Journal of Al-Nahrain University Vol.11(2), August, 2008, pp.136-144 Science

EXISTENCE AND UNIQUENESS FOR A CLASS OF SEMILINEAR INITIAL VALUE CONTROL PROBLEMS IN SUITABLE BANACH SPACES

Manaf A. Salah, Ahmed A. Yousif and Ranen Z. Ahmood
Al-Nahrain University, College of Science, Department of Mathematics & Computer Applications.

Abstract

The local existence and uniqueness of S-classical solution (semi-classical solution) for a class of semilinear initial value control problems in suitable Banach spaces have been discussed and proved. The theoretical results are depending on the theory of analytic semigroup and Banach contraction principle.

Keywords: S-classical solution (semi-classical solution), control problem in infinite dimensional spaces, fixed point theorem and analytic semigroup theory.

Introduction

Byszewski in 1991 [1], has study the local existence and uniqueness of the mild solution to the semilinear initial value problem:

$$\frac{dv}{dt} + Av(t) = f(t, v(t)) \quad \text{.................................(1)}$$
$$v(0) = v_0$$

Where A is the infinitesimal generator of a C_0 semigroup (strongly continuous semigroup) defined from $D(A) \subset X$ into X (X is suitable Banach space) and f is a nonlinear continuous map define from $[0,r)\times X$ into X. Eduardo [2] in 2001, has study the local existence and uniqueness of the of S-classical solution (Semi-classical Solution) to the problem defined in (1).

Definition :

A function $x \in C([0, r):X)$ is said to be an S-classical solution (semi-classical solution) to the semilinear initial value problem defined in (1), if $x(t)$ has the following form:

$$x(t)=T(t)x_0+\int_0^t T(t-s)f(s,x(s))\,ds \quad \text{...............(2)}$$

Satisfies the following conditions: $x(0)=x_0$, $\frac{d}{dt}x(t)$ is continuous on (0, r), $x(t) \in D(A)$ for all $t \in (0,r)$ and x(.) satisfies equation (1) on (0, r).

Manaf [3] in 2005, has study the local existence and uniqueness of the mild solution to the semilinear initial value control problem:

$$\left.\begin{array}{l} \frac{dx}{dt} + Ax(t) = f(t,x(t)) + \\ \int_{s=0}^{t} h(t-s)\,g(s,x(s))\,ds + (Bw)(t),\ t>0 \\ x(0)=x_0 \end{array}\right\} \quad \text{...............................(3)}$$

where A is the infinitesimal generator of a C_0 semigroup defined from $D(A)\subset X$ into X and f and g are a nonlinear continuous maps defined from $[0,r)\times X$ into X, h is the real valued continuous function defined from [0,r) into R where R is the real number and B is a bounded linear operator define from O into X. Where O is a Banach space and *w*(.) be the arbitrary control function is given in $L^p([0, r):O)$, a Banach space of control functions with $\|w(t)\|_O \le k_1$, for $0 \le t < r$.

Definition :

A continuous function x_w is said to be a mild solution to the semilinear initial value problem defined in (3) given by:

$$x_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[(Bw)(s) + f(s,x_w(s)) + \int_{s=0}^{t} h(s-\tau)\,g(s,x_w(\tau))\,d\tau\right]ds$$
$$\forall w \in L^p([0,r):O)$$

In the present paper, the S-classical solution (semi-classical solution) of the semilinear initial value control problem defined in (3) will be developed by the following definition:

Definition :

A function $v \in C([0,r):X)$ is said to be an S-classical solution (semi-classical solution) to the semilinear initial value problem defined in (3), if $v(t)$ has the following form:

$$v_w(t) = T(t)u_0 + \int_{s=0}^{t} T(t-s)\left[(Bw)(s) + f(s, v_w(s)) + \int_{s=0}^{t} h(s-\tau)g(s, v_w(\tau))\,d\tau\right]ds$$

$\forall w\ L^p([0,r):O)$

Satisfies the following conditions: $v_w(0) = x_o$, $\frac{d}{dt}v_w(t)$ is continuous on (0,r), $v_w(t) \in D(A)$ for all t $\in$(0,r) and $v_w(.)$ satisfies equation (3) on (0, r).

Throughout this paper X will be a Banach space equipped with the norm $\|.\|$ and the operator $A: D(A) \subset X \to X$ will be the infinitesimal generator of an analytic semigroup of bounded linear operators $\{T(t)\}_{t\ge 0}$ on X. For the theory of analytic semigroup, refer to Pazy [4], Jerome [5] and Balachandran [6]. The books of Pazy [4], Krien [7] and Fitzgibbon [8] contained therein, give a good account of important results. We mention here only some notation and properties essential to our purpose, In particular, we assume that $\{T(t)\}_{t\ge 0}$ is an analytic semigroup generated by infinitesimal generator A and 0$\in\rho$(A), (ρ(A) stands for resolvent set).In this case it is possible to define the fractional power $(-A)^{\alpha}$, for 0<α<1, as a closed linear operator with domain $D((-A)^{\alpha})$ is dense in X and the expression $\|x\|_{\alpha} = \|(-A)^{\alpha}x\|_X$,

Defines a norm on $D((-A)^{\alpha})$.Hereafter we represent by X_{α} the space $D((-A)^{\alpha})$ endowed with the norm $\|.\|_{\alpha}$.

Preliminaries

Definition :

A family $\{T(t)\}_{t\ge 0}$ of bounded linear operators on a Banach space X is called a semigroup on X if it satisfies the following conditions:

T(t+s)=T(t)T(s), $\forall t,s \ge 0$ T(0)=I ,(T(0) is the identity operator on X).

Definition :

A family $\{T(t)\}_{t\ge 0}$ is said to be an analytic semigroup if the following conditions are satisfy:

(i) $t \to T(t)$ is analytic in some sector Δ,where Δ is a sector containing the nonnegative real axis.

(ii) T(0)=I and $\lim_{t\downarrow 0}\|T(t)x - x\|_X = 0, \forall x \in X.$

(iii) T(t+s)=T(t)T(s), $\forall t,s \ge 0$.

Definition :

A semigroup $\{T(t)\}_{t\ge 0}$ on a Banach space X is called strongly continuous semigroup of bounded linear operators or (C_0 semigroup) if

The map $R^+ \ni t \longrightarrow T(t) \in L(X)$, satisfies the following conditions:

(i) $T(t + s) = T(t)T(s),\ \forall t,s \ge 0$.

(ii) T(0) = I.

(iii) $\lim_{t\downarrow 0}\| T(t)x - x \| = 0$, for every x$\in$X.

Definition :

If $-A$ is the infinitesimal generator of bounded analytic semigroup then the fractional power $A^{-\alpha}$ exist for α>0.

Definition :

Let $-A$ be the infinitesimal generator of an analytic semigroup T(t) if 0 $\in \rho$(A), then:

(a) T(x): X $\longrightarrow$ D(A^{α}), for every t > 0 and $\alpha \ge 0$.

(b) For every x$\in$D(A^{α}), we have $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.

(c) For every t $\ge$ 0, the operator $A^{\alpha}T(t)$ is bounded and $\|A^{\alpha}T(t)\| \le M_{\alpha}t^{-\alpha}$.

(e) Let 0<$\alpha\le$1 and x$\in$D(A^{α}) then $\|T(t)x - x\| \le C_{\alpha}t^{\alpha}\|A^{\alpha}x\|_X$,where C_{α} is the positive constant depend onα.

Definition :

Suppose X is a Banach space. A mapping T: X⟶X is said to be strict contraction, with strict contraction constant L, if $\|Tx - Ty\|_X \le L\|x - y\|_x$, $\forall$ x, y$\in$ X, where 0< L< 1.

Definition Banach contraction principle:

Let M is a closed nonempty set in the Banach space X over k, where k are a scalar field and the operator T: M ⟶ M is strict contraction operator then T has a unique fixed point.

Definition :

Let I be an interval, A function f: I⟶X, where X is a Banach space is said to be Hölder continuous with exponent ϑ, 0 < ϑ < 1on I, if there is a constant L such that $\|f(t) - f(s)\|_X \le L|t-s|^{\vartheta}$, for s, t $\in$ I.

Main Result

It should be notice that the local existence and uniqueness of S-classical solution (Semi-classical Solution) of the semilinear initial value control problem defined in (3) developed, by assuming the following assumptions:

A. $-A$ be the infinitesimal generator of bounded analytic semigroup $\{S(t)\}_{t\ge 0}$ and $0 \in \rho(-A)$.where the operator $-A$ define from $D(-A)\subset X$ into X, (X is a Banach space).

B. Let U be an open subset of $[0, r)\times X_\alpha$, for 0< r≤∞.Where X_α is a Banach space being dense in X.

C. For every $(t, x)\in U$, there exists a neighborhood $G \subset U$ of (t, x),the nonlinear maps f, g:$[0,r)\times X_\alpha$⟶X satisfy the locally Lipchitize condition with respect to second argument,

$\|f(t, u) - f(s, v)\|_x \le L_0\|v - u\|_\alpha$

$\|g(t, u) - g(s, v)\|_x \le L_1\|v - u\|_\alpha$,

for all (t, u) and (s, v) $\in$G.

D. For $t''>0$, $\|f(t,v)\|_x \le B_1$, $\|g(t,v)\|_x \le B_2$, for $0\le t\le t''$ and for every $v\in X_\alpha$.

E. For $t''' > 0$, $\|S(t) - I\|\,\|A^\alpha u_0\| \le \delta'$, Where $\delta' < \delta$, $0 \le t \le t'''$.

F. h is continuous function which at least $h\in L^1([0,r):R)$,Where R is the real number.

G. $w(.)$ be the arbitrary control function is given in $L^p([0,r):O)$, a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X and $\|w(t)\|_O \le k_1$, for $0 \le t < r$.

H. Let $t_1>0$ such that $t_1=\min\{t', t'', t''', r\}$,satisfy the condition

(H.i) $$t_1 \le \{[K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1} C_\alpha^{-1}(1-\alpha)\,(\delta\text{-}\delta')\}^{\frac{1}{1-\alpha}}$$

$$\Rightarrow t_1^{1-\alpha} \le [K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1} C_\alpha^{-1}(1-\alpha)\,(\delta\text{-}\delta').$$

I. There exist $C_2\ge 0$ and $0<\vartheta\le 1$ such that: $|h(t) - h(s)| \le C_3|t-s|^{\vartheta}$, for all t, s $\in[0, t_1]$.

J. There exist $R_0\ge 0$ and $0<\xi\le 1$ such that: $\|w(t) - w(s)\|_O \le R_0|t-s|^{\xi}$, for all t, s $\in[0, t_1]$.

Main Theorem:

Assume that hypotheses (A)-(J) are hold, then for every $v_0\in X_\alpha$, there exists a fixed number t_1, $0< t_1< r$, such that the initial value control problem defined in (3) has a unique S-classical solution $v_w \in C([0,t_1]:X)$, for every control function $w(.) \in L^p([0,t_1]:O)$.

Proof :

Without loss of generality, we may suppose r<∞, because we are concerned here with the local existence only.

For a fixed point $(0,v_0)$ in the open subset U of $[0,r)\times X_\alpha$, we choose δ>0 such that the neighborhood G of the point $(0,v_0)$ define as follow:

$G=\{(t, x)\in U: 0\le t\le t', \|v - v_0\|_\alpha \le \delta\}\subset U$ {since U is an open subset of $[0,r)\times X_\alpha$}.

It is clear that $\|A^\alpha S(t)\|\le C_\alpha t^{-\alpha}$, for t>0, see {theorem (1.8.7) in [4]}.

Where C_α is a positive constant depending on α and assume

$h_r = \int_0^r |h(s)|ds$.

Set Y= C([0,t$_1$]:X), then Y is a Banach space with the supremum norm: $\|y\|_Y = \sup_{0\le t\le t_1} \|y(t)\|_X$.

Let S_w be the nonempty subset of Y, define as follow:

$$S_w = \{x_w \in Y : x_w(0) = A^{\alpha}x_0,\ \|x_w(t) - A^{\alpha}x_0\|_X \le \delta,\ 0 \le t \le t_1\} \quad ....(4)$$

To prove the closedness of S_w as a subset of Y. Let $x_w^n \in S_w$, such that

$x_w^n \xrightarrow{P.C.} x_w$ as $n \longrightarrow \infty$, we must prove that $x_w \in S_w$, where (P.C) stands for point wise convergence.

Since $x_w^n \in S_w \Rightarrow x_w^n \in Y, x_w^n(0) = A^{\alpha}x_0$ and $\|x_w^n(t) - A^{\alpha}x_0\|_X \le \delta, 0 \le t \le t_1$.

Since $x_w^n \xrightarrow{U.C.} x_w$, hence $x_w \in Y$. where (U.C) stands for the uniform convergence, and also since $x_w^n \xrightarrow{U.C.} x_w$ $\Rightarrow \|x_w^n - x_w\|_Y \longrightarrow 0$, as $n \longrightarrow \infty$

$\|x_w^n - x_w\|_Y = \sup_{0\le t\le t_1} \left\|x_w^n(t) - x_w(t)\right\|_X \longrightarrow 0$,as $n \longrightarrow \infty$, {By $\|y\|_Y = \sup_{0\le t\le t_1} \|y(t)\|_X$}.

Implies that $\|x_w^n(t) - x_w(t)\|_X \longrightarrow 0$,as $n \longrightarrow \infty$, for every $0 \le t \le t_1$,

i.e., $\lim_{n\to\infty} x_w^n(t) = x_w(t),\ \forall\ 0 \le t \le t_1$ .... (5)

$\Rightarrow \lim_{n\to\infty} x_w^n(0) = x_w(0)$ {by (5)}

$\Rightarrow \lim_{n\to\infty} A^{\alpha}x_0 = x_w(0)$ {since $x_w^n \in S$}

$\Rightarrow A^{\alpha}x_0 = x_w(0)$

Notice that:

$\|x_w(t) - A^{\alpha} x_0\|_X = \|\lim_{n\to\infty} x_w^n(t) - A^{\alpha} x_0\|_X$, {by (5)}

$= \|\lim_{n\to\infty} x_w^n(t) - \lim_{n\to\infty} A^{\alpha} x_0\|_X = \|\lim_{n\to\infty} (x_w^n(t) - A^{\alpha} x_0)\|_X = \lim_{n\to\infty} \|x_w^n(t) - A^{\alpha} x_0\|_X$

$\Rightarrow \|x_w(t) - A^{\alpha} x_0\|_X \le \lim_{n\to\infty} \delta$

{since $x_w^n \in S_w$}

$\Rightarrow \|x_w(t) - A^{\alpha} x_0\|_X \le \delta$, for $0 \le t \le t_1$.

We have got S_w is closed subset of Y.

Now, define a map F_w: $S_w \longrightarrow Y$, given by:

$$(F_w x_w)(t) = S(t)A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}x_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}x_w(\tau))d\tau\right]ds + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)ds$$

...............................(6)

To show that $F_w(S_w) \subseteq S_w$, let x_w be arbitrary element in S_w and let $F_w x_w \in F_w(S_w)$.

To prove $F_w x_w \in S_w$ for arbitrary element x_w in S_w. From (4), notice that $F_w x_w \in Y$ {by the definition of the map F_w} And $(F_w x_w)(0) = A^{\alpha}x_0$ {by (6)}.

Now we have got

$$\left\|(F_w x_w)(t) = A^{\alpha}x_0\right\|_X =$$

$$\left\|S(t)A^{\alpha}x_0 - A^{\alpha}u_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}x_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}x_w(\tau))d\tau\right]ds\right\|_X$$

$\|(F_w x_w)(t) - A^{\alpha}x_0\|_X =$

$$\|S(t)A^{\alpha}x_0 - A^{\alpha}x_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s, A^{-\alpha}x_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau, A^{-\alpha}x_w(\tau))d\tau\right]ds +$$

$$\int_{s=0}^{t} A^{\alpha}S(t-s)f(s,x_0)\,ds - \int_{s=0}^{t} A^{\alpha}S(t-s)f(s,x_0)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left(\int_{s=0}^{t} h(s-\tau)g(\tau,x_0)\,d\tau\right)ds - \int_{s=0}^{t} A^{\alpha}S(t-s)\left(\int_{s=0}^{t} h(s-\tau)g(\tau,x_0)\,d\tau\right)ds \,\|_x$$

$$\|(F_w x_w)(t) - A^{\alpha}x_0\| = \|S(t)A^{\alpha}x_0 - A^{\alpha}x_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}x_w(s)) - f(s,x_0)\right]ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left(\int_{s=0}^{t} h(s-\tau)\left[g(\tau,A^{-\alpha}x_w(\tau)) - g(\tau,x_0)\right]d\tau\right)ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left(f(s,x_0) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,x_0)\,d\tau\right)ds\right\|_x$$

$$\|(F_w x_w)(t) - A^{\alpha}x_0\|_x \le \|S(t)A^{\alpha}x_0 - A^{\alpha}x_0\|_x + \int_{s=0}^{t} \|A^{\alpha}S(t-s)\|\,\|(Bw)(s)\|\,ds + \int_{s=0}^{t} \|A^{\alpha}S(t-s)\|\,\|f(s,A^{-\alpha}x_w(s)) - f(s,x_0)\|\,ds + \int_{s=0}^{t} \|A^{\alpha}S(t-s)\|\left(\int_{s=0}^{t} |h(s-\tau)|\,\|g(\tau,A^{-\alpha}x_w(\tau)) - g(\tau,x_0)\|\,d\tau\right)ds + \int_{s=0}^{t} \|A^{\alpha}S(t-s)\| \cdot \left(\|f(s,x_0)\| + \int_{\tau=0}^{s} |h(s-\tau)|\,\|g(\tau,x_0)\|\,d\tau\right)ds$$

After a series of simplifications and using the conditions C, D and E we have got:

$$\|(F_w x_w)(t) - A^{\alpha}x_0\|_x \le \delta' + \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha}K_0K_1\,ds + \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha}L_0\|A^{-\alpha}x_w(s) - x_0\|_{\alpha}\,ds + \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha}h_{t_1}L_0\|A^{-\alpha}x_w(s) - x_0\|_{\alpha}\,ds + \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha}[B_1 + h_{t_1}B_2]\,ds$$

By using the properties $\|x\|_{\alpha} = \left\|A^{\alpha}x\right\|_X$, we get:

$$\|(F_w x_w)(t) - A^{\alpha}x_0\|_x \le \delta' + {}_{\alpha}K_0K_1(1-\alpha)^{-1}t_1^{1-\alpha} + \delta C_{\alpha}L_0(1-\alpha)^{-1}t_1^{1-\alpha} + \delta C_{\alpha}L_1 h_{t_1}(1-\alpha)^{-1}t_1^{1-\alpha} + (B_1 + h_{t_1}B_2)C_{\alpha}(1-\alpha)^{-1}t_1^{1-\alpha}$$

$$\|(F_w x_w)(t) - A^{\alpha}x_0\|_x \le \delta' + [K_0K_1 + \delta L_0 + \delta L_1 h_{t_1} + (B_1 + h_{t_1}B_2)]\,C_{\alpha}(1-\alpha)^{-1}t_1^{1-\alpha}$$

$$\|(F_w x_w)(t) - A^{\alpha}x_0\|_x \le \delta' + [K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_0 + B_2)]C_{\alpha}(1-\alpha)^{-1}t_1^{1-\alpha}$$

By using condition (H.i), we get: $\|(F_w x_w)(t) - A^{\alpha}x_0\|_x \le \delta$, for $0 \le t \le t_1$.

So one can select $t_1>0$, such that:

$$t_1 = \min\{t', t'', t''', r, [\{K_0K_1 + (\delta L_0 + B_1) + h_{t_1}(\delta L_1 + B_2)]^{-1}C_{\alpha}^{-1}(1-\alpha)(\delta-\delta')\}^{\frac{1}{1-\alpha}}\}$$

Thus, we have that $F_w: S_w \longrightarrow S_w$

Now, to show that F_w is a strict contraction on S_w, this will ensure the existence of a unique S-classical solution to the semilinear initial value control problem. *

Let $x'_w, x''_w \in S_w$, then:

$$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_x = \left\|S(t)A^{\alpha}x_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}x''_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau,A^{-\alpha}x''_w(\tau))\,d\tau\right]ds - S(t)A^{\alpha}x_0 - \int_{s=0}^{t} A^{\alpha}S(t-s)(Bw)(s)\,ds - \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}x'_w(s)) + \int_{s=0}^{t} h(s-\tau)g(\tau,A^{-\alpha}x'_w(\tau))\,d\tau\right]ds\right\|_x$$

$$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_x =$$

$$\left\| \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}x''_{w}(s))+\int_{s=0}^{t} h(s-\tau)g(\tau,A^{-\alpha}x''_{w}(\tau))\,d\tau\right]ds - \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}x'_{w}(s))+\int_{s=0}^{t} h(s-\tau)g(\tau,A^{-\alpha}x'_{w}(\tau))\,d\tau\right]ds\right\|_{X}$$

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq$

$$\int_{s=0}^{t} \|A^{\alpha}S(t-s)\|\,\|f(s,A^{-\alpha}x''_{w}(s))-f(s,A^{-\alpha}x'_{w}(s))\|_X\,ds + \int_{s=0}^{t}\|A^{\alpha}S(t-s)\|\left[\int_{s=0}^{t}|h(s-\tau)|\left\|g(\tau,A^{-\alpha}x''_{w}(\tau))-g(\tau,A^{-\alpha}x'_{w}(\tau))\right\|d\tau\right]ds$$

By using the condition C and the properties $\|A^{\alpha}S(t)\| \leq C_{\alpha}t^{-\alpha}$ with $h_r = \int_0^r |h(s)|\,ds$,

We have got:

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq$

$$\int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha} L_0 \|A^{-\alpha}x''_{w}(s)-A^{-\alpha}x'_{w}(s)\|ds + \int_{s=0}^{t} C_{\alpha}(t-s)^{-\alpha} h_{t_1}L_1\|A^{-\alpha}x''_{w}(\tau)-A^{-\alpha}x'_{w}(\tau)\|ds$$

$\|(F_w x''_w)(t)-(F_w x'_w)(t)\|_X \leq C_{\alpha}(1-\alpha)^{-1}L_0 \|x''_w(s)-x'_w(s)\|_X\, t_1^{1-\alpha} + C_{\alpha}(1-\alpha)^{-1} h_{t_1} L_0 \|x''_w(\tau)-x'_w(\tau)\|_X\, t_1^{1-\alpha}$

$\|(F_w x''_w)(t)-(F_w x'_w)(t)\|_X \leq C_{\alpha}(1-\alpha)^{-1}L_0 \sup_{0\leq t\leq t_1}\|x''_w(t)-x'_w(t)\|_X\, t_1^{1-\alpha} + C_{\alpha}(1-\alpha)^{-1} h_{t_1} L_1 \sup_{0\leq t\leq t_1}\|x''_w(t)-x'_w(t)\|_X\, t_1^{1-\alpha}$

$\|(F_w x''_w)(t)-(F_w x'_w)(t)\|_X \leq C_{\alpha}(1-\alpha)^{-1}L_0 \|x''_w - x'_w\|_Y\, t_1^{1-\alpha} + C_{\alpha}(1-\alpha)^{-1} h_{t_1}L_1 \|x''_w - x'_w\|_Y\, t_1^{1-\alpha}$

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq (L_0 + h_{t_1}L_1)\, C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha} \|x''_w - x'_w\|_Y$

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq \frac{1}{\delta}\delta(L_0 + L_1 h_{t1})\, C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha} \|x''_w - x'_w\|_Y$

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq \frac{1}{\delta}[\delta L_0 + \delta h_{t1}L_1]\, C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha} \|x''_w - x'_w\|_Y$

$\|(F_w x''_w)(t)-(F_w x'_w)(t)\|_X \leq \frac{1}{\delta}[\delta L_0 + \delta h_{t1}L_1 + K_0K_1 + B_1 + h_{t1}B_2]\, C_{\alpha}(1-\alpha)^{-1} t_1^{1-\alpha} \|x''_w - x'_w\|_Y$

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq \frac{1}{\delta}[K_0K_1 + (\delta L_0 + B_1) + h_{t1}(\delta L_1 + B_2)]\, C_{\alpha}(1-\alpha)^{-1} \|x''_w - x'_w\|_Y\, t_1^{1-\alpha}$

$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq \frac{1}{\delta}[K_0K_1 + (\delta L_0 + B_1) + h_{t1}(\delta L_1 + B_2)]\, C_{\alpha}(1-\alpha)^{-1}[K_0K_1 + (\delta L_0 + B_1) + h_{t1}(\delta L_1 + B_2)]^{-1} C_{\alpha}^{-1}(1-\alpha)(\delta-\delta') \|x''_w - x'_w\|_Y$ {by the condition (H.i)}.

$$\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq \left(1-\frac{\delta'}{\delta}\right)\|x''_w - x'_w\|_Y \qquad ....(7)$$

Taking the supremum over $[0, t_1]$ of both sides to (7), we have got:

$\sup_{0\leq t\leq t_1}\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_X \leq \left(1-\frac{\delta'}{\delta}\right)\|x''_w - x'_w\|_Y$

$\|F_w x''_w - F_w x'_w\|_X \leq \left(1-\frac{\delta'}{\delta}\right)\|x''_w - x'_w\|_Y$, {by $\|y\|_Y = \sup_{0\leq t\leq t_1}\|y(t)\|_X$}

Thus, F_w is a strict contraction map from S_w into S_w and therefore by the Banach contraction principle there exist a unique fixed point x_w of F_w in S_w, i.e., there is a unique $x_w \in S_w$, such that $F_w x_w = x_w$. This fixed point satisfies the integral equation:

$$x_w(t) = S(t)A^{\alpha}x_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[f(s,A^{-\alpha}x_w(s)) + \int_{\tau=0}^{s} h(s-\tau)g(\tau,A^{-\alpha}x_w(\tau))d\tau\right]ds + \int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)ds, \text{ for } 0 \leq t \leq t_1,\ \forall\, w(.) \in L^p((0,t_1):O) \qquad ....(8)$$

For simplification, we set $\tilde{f}(t) = f(t, A^{-\alpha}x_w(t))$, $\tilde{g}(t) = g(t, A^{-\alpha}x_w(t))$. Then equation (8) can be rewritten as:

$$x_w(t)=S(t)A^{\alpha}x_0 + \int_{s=0}^{t} A^{\alpha}S(t-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau))d\tau\right]ds + \int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)ds, \text{ for } 0 \le t \le t_1,$$

$$\forall w(.) \in L^p((0,t_1):O) \quad \ldots\ldots (9)$$

To show that $\tilde{f}(t)$ is locally Hölder continuous on $(0, t_1]$.

We first show that $x_w(t)$ given by (9) is locally Hölder continuous on $(0, t_1]$.

Notice that, from the theorem (IV.7) in [4], it follows that for every $0 < \beta < 1 - \alpha$ and every $0 < h < 1$, we have:

$$\|(S(h) - I)A^{\alpha}S(t-s)\| \le C_{\beta}\, h^{\beta}\, \|A^{\alpha+\beta}S(t-s)\| \le Ch^{\beta}(t-s)^{-(\alpha+\beta)} \quad \ldots\ldots (10)$$

Which is useful for proving $x_w(t)$ given by (9) is locally Hölder continuous on $(0, t_1]$.

Next, we have for $0 < t < t + h \le t_1$

$$\|x_w(t+h)-x_w(t)\|_x = \|S(t+h)A^{\alpha}x_0 + \int_{s=0}^{t+h} A^{\alpha}S(t+h-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau\right]ds + \int_{s=0}^{t+h} A^{\alpha}S(t+h-s)Bw(s)ds - S(t)A^{\alpha}x_0 - \int_{s=0}^{t+h} A^{\alpha}S(t-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau\right]ds - \int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)ds\,\|_x$$

$$\|x_w(t+h)-x_w(t)\|_x = \|S(t+h)A^{\alpha}x_0 - S(t)A^{\alpha}x_0 + \int_{s=0}^{t} A^{\alpha}S(t+h-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau\right]ds + \int_{s=t}^{t+h} A^{\alpha}S(t+h-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau\right]ds + \int_{s=0}^{t} A^{\alpha}S(t+h-s)Bw(s)ds + \int_{s=t}^{t+h} A^{\alpha}S(t+h-s)Bw(s)ds - \int_{s=0}^{t} A^{\alpha}S(t-s)\left[\tilde{f}(s)+\int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau\right]ds - \int_{s=0}^{t} A^{\alpha}S(t-s)Bw(s)ds\,\|_x$$

$$\|x_w(t+h)-x_w(t)\|_x \le \|(S(h)-I)S(t)A^{\alpha}u_0\|_x + \int_{s=0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\left[\|\tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\,d\tau\right]ds + \int_{s=0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\|_X\,\|Bw(s)\|_X\,ds + \int_{s=t}^{t+h} \|A^{\alpha}S(t+h-s)\|_X\left[\|\tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\,d\tau\right]ds + \int_{s=t}^{t+h} \|A^{\alpha}S(t+h-s)\|_X\,\|Bw(s)\|_X\,ds$$

$$= I_1 + I_2 + I_3 + I_4 + I_5. \quad \ldots\ldots (11)$$

We estimate each of the terms of (11) separately.

$I_1 = \|S(h) - I)S(t)A^{\alpha}x_0\|_x \le C_{\beta}h^{\beta}\|A^{\alpha+\beta}S(t)\|_X\,\|x_0\| \le Ch^{\beta}\,\|x_0\|t^{-(\alpha+\beta)}$ {by using equation 10}

$I_1 \le M_1h^{\beta}$, where $M_1 = C\,\|x_0\|t^{-(\alpha+\beta)}$ depends on t for $0 \le t \le t_1$.

$$I_2 = \int_{s=0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\left[\|\tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\,d\tau\right]ds$$

$$I_2 \le \int_{s=0}^{t} (B_1 + h_{t_1}B_2)Ch^{\beta}(t-s)^{-(\alpha+\beta)}ds \le (B_1 + h_{t_1}B_2)Ch^{\beta}\int_{s=0}^{t}(t-s)^{-(\alpha+\beta)}ds,$$

{by using equation (10) and the condition D with $h_r = \int_0^r |h(s)|ds$ },

$$I_2 \le \frac{(B_1 + h_{t_1}B_2)Ch^{\beta}}{1-(\alpha+\beta)}t^{-(\alpha+\beta)+1} \le \frac{(B_1 + h_{t_1}B_2)Ch^{\beta}}{1-(\alpha+\beta)}t_1^{-(\alpha+\beta)+1}$$

$I_2 \le M_2\, h^{\beta}$, where $M_2 = \dfrac{(B_1 + h_{t_1}B_2)Ch^{\beta}t_1^{-(\alpha+\beta)+1}}{1-(\alpha+\beta)}$ is independent of t for $0 \le t \le t_1$.

$$I_3 = \int_{s=0}^{t} \|(S(h)-I)A^{\alpha}S(t-s)\|_X\,\|Bw(s)\|_X\,ds$$

$$I_3 \le \int_{s=0}^{t} Ch^{\beta}(t-s)^{-(\alpha+\beta)}K_0K_1\,ds \le Ch^{\beta}K_0K_1\int_{s=0}^{t}(t-s)^{-(\alpha+\beta)}ds$$

{by using equation (10) and the condition G},

$$I_3 \leq \frac{Ch^{\beta}K_0K_1}{1-(\alpha+\beta)} t^{1-(\alpha+\beta)} \leq \frac{Ch^{\beta}K_0K_1}{1-(\alpha+\beta)} t_1^{1-(\alpha+\beta)}$$

$I_3 \leq M_3\ h^{\beta}$, where $M_3 = \frac{Ch^{\beta}K_0K_1 t_1^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}$ is independent of t for $0 \leq t \leq t_1$.

$$I_4 = \int_{s=t}^{t+h} \| A^{\alpha}S(t+h-s)\|_X \left[\|\tilde{f}(s)\|_X + \int_{\tau=0}^{s} |h(s-\tau)|\,\|\tilde{g}(\tau)\|_X\, d\tau \right] ds$$

$$I_4 \leq (B_1 + h_{t_1}B_2)C_{\alpha} \int_{s=t}^{t+h} (t+h-s)^{-\alpha}\, ds \leq \frac{(B_1 + h_{t_1}B_2)C_{\alpha}}{1-\alpha} h^{1-\alpha}$$

{by using the condition D and the properties $\|A^{\alpha}S(t)\| \leq C_{\alpha}t^{-\alpha}$ with $h_r = \int_0^r |h(s)|ds$ },

$I_4 \leq M_4\ h^{1-\alpha}$, where $M_4 = \frac{(B_1 + h_{t_1}B_2)C_{\alpha}}{1-\alpha}$ is independent of t for $0 \leq t \leq t_1$,

So that $I_4 \leq M_4\, h^{\beta}$.

$$I_5 = \int_{s=t}^{t+h} \| A^{\alpha}S(t+h-s)\|_X\ \|Bw(s)\|_X\, ds$$

$$I_5 \leq C_{\alpha}K_0K_1 \int_{s=t}^{t+h} (t+h-s)^{-\alpha}\, ds \leq \frac{C_{\alpha}K_0K_1}{1-\alpha} h^{1-\alpha},$$

{by using the condition G and the properties $\|A^{\alpha}S(t)\| \leq C_{\alpha}t^{-\alpha}$},

$I_5 \leq M_5\ h^{1-\alpha}$, where $M_5 = \frac{C_{\alpha}K_0K_1}{1-\alpha}$ is independent of $t \in [0, t_1]$, so that $I_5 \leq M_5\, h^{\beta}$.

Combining (11) with these estimates, it follows that there is a constant C_1 such that:

$\|x_w(t+h) - x_w(t)\|_x \leq C_1\, h^{\beta} \leq C_1 |h^{\beta}|$

Where $C_1 = M_1 + M_2 + M_3 + M_4 + M_5$.

$\Rightarrow$Let $k=t+h \Rightarrow h=k-t$

$\Rightarrow \|x_w(k)-x_w(t)\|_x \leq C_1|k-t|^{\beta}$. For $0<t<k\leq t_1$. ....(12)

Therefore x_w is locally Hölder continuous on $(0, t_1]$.

Now, to show that $\tilde{f}(t)$ is locally Hölder continuous on $(0, t_1]$, we have, For $t > s$:

$\|\tilde{f}(t)-\tilde{f}(s)\|_X = \|f(t, A^{-\alpha}x_w(t)) - f(s, A^{-\alpha}x_w(s))\|_x$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \leq L_0\,[|t-s|^{\theta} + \| A^{-\alpha}x_w(t) - A^{-\alpha}x_w(s)\|_{\alpha}]$ for $0<\theta\leq 1$ {by using the condition C},

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \leq L_0\,[|t-s|^{\theta} + \|x_w(t) - x_w(s)\|_x]$ {by using the properties $\|x\|_{\alpha} = \|A^{\alpha}x\|_x$},

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \leq L_0\,[|t-s|^{\theta} + C_1|t-s|^{\beta}]$, {by using equation (12)}

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \leq L_0\,[|t-s|^{\gamma} + C_1|t-s|^{\gamma}]$, where $\gamma = \min\{\theta,\beta\}$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \leq L_0\,(1 + C_1)\,|t-s|^{\gamma}$

$\|\tilde{f}(t)-\tilde{f}(s)\|_X \leq C_2|t-s|^{\gamma}$, ...(13)

Where $C_2 = L_0\,(1 + C_1)$ is a positive constant.

Let $\tilde{h}(t) = \tilde{f}(t) + \int_{\tau=0}^{t} h(t-\tau)\tilde{g}(\tau)d\tau + Bw(t)$

To show that $\tilde{h}(t)$ is locally Hölder continuous on $(0, t_1]$. For $t > s$, we have:

$$\|\tilde{h}(t)-\tilde{h}(s)\|_X = \left\| \tilde{f}(t) + \int_{\tau=0}^{t} h(t-\tau)\tilde{g}(\tau)d\tau - \tilde{f}(s) - \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau)d\tau + Bw(t) - Bw(s) \right\|_X$$

$$\|\tilde{h}(t)-\tilde{h}(s)\|_X \leq \|\tilde{f}(t)-\tilde{f}(s)\|_X + \int_{\tau=0}^{t} |h(t-\tau)-h(s-\tau)|\ \|\tilde{g}(\tau)\|_X d\tau + \|B\,(w(t) - w(s))\|_X$$

After a series of simplifications and using the conditions I, D and J with equation (13), we have got:

$\|\tilde{h}(t)-\tilde{h}(s)\|_X \leq C_2|t-s|^{\gamma} + C_3\,B_2\,|t-s|^{\vartheta}\,t_1 + K_0R_0|t-s|^{\xi}$

$\|\tilde{h}(t)-\tilde{h}(s)\|_X \leq C_2|t-s|^{\wp} + C_3\,B_2\,|t-s|^{\wp}\,t_1 + K_0R_0\||t-s|^{\wp}$, where $\wp = \min\{\gamma, \vartheta, \xi\}$

$\|\tilde{h}(t)-\tilde{h}(s)\|_X \leq \{C_2 + C_3\,B_2\,t_1 + K_0R_0\}\,|t-s|^{\wp}$.

This show that $\tilde{h}(t)$ is locally Hölder continuous on $(0, t_1]$.

From the theorem (2.4.1) in [5], we infer that the function:

$$v_w(t) = S(t)x_0 + \int_0^t S(t-s)\tilde{h}(s)\,ds \text{, where}$$

$$\tilde{h}(t) = \tilde{f}(t) + \int_{\tau=0}^{t} h(t-\tau)\tilde{g}(\tau)\,d\tau + Bw(t).$$

$$v_w(t) = S(t)x_0 + \int_0^t S(t-s)\left[\tilde{f}(s) + \int_{\tau=0}^{s} \tilde{h}(s-\tau)\tilde{g}(\tau)\,d\tau + Bw(s)\right]ds$$

$$v_w(t) = S(t)x_0 + \int_0^t S(t-s)\left[f(s, A^{-\alpha}x_w(s)) + \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau, A^{-\alpha}x_w(\tau))\,d\tau + Bw(s)\right]ds \qquad (14)$$

For $0 < t \le t_1$, $\forall\, w \in L^p((0, t_1]:O)$.

is X_a –valued, that the integral terms in (14) are functions in $C^1((0,t_1]:X)$ and that $v_w(t) \in D(A), \forall t \in (0,t_1]$. Operating on both sides of equation (14) with A^{α}, we have got:

$$A^{\alpha}v_w(t) = S(t)A^{\alpha}x_0 + \int_0^t S(t-s)A^{\alpha}\left[f(s, A^{-\alpha}x_w(s)) + \int_{\tau=0}^{s} h(s-\tau)\tilde{g}(\tau, A^{-\alpha}x_w(\tau))\,d\tau + Bw(s)\right]ds \qquad (15)$$

From equation (8), implies that $A^{\alpha} v_w(t) = x_w(t)$, i.e., $v_w(t) = A^{-\alpha} x_w(t)$,

For $0 < t \le t_1, \forall\, w(.) \in L^p((0, t_1]:O)$, and hence that $v_w(t)$ is a C^1 function on $[0,t_1]$. So we have got a unique S-classical solution $v_w \in C([0,t_1]:X)$.

References

[1] L.Byszewski "Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem", J. Math. Anal. Appl. , Vol 162, no. 2, 1991, pp. 494-505.

[2] Eduardo M. "Existence results for a class of semi-linear evolution equations" in electronic journal of differential equations, Vol. 24, 2001, pp. 1-14.

[3] Manaf A. Salah "Solvability and Controllability of Semilinear Initial Value Control Problem via Semigroup Approach", MSC thesis in mathematics, Al- Nahrain University, Baghdad, Iraq, 2005.

[4] A.Pazy "Semigroups of linear operators and applications to partial differential equations", Springer-Verlag, 1983.

[5] Jerome A. "Semigroup of linear operators and applications", Oxford mathematical monographs, 1985.

[6] Krishnan Balachandran "Regularity of solutions of Sobolev type semilinear integrodifferential equations in Banach spaces",Electronic Journal of differential equations, Vol. 114, 2003, pp.1-8.

[7] Krein "Linear differential equations in Banach space", American mathematical society, 1971.

[8] W.E.Fitzgibbon "Semilinear functional differential equations in Banach space", Nonlinear Analysis; Theory, Methods and Applications, Vol.4, 1980, pp.745-760.

[9] Klaus-Jockon Engel "One- parameter semigroups for linear evolution equations", 2000.

[10] Dieudonne "Foundation of modern analysis", Dieudonne.J, Academic press,New york, 1960.

الخلاصة

الوجود المحليّ ووحدانية الحَلّ نصف كلاسيكي (حَلّ نصف كلاسيكي) لصنف مسألة سيطرة شبه خطية في فضاء باناخ مناسب نوقشتْ وأثبتتْ. إنّ النَتائِجَ النظريةَ تَعتمدُ على نظريةِ نصفِ المجموعةَ التحليليةَ ومبدأ إنكماشِ باناخ.