AN INTEGRAL EQUATION FOR ANALYSIS OF TWO DIMENTIONAL THIN STRUCTURE PROBLEM

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Abstract

The displacement of two-dimensional thin structure problem is formulated in terms of integral equations, and an efficient numerical method is developed based on boundary element method. Good results are obtained by studying simple test problem.

Introduction

The numerical analysis of thin structures, especially thin structures with ultrathin thickness has long been a difficult problem in history. Earlier methods for thin structures analysis are based on Finite Element Method (FEM) theories, in which, there are two disadvantages of such theories. First, their applications are limited to case of simple geometry. Second, the special formulation base on different kinds of assumptions may be only correct for some rang of thickness. The Boundary Element Method (BEM) is a numerical method to solve the boundary value problem by the Boundary Integral Equation (BIE) method. It‘s name appeared in the late seventies in an attempt to make analogy with (FEM). In the (BEM), only the boundary of the domain to be analyzed has to be discretised, in which (FEM) is implemented, and the subdivision of the whole domain is required. However, only linear elastic and homogeneous domains can be analyzed if the boundary discretization is used, non- linear problems can be analyzed by the (BEM) but require an additional discretization of the domain for the integrals evaluation(for more details, see [1-4])

An extensive searches effort has been made in (BIE) formulations of problems and their numerical solution schemes. Today, the BIE/BEM has gained a great deal of application in many fields such as optimization and inverse problems.

In this paper, we consider the problem of determining the displacement field throughout a finite plane domain. The formulation in terms of integral equations and new numerical approach using BEM for solving such problem is presented.

The Problem Formulation

The potential problem was first formulated in terms of a direct (BIE) and solved numerically thirty years ago. We are introducing here, the in the function

$\Omega = x\Phi + \Psi$

Where Φ and Ψ are harmonic functions. For more details see [6].

In this paper, we are presenting a different formulation slightly from that proposed in [6], in which the dangers of an ill-conditioning could happened, so the logarithmic potential representation is ascribed to $\partial\Phi/\partial y$ rather than to $\partial\psi/\partial y$, avoiding the dangers of ill-conditioning which are characteristic of non-singular Fredholm integral equations of the first kind.

We introduce the simple-layer logarithmic potential representation

$$\Phi(\rho) = \int_{\delta B} \log\int \rho - q \int \sigma(q) dq \;\ldots\ldots\ldots\ldots\ldots (1)$$

for Φ at a point p, in $B + \partial B$, together with a similar representation

$$\frac{\partial\psi(P)}{\partial y} = \int_{\partial B} \log / p - q / s(q) dq \;\ldots\ldots\ldots\ldots\ldots (2)$$

for $\partial\psi/\partial y$, here q represent a point on the boundary ∂B, dq is the differential increment of the boundary at q, and are simple source densities. The corresponding conjugate harmonic functions (see [6] and [7]) are:

$$\Phi(p) = \int_{\partial B} O / P - q / \sigma(q) dq \;\ldots\ldots\ldots\ldots\ldots (3)$$

and

$$\frac{\partial\psi(P)}{\partial y} = \int_{\partial B} \log / p - q / s(q) dq \;\ldots\ldots\ldots\ldots\ldots (4)$$

where $\theta(p-q)$ is the angle between the vector $p-q$ an some fixed direction, e.g. the x-axis. The derivative of with respect to x, in the domain B, is:

$$\frac{\partial\psi(p)}{\partial x} = \int_{\partial B} \log_x |p-q|\,\sigma(q)\sigma(p) \;\ldots\ldots\ldots(5)$$

where $\log_x |p-q|$ denotes the x-derivative of $\log|p-q|$ at the point p. On the boundary ∂B, assumed to be smooth at p, this becomes:

$$\frac{\partial\Phi(p)}{\partial y} = \int_{\partial B} \log_y |p-q|\,\sigma(q)dq + ny^t(p)\sigma(p) \;\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where the prime denotes differentiation along the normal to ∂B directed into B.

Similarly, we have

$$\frac{\partial\Phi(p)}{\partial y} = \int_{\partial B} \log_y |p-q|\,\sigma(q)\,dq \;\ldots\ldots\ldots(7)$$

at a point p in B, and

$$\frac{\partial\Phi(p)}{\partial y} = \int_{\partial B} \log_y |p-q|\,\sigma(q)dq + ny^t(p)\sigma(p) \;\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

when p lies on ∂B, provided that θ is declined in an appropriable manner [6] and [8].

The representations (3) and (4) are continuous in $B+\partial B$, as are the logarithmic potentials (1) and (2). Therefore, given u, v as the displacement component in the directions of Cartesian coordinate axes x, y on ∂B, is represented in terms of the Airy stress function Ω, see [5]:

$$2\mu u = (1-r)H - \frac{\partial\Omega}{\partial x} \;\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$2\mu v = (1-r)\bar{H} - \frac{\partial\Omega}{\partial y}$$

where H and $\bar{H}$ are conjugate harmonic functions such that

$$\frac{\partial H}{\partial x} = \frac{\partial \bar{H}}{\partial y} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

and they correspond respectively to the real and imaginary parts of an analytic function, when the equality in (10) is simply one of the Cauchy-Riemann equations (the other being $\frac{\partial H}{\partial y} = \frac{\partial \bar{H}}{\partial x}$).

Combine the expressions (1)-(4) with (6)-(8) yields a pair of coupled integral equations for the two source densities σ and ς.

$$2\mu = (I-2r)\int_{\partial B} \log|p-q|\,\sigma(q)dq - x\int_{\partial B} \log_x |p-q|\,\sigma(q)dq + nx'(p)\sigma(p) - \int_{\partial B} \theta|p-q|\,s(q)dq + \alpha \;\ldots\ldots\ldots\ldots\ldots(11)$$

$$2\mu v = 2(I-r)\int_{\partial B} O|p-q|\,\sigma(q)dq - x\int_{\partial B} \log_y |p-q|\,\sigma(q)dq + ny(p)\sigma(p) - \int_{\partial B} \log|p-q|\,s(q)dq + \beta$$

where a and are unknown constants, defined as a = (1 − v)b and B = (1 − v)c, with b and c are arbitrary constants.

By solving equations (11), for $\sigma, \varsigma, \alpha$ and β, we may compute Φ, $\frac{\partial\Phi}{\partial y}$, etc.,

From (1), (3), etc., and hence obtain u and y at any point p in B.

The Solution Approach

In general, the analytical solution to the boundary integral equations (11) of complicated shape is very difficult to be obtain and numerical method which reduces

The integral equation to linear algebraic equations has to be used. In order to solve the integral equation numerically, the boundary will be discretized into a series of elements over which displacements are written in terms of their values at a series of points. Writing the discretized form of (11) for every point, a system of linear algebraic equations is obtained. Once the boundary condition is applied, the system can be solved to obtain all the unknown values and consequently an approximate solution to the boundary values is obtained. Also, it is ell known in the BIE/BEM literatures, will degenerate when it is applied to cracks or thin voids in structures because of the closeness of the two crack surfaces [5] and [8].

In order to implement our formulation numerically, we divide the boundary into n intervals, not necessarily equation in each of the two crack surfaces [5] and [8].

In order to intervals, not necessarily equation in each of which we approximate the source densities and by constants. Denoting these constants by and n, we thus approximate $\Phi(\rho)$ by:

$$\Phi(\rho) = \int_{\delta B} \log|p-q|\sigma(q)dq \ ....(12)$$

where $\int_{\delta B}$ indicates integration over the interval. Similarly we approximate:

$$\frac{\partial\psi(P)}{\partial y} = \int_{\partial B} \log/p-q/s(q)dq \ ....(13)$$

and $\varphi(p)$ by:

$$\bar{\Phi}(p) = \sum_{j-1}^{n} \sigma_j \int_j O(p-q)dq \ ....(14)$$

and

$$\frac{\partial\psi(p)}{\partial x} = \sum_{j-1}^{n} \xi_j \int_j \theta(p-q)dq \ ....(15)$$

when P lies in B, we approximate

$$\frac{\partial\Phi(p)}{\partial x} = \sum_{j=1}^{n} \sigma_j \int_j \log|p-q|dq \ ....(16)$$

Also, on the boundary ∂B, at a point q in the i-th interval, we approximate similarly.

$$\frac{\partial\Phi(q_1)}{\partial x} = \sum_{j-1}^{n} \sigma_j \int_j \log_x|q_1-q|dq + nx(q_1)\sigma_1 \ ....(17)$$

With these approximations, equations (11), applied at one "nodal" point q in each interval, i = 1, 2, ..., n, yield 2n simultaneous linear equations in the 2n + 2 unknowns σ_j and j = 1,2,…,n, and a and B. Two further equations can be obtained by discretizing given conditions to have

$$\sum_{j-1}^{n} \sigma_j / h_j = O \text{ and } \sum_{j-1}^{n} \xi_j h_j = O \ .....(18)$$

where h is the length of the jth interval of. Substitution of the solution of the resuling system of equations back in to the discrete form of equations (11) yields approximations and to the displacement components u and y at any point p in B.

Numerical Verification

To verify the method developed above, a program is developed, and simple test problem is studied in which our approximated solution compared with the exact FEM solutions.

A thin plate under pressure show a in the figure below is studied, see [5]. We assume the length of the plate in z-direction is iarge so that the problem can be simplified as a plate strain problem, with the constant length in x-direction, while the thickness is slightly change from position to another. However, it is of more interest here to verify the validity and effectiveness of the developed BEM approach for such two-dimensional thin structures. The boundary of the plate is discretized with only four-quadratic boundary elements with given constant length, and the other two

Element with thickness varying between upper and lower bound which are known.

On node 1; 3 displacement in y direction is constrained. on node 2, displacement component in both x, y direction are constrained. The displacement of node 5 is in x direction. The displacement result deteriorate quickly as the thickness decreases. whereas the stress component result are even more accurate, in which, almost reproducing the exact values for all range thickness values. The prove that the developed analytical work in the BEM procedure is effective. The finite element analysis of this simple problem was also attempted, but it was soon found out that the number of the two-dimensional finite element were so large that the task quickly exceed the capacity of the computer used.

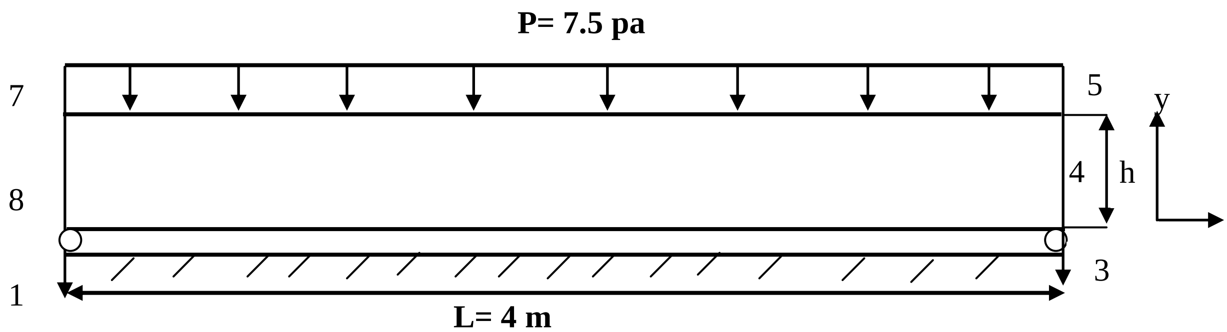

Fig. (1): Thin plate under constant pressure p (2-D plain strain model, shear modulus Pa, Poisson's ratio y = 0.2).

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