+T^*B=C$

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Abstract

In this paper we study the solvability of the operator equation

$AT+T^*B=C$

for linear bounded operators on Hilbert space, where T is the unknown operator.

First we show this equation may not have a solution in B(H), and then we show that this equation have a simple solution in case A,B are Hermitian or Skew-Hermitian operators. Finaly we study the solvability of the operator equation $AT+T^*B=f(A)$ in case A and B are normal operators.

Introduction

Let H be a separable complex Hilbert space, and let B(H) be the algebra of bounded linear operators on H. For given $A,C\in B(H)$ many mathematicians interested in finding the solution $T\in B(H)$ of the equation $A^*T+T^*A=C$. This equation is considered for matrices over a finite field [9].

We mention similar matrix equations, which have applications in control theory. These equations are investigated for matrices over fields, mostly R or C. The equation $CX - XA^T = B$ is the Sylvester equation [10]. One special and important case is the Lyapunov equation $AX + XA^T = B$ [3]. Also, the generalized Sylvester equation $AV + BW = EVJ+R$ with unknown matrices *V* and *W*, has many applications in linear systems theory (see [6]).

Dragan S. Djordjevic´ in [5] deals with extension of results from [9] to infinite dimensional settings.

Dragan S.Cvetkovic-Ilic in [4] generalized the results of Dragan S. Djordjevic´ in [5] to the operator equation $AT+T^*B=C$. He investigated the solvability of this equation under some conditions and described the set of the solutions.

In this paper we proved that this equation may not have a solution in B(H), and then we show that this equation have a simple solution in case A,B are Hermitian or Skew-Hermitian operators. Finaly we study the solvability of the operator equations $AT+T^*B=p(A)$ and $AT+T^*B=f(A)$ in case A and B are normal operators.

1. About the solution of $AT+T^*B=C$

Let A be a bounded linear operator on the separable complex Hilbert space H. A is called Hermitian operator if $A^*=A$ and A is skew-Hermitian if $A^*= -A$, where A^* is the adjoint of A.

Recall that the spectrum of A, denoted by $\sigma(A)=\{\lambda\in\mathbb{C}: A-\lambda I$ is not invertible). It is known that if A is Hermitian, then $\sigma(A)$ consists of real numbers and if A is skew-Hermitian, then $\sigma(A)$ consists of pure imaginary numbers, i.e. $\sigma(A)\subseteq iR$, where R is the set of real numbers ([2],[8]).

The following theorem shows that the equation $AT+T^*B=C$ for a fixed A, B have no solution.

Theorem 1.1:

Let $A, B\in B(H)$ such that $A-B^*$ is not invertible. Then, the equation $AT+T^*B=iI$ has no solution in B(H).

Proof:

Assume that $AT+T^*B=iI$ for some $T\in B(H)$. Then,

$$T^*A^*+B^*T=-iI$$

Hence, $AT+T^*B-T^*A^*-B^*T=2iI$

Therefore, $(A-B^*)T+T^*(B-A^*)=2iI$

And this implies $(A-B^*)T-T^*(A^*-B)=2iI$

If we put $K=(A-B^*)$, then

$$KT-T^*K^*=2iI$$

And hence, $KT-(K\,T)^*=2iI$

This implies $KT=iI+(K\,T)^*+iI$

If we put $C=iI+(K\,T)^*$, then $C=K\,T-iI$. Note that C is Hermitian, in fact

$$C^*=(iI+(K\,T)^*)^*=-iI+KT=C.$$

Hence, as stated in([3]), the spectrum of C, $\sigma(C)$ consists of real numbers. On the other hand, since $A-B^*$ is not invertible, then it can be checked easily that KT is not invertible, and hence $0\in\sigma(KT)$.

But $C=K\,T-iI$, then by the spectral mapping theorem ([12]), $-i\in\sigma(C)$, which is a contradiction.

In a similar way we prove the following:

Theorem 1.2:

Let A, $B\in B(H)$ such that $A+B^*$ is not invertible. Then, the equation $AT+T^*B=I$ has no solution in B(H).

The following proposition shows that if the assumptions of non-invertiblity are drooped in Theorem 1.1 and Theorem 1.2, then the operator equation AT+T*B=C may have a solution.

Proposition 1.3:

Let A and B are Hermitian operators. Then the equation $AT+T^*B=C$ has a solution $T\in B(H)$ if $C=(A+B)^2$.

Proof:

If $C=(A+B)^2$ and A,B are Hermaitian, then it is easily seen that $T=A+B$ is a solution for the equation $AT+T^*B=C$.

In a similar manner, one can prove the following:

Proposition 1.4:

Let A and B are skew-Hermitian operators. Then the equation $AT+T^*B=C$ has a solution in B(H) if $C=A^2-B^2$.

2.On The Operator Equation AT+T*B=C when A and B are Normal Operators.

Let $A\in B(H)$, A is called normal if $AA^*=A^*A$, ([12]). If $p(\lambda)=a_0+a_1\lambda+\cdots+a_n\lambda^n$ is a complex polynomial, where a_i, i=0,1,…,n, a complex number, we define

$$p(A)=a_0I+a_1A+\cdots+a_nA^n$$

Recall that an element $\lambda\in\sigma(A)$ is called an eigenvalue of A ($\lambda\in\prod_o(A)$) if there exists a non-zero vector $x\in H$ such that $Ax=\lambda x$. And λ is called an approximate eigenvalue of A ($\lambda\in\prod(A)$) if there exists a sequence $\{u_n\}$ of unit vectors in H such that $\|(A-\lambda I)u_n\|\to 0$.

Equivalently, for each $\varepsilon>0$ there exists a unit vector $x\in H$ such that $\|Ax-\lambda x\|<\varepsilon$, ([4]). It is known that if A is a normal operator, then every element in the spectrum is an approximate eigenvalue, ([4]).

In this section, we study the equations $AT+T^*B=p(A)$ and $AT+T^*B=p(B)$ for normal operators A and B. Before we state our main result of this section, we need the following:

Lemma 2.1[1]:

Let $T\in B(H)$, and let p be any polynomial. If λ is any approximate eigenvalue for T, then $p(\lambda)$ is an approximate eigenvalue for p(T).

Now, suppose that A, B are two normal operators. The following theorem shows that if the equation $AT+T^*B=p(A)$ has a solution and if $\lambda\in\sigma(A)=\sigma(B)$, $\lambda\neq 0$, then $\frac{p(\lambda)}{\lambda}$ is a real number.

Theorem 2.2:

Let A, $B\in B(H)$ be two normal operators such that $\sigma(A)=\sigma(B)$, and let $p(x)=a_nx^n+a_{n-1}\lambda^{n-1}+\cdots+a_0$ be a polynomial. Then:

(1) If the equation $AT+T^*B=p(A)$ has a solution T in B(H), then for each $\lambda\in\sigma(A)$, if $\lambda=0$ then $p(\lambda)=0$ and otherwise $\frac{p(\lambda)}{\lambda}$ is a real number.

(2) If the equation $A^*T+T^*B^*=p(A)$ has a solution T in B(H), then for each $\lambda\in\sigma(A)$, $\lambda p(\lambda)$ is a real number.

Proof:

(1) Let $\varepsilon>0$ and let $\lambda\in\sigma(A)=\sigma(B)$. Suppose that T=0, then $p(A)=AT+T^*B=0$. By the spectral mapping theorem $p(\sigma(A))=\sigma(p(A))=\sigma(0)=\{0\}$. It follows that $p(\lambda)=0$. Hence, if $\lambda=0$, then $p(\lambda)=0$, and otherwise $\frac{p(\lambda)}{\lambda}=0\in R$.

Suppose that $T\neq 0$. Since λ is an approximate eigenvalue for A and B, then there exists a unit vector x in H such that,

$$\|T\|\|Ax-\lambda x\|<\varepsilon, \|x\|=1 \quad ....(2.1)$$

and

$\|Bx - \lambda x\|\|T\| < \varepsilon t, t > 0, \|x\| = 1$ .... (2.2)

Hence by lemma 2.1, p(λ) is an approximate eigenvalue for p(A). Thus,

$\|p(A)x - p(\lambda)x\| < \varepsilon t_1, t_1 > 0, \|x\| = 1$

By Schwarz inequality, $|< p(A)x - p(\lambda)x, x >| < \varepsilon t_1$ and hence

$|< p(A)x, x > - p(\lambda)| < \varepsilon t_1$ .... (2.3)

On the other hand, $p(A)-(AT+T^*B)=0$ implies $<p(A) -(AT+T^*B)x,x>=0$.

And hence,

$|< p(A)x, x > - < Tx, A^*x > - < Bx, Tx >| = 0$ .... (2.4)

Now, the normality of A implies $\|Ay\| = \|A^*y\|$ for all y∈H. Hence from equation (2.1) we get

$\|T\|\|A^*x - \bar{\lambda}x\| < \varepsilon, \|x\| = 1$ .... (2.5)

From equations (2.2), (2.5) and by using Schwarz inequality, one can get

$|< Bx, Tx > - \lambda < x, Tx >| < \varepsilon t$ .... (2.6)

and

$|< Tx, A^*x > - \lambda < Tx, x >| < \varepsilon$ ...... (2.7)

By adding equations (2.3), (2.4), (2.6), (2.7) we get

$|p(\lambda) - \lambda(T + T^*)| < \varepsilon(1 + t + t_1)$

Since $T+T^*$ is Hermitian, then it is easily seen that $|< (T + T^*)x, x >|$ is real number say r. Thus

$|p(\lambda) - \lambda r| < \varepsilon(1 + t + t_1) \quad, r \in \mathrm{R}$

for each ε>0. It follows that p(λ)−λr=0 and hence p(λ)=λr, r∈R. It is clear that if λ=0, then p(λ)=0, otherwise $\frac{p(\lambda)}{\lambda}$=r, r∈R.

(2) Assume that $p(A)= A^*T+T^*B^*$ for some T∈B(H). As in part (1), we can assume T≠0.

$\|T\|\|Ax - \lambda x\| < \varepsilon, \|x\| = 1$

$\|Bx - \lambda x\|\|T\| < \varepsilon t, t > 0, \|x\| = 1$

$\|p(A)x - p(\lambda)x\| < \varepsilon t_1, t_1 > 0, \|x\| = 1$

Thus, the equations (2.3), (2.4), (2.6), (2.7) become

$|< p(A)x, x > - p(\lambda)| < \varepsilon t_1$

$|< p(A)x, x > - < Tx, Ax > - < B^*x, Tx >| = 0$

$|< B^*x, Tx > - \bar{\lambda} < x, Tx >| < \varepsilon t$

$|< Tx, Ax > - \bar{\lambda} < Tx, x >| < \varepsilon$

By the adding the last four equations we get

$|p(\lambda) - \bar{\lambda}(T + T^*)| < \varepsilon(1 + t + t_1)$

Thus, we have

$|p(\lambda) - \bar{\lambda}r| < \varepsilon(1 + t + t_1)$

Therefore, $p(\lambda)-\bar{\lambda}r=0$, r∈ R, hence λp(λ)∈R.

Remark 2.3:

In a similar way we prove the following:

Let A, B∈B(H) are two normal operator such that σ(A)=σ(B), and let $p(x)= a_nx^n+ a_{n-1}\lambda^{n-1}+ \cdots+a_0$ be a polynomial. Then:

(1) If the equation $AT+T^*B=p(B)$ has a solution T in B(H), then for each λ∈σ(B), if λ=0 then p(λ)=0 and otherwise $\frac{p(\lambda)}{\lambda}$ is a real number, i.e, an element in R.

(2) If the equation $A^*T+T^*B^*=p(B)$ has a solution T in B(H), then for each λ∈σ(B), λp(λ) is a real number.

3. Operator Equation AT+T*B=C and Analytic Operators

Let f be a complex analytic function defined on the ball $B_r=\{z\in C: |z|<r\}$ where r > 0. Let A∈B(H) such that $\|A\|$<r. By Taylor theorem $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and this series converges in $|z|$<r. It is known that $\sum_{n=0}^{\infty} a_n z^n$ converges in B(H), ([11]), and one can define

$$f(A) = \sum_{n=0}^{\infty} a_n A^n$$

Since $|\sigma(A)|<\|A\|$, where $|\sigma(A)|$ denotes the spectral radius of A, then $\sigma(A)\subset B_r$. In particular, if λ is an eigenvalue for A, then $\sum_{n=0}^{\infty} a_n \lambda^n$ that converges to *f(λ)* is defined. The operator f(A) is called an analytic operator [12].

Before we give one of our main results in this paper, we need the following:

Lemma 3.1[1]:

Let f and A be as above, and let λ be an eigenvalue for A, then f(λ) is an eigen value for f(A). And if Ax=λx, then f(A)x=f(λ)x.

The following theorem shows that if the equation $AT+T^*B=f(A)$ has a solution and if $\lambda\in\Pi_o(A)= \Pi_o(B)$, λ≠0, then $\frac{p(\lambda)}{\lambda}$ is a real number.

The next theorem gives necessary conditions for the equation $AT+T^*B=f(A)$ to have a solution.

Theorem 3.2

Let A, B∈B(H) be two normal operators such that $\Pi_o(A)= \Pi_o(B)$ and A,B have the same eigenvectors , and let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be an analytic function in the ball B_r such that that $\|A\|$<r.

(1) If the equation $AT+T^*B=f(A)$ has a solution in B(H), then for each eigenvalue λ of A, if λ=0, f(λ)=0 and otherwise $\frac{f(\lambda)}{\lambda}$ is a real number.

(2) If the equation $A^*T+T^*B^*=f(A)$ has a solution T in B(H), then for each eigenvalue λ of A λf(λ) is a real number.

Proof:

(1) Let λ be an eigenvalue for A, and let x be the corresponding eigenvector. Thus, Ax=λx, we may assume $\|x\| = 1$. By 3.1 f(A)x=f(λ), hence <f(A)x,x>−f(λ)x=0. Moreover, $f(A)-(AT+T^*B)=0$, hence $<f(A)x,x> - <Tx,A^*x> - <Bx,Tx>=0$. Since A is normal $A^*x = \bar{\lambda}x$, hence $<Tx, A^*x - \bar{\lambda}x>=0$.

And since $\Pi_o(A)= \Pi_o(B)$ then, Bx=λx, thus <Bx−λx,Tx>=0.

It follows now from these equations that $f(\lambda)-\lambda(<(T+T^*)x,x>)=0$ and f(λ)=λr, where $r=<(T+T^*)x,x> \in R$.

(2.0) Let λ be an eigenvalue for A, and let x be the corresponding eigenvector. Thus, Ax=λx, we may assume $\|x\| = 1$. By 3.1 f(A)x=f(λ), hence <f(A)x,x>−f(λ)x=0. Moreover, $f(A)-(A^*T^*+T^*B^*)=0$, hence $<f(A)x,x> - <Tx,Ax> - <B^*x,Tx>=0$. Since B is normal and since $\Pi_o(A)= \Pi_o(B)$ then $B^*x = \bar{\lambda}x$, hence $<B^*x - \bar{\lambda}x$,Tx>=0. From Ax=λx, one can get thus <Tx,Ax−λx>=0.

It follows now from these equations that $f(\lambda)-\bar{\lambda}(<(T+T^*)x,x>)=0$ and $f(\lambda)=\bar{\lambda}r$, where $r=<(T+T^*)x,x> \in R$, hence λf(λ)∈R. .

Remark 3.3:

In a similar way we prove the following:

Let A, B∈B(H) be two normal operators such that $\Pi_o(A)= \Pi_o(B)$ and A,B have the sam eigenvectors, and let $f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$ be an analytic function in the ball B_r such that that $\|B\|$<r.

(1) If the equation $AT+T^*B=f(B)$ has a solution in B(H), then for each eigenvalue λ of A, if λ=0, f(λ)=0 and otherwise $\frac{f(\lambda)}{\lambda}$ is a real number.

(2) If the equation $A^*T+T^*B^*=f(B)$ has a solution T in B(H), then for each eigenvalue λ of A λf(λ) is a real number.

References

[1] Adil G. Naoum and Amina R. Al-Kabbani, 2007, On Operator Equation $TA-AT^*=C$, Dirasat, pure science, volume 34, No. 1, 113-118.

[2] Berberian, S. 1976. Introduction to Hilbert Space, Chelsea Publishing Company, New York.

[3] D. C. Sorensen and A. C. Antoulas,2002, The Sylvester equation and approximate balanced reduction, Linear Algebra Appl. 351-352 (671–700).

[4] Dragan S.Cvetkovic-Ilic. 2008. The solutions of some operator equations, J. Korean Math. Soc. 45, No. 5, 1417-1425.

[5] Dragan S. Djordjevic´. 2005, Explicit Solution of The Operator Equation A*X + X*A = B, Supported by Grant No. 144003 of the Ministry of Science, Republic of Serbia.

[6] G. R. Duan,2004, The solution to the matrix equation AV +BW = EV J +R, Appl. Math. Letters, 17, 1197–1202.

[7] Halmos, P. 1982. A Hilbert Space Problem Book. Springer Verlag, New York, INC.

[8] Halmos, P. 1957. Introduction to Hilbert Space and Theory of Multiplicity, Chelsea Publishing Company, N.Y., New York.

[9] J. H. Hodges, 1957, Some matrix equations over a finite field, Ann. Mat. Pura Appl. IV. Ser. 44, 245–250.

[10] P. Kirrinnis, 2001, Fast algorithms for the Sylvester equation $AX - XB^T=C$, Theoretical Compute Science, 259, 623–638.

[11] Rajavi, H. and Rosenthal, P. 1973. Inveriant Subspace, Springer-Verlag, Berlin, Heidelberg, New York.

[12] Taylor, A. and Lay, D. 1980. Introduction to Functional Analysis, John Wiley and Sons, London.

الخـــلاصـــة

ليكن H فضاء هلبرت وليكن $B(H)$ جبر المؤثرات الخطية المقيدة على H. ليكن A,B,C عناصر في $B(H)$. المعادلة التي بالصيغة $CX - XA^T = B$ والتي تدعى معادلة سلفستر درست باهتمام من قبل العديد من الباحثين لما لها من تطبيقات مهمة، في الاونة الاخيرة ظهرت دراسة نوع آخر من الدوال التي بدأت تحظى باهتمام اكثر فاكثر، وهي على الصيغة $AT+T^*B=C$ حيث T^* هو المؤثر المرافق الى T، في هذا البحث اثبتنا ان هذه المعادلة يمكن ان يكون ليس لها حل في جبر المؤثرات الخطية المقيدة $B(H)$. وايضا بينا ان لهذه المعادلة حل اذا كان كل من A وB مؤثرا هيرماتيا، واخيرا درسنا حل المعادلات $AT+T^*B=p(A)$ و $AT+T^*B=f(A)$ عندما يكون كل من A و B مؤثرا سويا.