A not of Modules with (f.S*) Property

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Abstract

Let R be an associative ring with identity and M be unital non zero right R-module. In this work,we introduce (f.S*) property as a generilization of (S*) property .A module M is said to satisfy the property (f.S*) if for every finitely generated submodule N of M there exists a direct summand K of M such that K≤N and N/K is cosingular. A ring R satisfies (f.S*) if the (right) R-module R satisfies (f.S*), and study the concept of module that satisfies the property of (f.S*) we was proved in theorem (3.1) that every right R-module M is satisfies (f.S*) if and only if every finitily generated submodule is direct sum of injective module and a cosingular module .Also we investigate some of their properties that are relevant with our work .

1. Introduction and Preliminaries :

Let R be an associative ring with identity and module M a non zero unital right R-modules. A submodule N of a module M is called a small submodule of M , denoted by N << M , if N + L ≠ M for any proper submodule L of M[1].

$Z^*(M) = \{m \in M : mR \text{ is small}\}$. Equvelently $Z^*(M) = M \cap \text{Rad } E(M)$ where E(M) is the injective hull of M. $Z^*(M) = M \cap \text{Rad } E(M) = M \cap \text{Rad}(E_1)$ for every injective module$E_1 \geq M$ [2] ,Z*(M) is called the co-singular submodule of M . For any module M , $\text{Rad}(M) \leq Z^*(M)$.

The following lemmas give some properties of a co-singular submodule of M which are needed later in this paper .

***Lemma 1:**[2]*

Let R be a ring and let $\varphi : M \to M'$ be a homomorphism of R- modules M , M′. Then $\varphi(Z^*(M)) \leq Z^*(M')$.

***Lemma 2:** [3]*

Let R be a ring. Then $M.Z^*(R) \leq Z^*(M)$ for any R-module M .

***Lemma 3**: [3]*

Let N be a submodule of an R-module M. Then $Z^*(N) = N \cap Z^*(M)$.

***Lemma 4:**[3]*

Let $M_{i(i\in I)}$ be any collection of R-modules and let $M = \oplus_{i\in I}M_i$. Then $Z^*(M) = \oplus_{i\in I}Z^*(M_i)$.

***Lemma 5:** [3]*

Let M be any R-mdule. Then $Z^*(M) = \sum Z^*(N)$ where the sum is taken over all finitely generated (cyclic) submodules of M .

An R-module M is called cosingular module if Z*(M) = M .And R is called right cosingular if the (right) R-module R is cosingular [3].

An R-module M is said to satisfy the property (S*) if every submodule N of M is cosingular of a direct summand of M [3]. Equivalently ,M satisfies (S*) if for every submodule N of M there exists a direct summand K of M such that K≤N and N/K is cosingular [3] . A ring R satisfies (S*) if the (right) R-module R satisfies (S*).

Recall that an R-module M is called lifting if for every submodule N of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ [4] Let N,Lbe submodules of a module M then N supplement of L in M if M=N+L and $N \cap L << N$[4] .

In this paper we introduce (f.S*) property as a generalization of (S*) property and study the concept of module that satisfy the property of (f.S*) and the relaion btween this kind of modules and f.lifting modules.We proved that every right R-module M satisfies (f.S*) if and only if every finitely generated submodule is direct sum of injective module and a cosingular module .Also we investigate some of their properties that are relevant with our work .

2. Modules with (f.S*) property :

In this section the (f.S*) property will be introduced as a generilization of (S*) property

Definition(2. 1) :

An R-module M is said to satisfy the property (f.S*) if for every finitely generated submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular. A ring R satisfies (f.S*) if the (right) R-module R satisfies (.f S*).

One can defineC.S* modules as follows. An R-module M is said to satisfis (C.S*) prportey if for every cyclic submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular.

Also we can show that (C.S*) and (f.S*) property are equivalent definitions.

It is clear that every (f.S*) is (C.S*). conversely ,let N be a finitely generated submodule of M then

$N = Rx_1 + Rx_2 + \dots + Rx_n$, $x_1, x_2, \dots, x_n \in N$,

Rx_i is cyclic for all i = 1,2,... ,n. then for all i = 1,2,... ,n. there exists $L_i \leq Rx_i$ such that $M = L_i \oplus K_i$ for some $K_i \leq M$ .and Rx_i / L_i is cosingular ,hence

$M = L_1 + L_2 + \dots + L_n + K$ where

$K = K_1 + K_2 + \dots + K_n$, $L_1 + L_2 + \dots + L_n \leq N$ and $N/L_1 + L_2 + \dots + L_n = Rx_1 + Rx_2 + \dots + Rx_n / L_1 + L_2 + \dots + L_n$
$\cong Rx_1 / L_1 + Rx_2 / L_2 \dots \dots + Rx_n / L_n$

Now ,

$Z^*(Rx_1 + Rx_2 + \dots + Rx_n / L_1 + L_2 + \dots + L_n) =$
$Z^*(Rx_1 / L_1 + Rx_2 / L_2 \dots \dots + Rx_n / L_n) =$
$Z^*(Rx_1 / L_1) + \dots \dots \dots + Z^*(Rx_n / L_n)$ (lemma 4) then

$Z^*(Rx_i / L_i) = Rx_i / L_i$ for all i = 1,2,... ,n.i.e Rx_i / L_i is cosinguler for all i = 1,2,... ,n then M has (f.S*) prportey.

Recall that an R-module M is called f.lifting if for every finitely generated submodule N of M there is a deco-mposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$ [5] The following follow immediately from the definitions.

Remarks(2.2) :

1) Every cosingular module satisfies (f.S*)
2) For any ring R if R is cosingular , then any R-module M satisfies (f.S*).
3) Every semi-hollow modules satisfies (f.S*).
4) Every f.lifting modules satisfies (f.S*).

Proof:

(1) and (2) clear.

(3) Let N be a finitely generated submodule of R-module M ,then N is small of M , hence $M = \{0\} \oplus M$ and. $Z^*(N) = N$

(4) let M be a f.lifting R-module . Then for every finitely generated submodule N of M, there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2 << M$.

Since $M = M_1 \oplus M_2$ .Then $N \cap M << M$.
Then $N = M_1 \oplus (N \cap M_2)$ i.e
$N/M_1 \cong (N \cap M_2) << M$ .Then
$Z^*(N/M_1) = N/M_1$. Then the module satisfies (f.S*)

Lemma (2.3):

Let M be an R-module that satisfies (f.S*). Then any submodule of M satisfies (f.S*).

Proof :

Let N be submodule of M and K a finitely generated proper submodule of N hence there existe $L \leq K$ such that Lis a direct summand of M . Then $M = L \oplus W$ for some $W \leq M$ and $Z^*(K/L) = K/L$ now $N = W \oplus (N \cap L)$, since $N \cap L \leq K \cap L \leq K$ then $Z^*(K/ N \cap L) = Z^*(K/L) = K/L = K/ N \cap L$.

Ozcan in [3] proved that an R-module M is lifting if it is satisfies (S*) and $Z^*(M) << M$, we prove the following for (f.S*).

Lemma (2.4):

Let M be a module that satisfies (f.S*) and such that Z*(M) is small in M Then M is f.lifting module.

Proof :

Let M be a module with (f.S*) then for every finitely generated proper submodule N of M there exists a direct summand K of M such that $K \leq N$ and N/K is cosingular. let L be a submodule of $M = L \oplus K$. $N \cap M = N \cap (K \oplus L)$ Then $N = K \oplus (N \cap L)$, $N/K \cong N \cap L$ but N/K cosingular then

$N \cap L = Z^*(N \cap L) \leq Z^*(M) << M$ (lemma 1).then $N \cap L << M$ .Hence M is f.lifting module.

The following rsulets appeard in [3] for (S*) module without proof we give similar results for (f.S*) module.

Lemma(2.5):

Let M be an R-module. The following statements are equivalent.

1) M satisfies (f.S*),

2) For every finitely generated proper submodule N of M, M has a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is cosingular,
3) For every finitely generated proper submodule N of M, N has a decomposition $N = A \oplus B$ such that A is a direct summand of M and B is cosingular.

Proof :

(1) $\Rightarrow$ (2) let N be a finitely generated proper submodule of M, then there exsist $A \leq_{\oplus} M$, $M = A \oplus B$, $A \leq N$ and N/A is cosingular,
$N \cap M = N \cap (A \oplus B)$ then
$N = A \oplus (N \cap B)$, $N/A \cong N \cap B$, i.e $N \cap B$ is cosingular.

(2) $\Rightarrow$ (1) let N be a finitely generated proper submodule of M then by (2) M has a decomposition $M = A \oplus B$, $A \leq N$ and $N \cap B$ is cosingular .
But $N \cap B \cong N/A$. Hence N/A is cosingular then M satisfies (f.S*) .

(2) $\Rightarrow$ (3) let N be a finitely generated proper submodule of M. there exist a direct summand A of M ,$M = A \oplus B$, $A \leq N$,
$N = A \oplus B \cap N = A \cap N \oplus B = A \oplus B$ .Let i: $B \cap N \rightarrow B$ be the inclusion homomorphism then
$i(Z^*(B \cap N)) \leq Z^*(B)$,[6]
hence $Z^*(B \cap N) \leq Z^*(B)$,but $B \cap N$ is cosingular hence
$Z^*(B \cap N)= B \cap N \leq Z^*(B)$, $B \cap N = B \cap (A \oplus B)=B \leq Z^*(B)$. Hence B is cosingular .

(3) $\Rightarrow$ (1) let N be a finitely generated proper submodule of M. such that
$N = A \oplus B$, $A \leq_{\oplus} M$, B is cosingular . Then $M = A \oplus L$ for some $L \leq M$ But
$N/A \cong B$. Then N/A is cosingular .

Remark:(2.6) [6] :

Let M be a finitely generated R-module and $N,L \leq M$ such that L is a supplement of N in M then L is finitely generated submodule**.**

Lemma (2.8):

Let M be a finitely generated R-module that satisfies (f.S*). Suppose that there exists a supplement of Z*(M) in M. Then there is a decomposition $M = A \oplus B$ such that A is a f. lifting module and B is cosingular.

Proof :

Let A be a supplement of Z*(M) in M hence $M = A + Z^*(M)$, and $A \cap Z^*(M) << A$. Then
$Z^*(A) = Rad(A) << A$.Since M satisfies (f.S*) and A is a supplement of Z*(M) ,M is finitely generated R-module then A is finitely generated, hence there exists a direct summand K of M such that $K \leq A$,
$A / K = Z^*(A/K)$. for some submodule B of M such that $M = K \oplus B$.
Then, $A \cap B = Z^*(A \cap B) \leq Z^*(A)$. Since $Z^*(A) << A$ and $A \cap B$ is a direct summand of A then $A \cap B = 0$. Hence
$M = A \oplus B$. By Lemma (2.4) and Lemma (2.4), A is a f.lifting module. now,
$M = A \oplus Z^*(M) = A \oplus Z^*(A) + Z^*(B)$,
$M = A \oplus Z^*(B)$ and hence $Z^*(B) = B$.

3.f.H ring :

In this section we introduce the definition of f.H.ring and give a proposition as a dule of proposition (4.3) in [3].

Aring R is called H ring if every injective R-module is lifting [7] we introduse the following :

Definition(3.1):

Aring R is called f.H ring if every finitely generated injective R-module is f.lifting .

Proposition (3.2):

Let R be a ring. An injective R-module M satisfies (f.S*) if and only if every finitely generated proper submodule of M is a direct sum of injective module and a cosingular module.

Proof *:*

Suppose that M satisfies (f.S*). Let N be a finitely generated proper submodule of M. There exist submodules K,K′ of M such that $M = K \oplus K'$, $K \leq N$ and N/K is cosingular.

Then $N = K \oplus (N \cap K')$ where K is finitely generated injective and $N \cap K'$ is cosingular because $N \cap K' \cong N/K$. Conversely, suppose that every submodule of M is a direct sum of a injective module and a cosingular module. Let L be proper submodule of M.

Then $L = L_1 \oplus L_2$ for some injective module L_1 and cosingular module L_2. Clearly L_1 is a direct summand of M and $L/L_1 = Z^*(L/L_1)$ because $L/L_1 \cong L_2$.

Theorem (3.3):

For any R-module M The following are equivalent.

1)Every finitely generated R-module satisfies (f.S*),

2) Every finitely generated injective right R-module satisfies (f.S*),

3) Every finitely generated R-module is a direct sum of injective module and a cosingular module.

Proof :

(1) $\Leftrightarrow$ (2) It is clear

(2) $\Leftrightarrow$ (3) by Proposition (3.2).

We well introduse the following reselt which is similar to that appeared in [3].

<u>Theorem (3.4):</u>

The following statements are equivalent for a ring R.

1) R is a right f.H-ring,

2) For every finitely generated injective right R-module M, Rad(M) << M and every right R- module satisfies (f.S*).

Proof :

(1) $\Rightarrow$(2) If R is a right f. H-ring, then every finitely generated injective R-module is f.lifting hence (f.S*).

(2) $\Rightarrow$ (1) Let M be a finitely generated injective R-module Then Rad(M) =Z*(M)[3]. but Z*(M) << M,hence M is
f. lifting module by lemma (2.5) .Hence M satisfies (f.S*).

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الخلاصة

لتكن R حلقة تجميعية ذات عنصر محايد وليكن M مقاسا احاديا غير صفري ايمن معرفا على R في هذا البحث سنقدم الخاصية (f.S*) كتعميم للخاصية (S*) يقال بان المقاس M يمتلك خاصية (f.S*) اذا كان لكل مقاس جزئي N من M يوجد مركبة مجموع مباشر K من M بحيث K≤N و N/K تكون منفرد مضاد حيث ان الغرض الرئيس من هذا البحث هو دراسة المقاسات التي تحقق الخاصية (f.S*).في المبرهنة (3.1) M يمتلك خاصية (f.S*) اذا وفقط اذا كان كل مقاس جزئي منته التولد يكون مركبة مجموع مباشر من مقاس اسقاطي ومقاس غير منفرد كذالك سنراجع العديد من الخواص التي لها علاقة بموضوع البحث 0