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STABILITY CRITERIA FOR SOLVING A CLASS OF OF DIFFERENCE EQUATIONS

Alaudin N. Ahmed
Department of Mathematics and Computer Applications, College of Science,
Al-Nahrain University.

Abstract
In this paper, stability results are obtained on a class of Explicit Multi-Stage Runge-Kutta methods of arbitrary order of accuracy suitable for solving linear and nonlinear Difference Equations obtained from discretizations ODE's and PDE's.

Introduction
It is common practice in solving Differential Equations (DEs) to discretize firs the spatial variables to obtain a system with variables which may be thought of as occurring discretely. That is, we cannot really observe organisms continuously, so we just monitor the quantities of interest at discrete intervals. In this case, it is usually required It discuss the stability measure, which have recently been introduced by a number of authors.

Stability criteria have also been derived for such equations which involves discretizations process which leads to several well known difference schemes. Thus besides the question of existence of solutions, stability behaviors of solutions o: difference schemes are also of fundamental importance, because these behaviors an related to the question of growth of numerical errors. The stability problem has been treated by several authors ([l], [2] & [6]). The techniques used to derive stability criteria in these studies include Cronwall type inequalities, general solutions, Laplace transforms comparison theorems, etc.

In this paper, we intend to give a brief introduction for the setup and bask properties of the explicit multi-stage Runge-Kutta methods and then new results are obtained to prove strong stability for a class of well-posed problems u' = Lu, where the operator L is linear, improving and simplifying the proofs for the results in [6].

Explicit Methods:
Consider the ordinary differential equation:

$$u'(t) = L(t)u(t) \qquad \text{.......................................(1)}$$

is discretized by a finite difference "FD" or finite element "FE" approximation, see [3], & [7]. Let R be the set of reals and N the set of nonnegative integers. Consider the following discrete difference equation of the form:

$$u_i^{(j+1)} = a_j u_{i-1}^{(j)} + b_j u_i^{(j)} + c_j u_{i+1}^{(j)} + g_i^{(j)} + G(j, u_i^{(j)}) \qquad \text{......................(2)}$$

where i = l, 2, …, n; $j \in N$; $\{a_j\}$, $\{b_j\}$ and $\{c_j\}$ are real sequences; $g = \{g_i^{(j)}\}$ is a real function defined for i = l,2,..,,n and $j \in N$, and G is a real function. We will also assume that side conditions:

$$u_0^{(j)} = h_j \in R,\ j \in N$$
$$u_{n+1}^{(j)} = q_j \in R,\ j \in N \qquad \text{...(3)}$$
$$u_i^{(0)} = \tau_i \in R,\ i \in R, i = 1, 2, \dots, n$$

are imposed. Let:

$\psi = \{(i, j) \mid i = 0, 1, \dots, n + 1, j \in N\}$

A solution of (2) & (3) is a discrete function $u = \{u_i^{(j)}\}_{(i,j) \in \psi}$ which satisfies the functional! relation (2) and also the side conditions (3).

For sufficiently small step, the total variation, denoted by "TV" of discrete solution u^n does not increase, i.e., the following holds: $TV(u^{n+1}) \le TV(u^n)$, where;

$$TV(u^n) := \sum_j \left| u_{j+1}^n - u_j^n \right| \qquad \text{..........................(4)}$$

and the objective of the discretization is to maintain the stability (4) while achieving higher-order accuracy, perhaps with a modified restriction:

$$\Delta t \le c\Delta t_{FE} \qquad \text{...(5)}$$

called Courant-Friedrichs-Levy (CFL).

Stability Criteria
In [5], a general Runge-Kutta methods for (1) is written in the form:

$$u^{(0)} = u^n,$$
$$u^{n+1} = \sum_{k=0}^{i-1}\left(\alpha_{i,k}u^{(k)} + \Delta t\beta_{i,k}Lu^{(k)}\right), \alpha_{i,k} \geq 0,\ i = 1,\ldots,m$$
$$u^{n+1} = u^{(m)} \quad \text{......................(6)}$$

Clearly, if all the $\beta_{i,k}$'s are nonnegative, $\beta_{i,k} \geq 0$, then since by consistency $\sum \alpha_{i,k} = 1$, it follows that u^{n+1} is given by a convex combinations of forward Euler operators, with suitably scaled Δt's replaced by $\frac{\beta_{i,k}}{\alpha_{i,k}}\Delta t$. If some $\beta_{i,k}$'s are negative, we need more storage requirement. However, as shown in [4], it is not always possible to avoid negative $\beta_{i,k}$, therefore, we would like to overcome this problem.

We begin our stability study for Rimge-Kutta approximation of (1) with the order forward-Euler scheme (with <.,.> denoting the usual Euclidean inner product) which is equivalent to (2)

$u^{n+1} = u^n + \Delta t_n L(t^n)u^n$

based on variable lime-steps, $\sum_{j=0}^{n-1}\Delta t_j$ Taking L^2 norms on both sides one finds:

$$|u^{n+1}|^2 = |u^n|^2 + 2\Delta t_n \mathrm{Re}(L(t^n)u^n, u^n) + (\Delta t^n)|L(t^n)u^n|^2$$

and hence strong stability holds, $|u^{n+1}| \leq |u^n|$, provided the following restriction on the time state, Δt_n is met, $\Delta t_n \leq -2\mathrm{Re}(L(t^n)u^n, u^n)/|L(t^n)u_n|^2$.

Following Levy and Tadmor [l3], we therefore make the assumptions:

Assumption 1: There exists $\eta(t) > 0$ such that:

$$\eta(t) = \inf_{|u|=1} \frac{\mathrm{Re}(L(t)u, u)}{|L(t)u|^2} > 0 \quad \text{....................(7)}$$

we conclude that L's, the forward Euler scheme is strongly stable:

$\|I + \Delta t_n L(t^n)\| \leq 1$ iff $\Delta t_n \leq 2\eta(t^n)$

Our aim is to show that the general m-stage, m-th order accurate Runge-Kutta scheme is strongly stable under the same CFL condition.

Observe that the constant, n is an upper bound in the size of L; indeed, by Cauchy-Schwartz $\eta(t) \leq |L(t)u|\,|u| / |L(t)u|^2$, and hence:

$$\|L(t)\| = \sup_u \frac{|L(t)u|}{|u|} \leq \frac{1}{\eta(t)} \quad \text{.........................(8)}$$

Consider the fourth order Runge-Kutta approximation of

$$k^1 = L(t^n)u^n \quad \text{..(9)}$$

$$k^2 = L(t^{n+\frac{1}{2}})\left(u^n + \frac{\Delta t_n}{2}k^1\right) \quad \text{.................(10)}$$

$$k^3 = L(t^{n+\frac{1}{2}})\left(u^n + \frac{\Delta t_n}{2}k^2\right) \quad \text{................(11)}$$

$$k^4 = L(t^{n+1})\left(u^n + \Delta t_n k^3\right) \quad \text{...................(12)}$$

$$u^{n+1} = u^n + \frac{\Delta t_n}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{....(13)}$$

Starting with the second order and higher Runge-Kutta intermediate steps depend on the time variation of L(.), and hence we require minimal smoothness in time, making:

<u>**Assumption 2 (Lipschitz regularity):**</u> We assume that L(.) is Lipschitz. Thus, there exists a constant $k > 0$, such that:

$$\|L(t) - L(s)\| \leq \frac{k}{\eta(t)}|t - s| \quad \text{....................(14)}$$

The result along these lines was introduced by [6,main theorem], staling the strong stability of the constant coefficients s-order Runge-Kutta scheme under CFL condition $\Delta t \leq C_s\eta(t^n)$. We are now ready to make our main result, we are state the following:

<u>**Proposition:**</u>

Consider the systems of ODE's (6)-(7), with Lipschitz continuous coefficients (14). Then the fourth-order Rungc-Kutta scheme (9-13) is stable under CFL condition:

$$\Delta t_n \leq 2\eta(t^n) \quad \text{...(15)}$$

and the following estimate holds:

$$|u^n| \leq e^{3Kt_n}|u^0| \quad \text{....................................(16)}$$

Here we improve in both simplicity and generality.

Proof:

We proceed in two steps. We first freeze the coefficients at $t = t^n$, considering (here we abbreviate $L^n = L(t^n)$)

$$j^1 = L^n u^n \quad \text{...(17)}$$

$$j^2 = L^n\left(u^n \frac{\Delta t_n}{2} j^1\right)$$

$$\equiv L^n\left(I+\frac{\Delta t_n}{2}L^n\right)u^n \;\ldots\ldots\ldots\ldots\ldots (18)$$

$$j^3 = L^n\left(u^n \frac{\Delta t_n}{2} j^2\right)$$

$$\equiv L^n\left(I+\frac{\Delta t_n}{2}L^n\left(I+\frac{\Delta t_n}{2}L^n\right)\right)u^n \;\ldots\ldots (19)$$

$$j^4 = L^n(u^n + \Delta t_n j^3) \;\ldots\ldots\ldots\ldots\ldots (20)$$

$$v_{n+1} = u_n + \frac{\Delta t_n}{6}(j^1 + 2j^2 + 2j^3 + j^4)$$

Thus, $v^{n+1} = P_4(\Delta t_n L^n)u^n$, where following (5)

$$P_4(\Delta t_n L^n) = \frac{3}{8}I + \frac{1}{3}(I + \Delta L) + \frac{1}{4}(I + \Delta L)^2 + \frac{1}{24}(I + \Delta L)^4$$

Since the CFL condition (5) implies the strong stability of forward-Euler, i.e., $\|I + \Delta t_n L^n\| \le 1$, it follows that $\|P_4(\Delta t_n L^n)\| \le \frac{3}{8}+\frac{1}{3}+\frac{1}{4}+\frac{1}{24}=1$.

Thus:

$$|v^{n+1}| \le |u^n| \;\ldots\ldots\ldots\ldots\ldots (22)$$

Next, we turn to include the time dependence. We need to measure the difference between the exact and the 'frozen' intermediate values the k's and the j's. We have:

$$k^1 - j^1 = 0 \;\ldots\ldots\ldots\ldots\ldots (23)$$

$$k^2-j^2 = \left[L(t^{n+\frac{1}{2}})-L(t^n)\right]\left(I+\frac{\Delta t_n}{2}L^n\right)u^n \;.(24)$$

$$k^3-j^3 = L(t^{n+\frac{1}{2}})\frac{\Delta t_n}{2}(k^2-j^2)+\left[L(t^{n+\frac{1}{2}})-L(t^n)\right]\frac{\Delta t_n}{2}j^2 \;\ldots\ldots (25)$$

$$k^4 - j^4 = L(t^{n+1})\Delta t_n(k^3-j^3)+\left[L(t^{n+\frac{1}{2}})-L(t^n)\right]\Delta t_n j^3 \;\ldots\ldots (26)$$

Lipschitz continuity (14) and the strong stability of forward Euler imply:

$$|k^2 - j^2| \le \frac{K\Delta t_n}{2\eta(t^n)}|u^n| \le K|u^n| \;\ldots\ldots\ldots (27)$$

Also, since $\|L^n\| \frac{1}{\eta(t^n)}$. We find from (18) that $|j^2| \le |u^n|/\eta(t^n)$, and hence (25) followed by (27) and the CFL condition (15) imply:

$$|k^3 - j^3| \le \frac{\Delta t_n}{2\eta(t^n)}|k^2 - j^2| + \frac{K\Delta t_n}{2\eta(t^n)}\,\frac{\Delta t_n}{2\eta(t^n)}|u^n|$$

$$\le 2K\left(\frac{\Delta t_n}{2\eta(t^n)}\right)^2|u^n|$$

$$\le 2K|u^n| \;\ldots\ldots\ldots\ldots\ldots (28)$$

Finally, since by (19), j^3 does not exceed $|j3| \le \frac{1}{\eta(t^n)}\left(I+\frac{\Delta t_n}{2\eta(t^n)}L^n\right)|u^n|$, we find from (26) followed by (28) and the CFL condition (15):

$$|k^4 - j^4| \le \frac{\Delta t_n}{\eta(t^n)}|k^3 - j^3| + \frac{K\Delta t_n}{2\eta(t^n)}\left(I+\frac{\Delta t_n}{2\eta(t^n)}L^n\right)|u^n|$$

$$\le K\left(\left(\frac{\Delta t_n}{\eta(t^n)}\right)^3+\left(\frac{\Delta t_n}{\eta(t^n)}\right)^2\right)|u^n|$$

$$\le 12K|u^n| \;\ldots\ldots\ldots\ldots\ldots (29)$$

we conclude that:

$$u^{n+1} = v^{n+1} + \frac{\Delta t_n}{6}[2(k^2-j^2) + 2(k^3-j^3) + (k^4-j^4)]$$

is upper bounded by, consult (22), (270)-(29)

$$|u^{n+1}| \le |v^{n+1}| + \frac{\Delta t_n}{6}[2K|u^n| + 4K|u^n| + 12K|u^n|]$$

$$\le (1 + 3K\Delta t_n)|u^n|$$

and the results follows. ■

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