

ON THE EXISTENCE OF A SOLUTION TO THE DUAL PARTIAL DIFFERENTIAL EQUATION OF DYNAMIC PROGRAMMING FOR GENERAL PROBLEMS OF BOLZA AND LAGRANGE

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Abstract

A main theorem which deals with the existence of a minimum solution to the dual partial differential equation of dynamic programming for optimal control problems of Bolza and Lagrange is proved. An example illustrates the value of this theorem is given. Properties of the value function and dual value function for problems of Bolza and Lagrange are described. Moreover, for these problems the existence of a maximum solution to the partial differential equation of dynamic programming, which satisfies the Lipschitz condition and which is also the value function is presented.

Keywords: Bolza problem, dynamic programming, dual value function, sufficient conditions, Hamilton-Jacobi equation, Lagrange problem.

1. Introduction

We consider the optimal control problem of Bolza:

$$\text{minimize } J(x, u) = \int_a^b L(t, x(t), u(t))dt + \lambda(x(b)), \tag{1}$$

where an absolutely continuous trajectory $x : [a, b] \rightarrow \mathbb{R}^n$ and the Lebesgue measurable control function $u : [a, b] \rightarrow \mathbb{R}^m$ are subject to the non-linear controlled state-space system :
 $\dot{x}(t) = f(t, x(t), u(t))$, a.e. in $[a, b]$, (2)
 $u(t) \in U(t), t \in [a, b]$, (3)
 $x(a) = c$, (4)
 where $f : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n, L : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}, \lambda : C \rightarrow \mathbb{R} \cup \{+\infty\}$ are given functions, C is a subset of \mathbb{R}^n, c is a point in the state space \mathbb{R}^n , and $U : [a, b] \rightarrow \mathbb{R}^m$ is a multifunction (i.e. $U(t)$ is a subset of \mathbb{R}^m for each t in $[a, b]$).

Let L be the collection of Lebesgue measurable subsets of $[a, b]$ and let B be Borel subsets of \mathbb{R}^m . $L \times B$ denotes the σ -algebra of subsets of $[a, b] \times \mathbb{R}^m$, generated by products of sets in L and B . In order that problem (1)-(4) make sense, throughout the paper we assume the following basis hypothesis:

(A) For each s in \mathbb{R}^n , the functions $(t, u) \rightarrow L(t, s, u), (t, u) \rightarrow f(t, s, u)$ are $L \times B$ -

measurable. There exists functions K_1, K_2 in $L^1(a, b)$ such that for t in $[a, b], u$ in $U(t)$, and s_1, s_2 in \mathbb{R}^n ,
 $\|L(t, s_1, u) - L(t, s_2, u)\| \leq K_1(t) \|s_1 - s_2\|,$
 $K_1(t) > 0.$
 $\|f(t, s_1, u) - f(t, s_2, u)\| \leq K_2(t) \|s_1 - s_2\|,$
 $K_2(t) > 0.$

The set $\{(t, u) \in [a, b] \times \mathbb{R}^m, u \in U(t)\}$ is $L \times B$ -measurable. The function λ is lower semicontinuous and not identically to $+\infty$.

A pair $x(t), u(t)$ is admissible if it satisfies (2), (3), $t \rightarrow L(t, x(t), u(t))$ is a summable, and $\ell(x(b))$ is finite; then the corresponding trajectory $t \rightarrow x(t)$ will be called admissible.

For the problem (1)-(4), if we take the function $x \rightarrow \ell(x)$ equal to zero, we obtain that the Lagrange control problem. And these two problems Bolza and Lagrange are equivalent in that each one can be formulated as one of the other form [12].

It is well-known that in classical dynamic programming (CDP) the whole family of problems with fixed initial points is considered. For one problem the initial point is fixed, but when a family of problems with different initial points are considered, the solution to these problems are dependent on their initial points. This dependence is called the value function. Whenever the value function is differentiable it satisfies a first order partial differential equation called the

partial differential equation of dynamic programming (PDEDP) known as the Hamilton-Jacobi (H-J) equation [6,9,12]. A sufficient condition for optimality can be phrased in terms of a continuously differentiable solution of the PDEDP (see, Th.2). The method of CDP encounters the difficulty that for many problems the value function is not differentiable everywhere.

For the Lagrange problem in 1976, R. Gonzales proved that there exists a maximum solution to the H-J equation, which satisfies the Lipschitz condition and which is also the value function (see, section 2).

For the problem (1)-(4), two methods of construction of fields of extremals (concourses of flights) were described in 1988 by A. Nowakowski, and as a consequence of this, sufficient conditions for optimality in a form similar to Weierstrass's were formulated. In remarks, there were given relations of these two theories to the dynamic programming technique. The first (classical) is the following: (see, [6,9,11,12]) define the value function $S(t, x)$ in a set $T \subset \mathbb{R}^{n+1}$ (being a set covered by graphs of trajectories of the field-concourses of flights) as

$$S(t, x) = \inf \left\{ \int_t^b L(\tau, x(\tau), u(\tau)) d\tau + \ell(x(b)) \right\}, \dots\dots\dots (5)$$

where the infimum is taken over all admissible pairs $x(\tau), u(\tau), \tau \in [t, b]$, whose trajectories start at $(t, x) \in T$, and graphs are contained in T . By [1,Th.4], the existence of value function (5) is determined by existence of a concourse of flights (field of extremals). Further, it was concluded that if $S(t, x)$ is differentiable then it satisfies the partial differential equation

$$S_t(t, x) + H(t, x, S_x(t, x)) = 0, \dots\dots\dots (6)$$

where $H(t, x, y) = y f(t, x, u(t, x)) + L(t, x, u(t, x))$ and $u(t, x)$ is an optimal feedback control, and the PDEDP

$$\min_{u \in U(t)} \{ S_t(t, x) + S_x(t, x) f(t, x, u) + L(t, x, u) \} = 0 \dots\dots\dots (7)$$

A. Nowakowski in [1, remark 4.2] suggested the second nonclassical approach to dynamic programming, the domain of exploration was carried out from the (t, x) -space to the space of multipliers $((t, y^0, y)$ -space), then another function was defined the-dual value function-

$S_D(t, p)$ in a set $P \subset \mathbb{R}^{n+2}$ of the dual space $(t, y^0, y) = (t, p), y^0 \leq 0,$

$$S_D(t, p) = \inf \left\{ -y^0 \int_t^b L(\tau, x(\tau), u(\tau)) d\tau - y^0 \ell(x(b)) \right\}, \dots\dots\dots (8)$$

where the infimum is taken over admissible pairs $x(\tau), u(\tau), \tau \in [t, b]$, whose trajectories start at $(t, x(t, p))$ ($x(t, p)$ will be defined in section 3) and their graphs are contained in T (defined above).

By [1, Corollary 5'], the existence of $S_D(t, p)$ is determined by the existence of a concourse of flights. Next, a new function was defined:

$$V(t, p) = -S_D(t, p) - x(t, p) y = V_{y^0}(t, p) y^0 + V_y(t, p) y = V_p(t, p) p, (t, p) \in P, \dots (9)$$

$(-S_D(t, p) = V_{y^0}(t, p) y^0, -x(t, p) = V_y(t, p))$, which satisfies the partial differential equation

$$V_t(t, p) + H(t, -V_y(t, p), p) = 0, \dots\dots\dots (10)$$

where $H(t, v, p) = y^0 L(t, v, u(t, p)) + y f(t, v, u(t, p))$ and $u(t, p)$ is a dual optimal feedback control, and the dual partial differential equation of dynamic programming (DPDEDP)

$$\max \{ V_t(t, p) + y f(t, -V_y(t, p), u) + y^0 L(t, -V_y(t, p), u) : u \in U(t) \} = 0, \dots (11)$$

In 1992, A. Nowakowski got some properties of the classical value function $S(t, x)$ by studying (8),(10),(11) directly. And he established that when examining (8)-(11) instead of (5)-(7), needed not require that the set T has nonempty interior, or that $S(t, x)$ is differentiable in T [2, Remark 2.1]. Moreover, he did not require that $y^0 = -1$ or that the problem is calm, which is essential in the classical setting (5)-(7)[5,7]. Further, if problem (1)-(4) can be solved using (6),(7), i.e., [12, Th.7.1, Ch.4], then it can be solved also by dual dynamic programming (DDP), i.e., Th. 4 of section 3.

From all the above, it can be seen that the solution to the DPDEDP (11) for the problem (1)-(4), is essential in the study of optimality. Therefore, it is found to be a reasonable justification to accomplish the study of this paper.

The aim of this paper is to study the existence solution to the DPDEDP (11), and we shall proved in the main theorem (Th.5), the function $V(t, p), (t, p) \in P$ (9) for the Bolza problem (1)-(4) is a minimum element of the set W (see, def.2) of all Lipschitz solutions to

the dual partial differential inequality of dynamic programming (DPDIDP) for problem (1)-(4). To demonstrate the value of Th.5, we will be gave two examples, these examples can also be solved by the nonclassical field theory method in [1], and Th.4, but could not be treated by any other known method.

2. The Existence of a Maximum Solution to the Hamilton-Jacobi Equation

In this section the properties of the classical value function for the Bolza problem (1)-(4) is described. We shall assume throughout this section, the following basis hypothesis :

(Z) $(t,x, u) \rightarrow f(t, x, u)$ and $(t, x, u) \rightarrow L(t, x, u)$ are continuous and bounded functions on $[a, b] \times \mathbb{R}^n \times U$; they are Lipschitz functions with respect to t, x, u ; $x \rightarrow \lambda(x)$ is Lipschitz functions with respect to x . U is a compact subset of \mathbb{R}^m .

Let $T \subset [a, b] \times \mathbb{R}^n$ be a set with a non-empty interior, covered by graphs of admissible trajectories, i.e., for every $(t_0, x_0) \in T$ there exists an admissible pair $x(\cdot), u(\cdot)$, defined in $[t_0, b]$, such that $x(t_0) = x_0$ and $(s, x(s)) \in T$ for $s \in [t_0, b]$. Further, let $S(t, x)$ be as in (5) but with T defined here.

One of the most important properties of the classical value function is stated in theorem 1:

Theorem 1:

If the functions $(t, x, u) \rightarrow f(t, x, u)$, $(t, x, u) \rightarrow L(t, x, u)$ and $x \rightarrow \lambda(x)$ satisfy assumptions (Z) from the Bolza problem (1)-(4), then the classical value function $(t, x) \rightarrow S(t, x)$ (5) satisfies a Lipschitz condition and is the solution to the PDEDP (7) for a.e. $(t, x) \in T$, with the boundary condition $S(b, x) = \lambda(x), (b, x) \in T$.

Proof:

See [12, Ch.4]. ■

Theorem 2:

Let $(t, x) \rightarrow G(t, x)$ be the solution of class C^1 to the PDEDP: $G_t(t,x)+H(t,x,G_x(t,x))=0, (t, x) \in T_0$, with the boundary condition, $G(b, x) = \lambda(x), (b, x) \in T_0$, where $T_0 \subset T$ is an open set, and the Hamiltonian $H(.,.,.)$ be as defined in (6).

If $x = x(t)$ and a pair $x(\cdot), u(\cdot)$, defined in $[a, b]$, $x(a) = c$, is admissible and such that

$$G_t(t, x(t)) + G_x(t, x(t)) f(t, x(t), u(t)) + L(t, x(t), u(t)) = 0,$$

then the pair $x(\cdot), u(\cdot)$ is optimal, and also $G(t, x) = S(t, x), (t, x) \in T_0$, where $S(\cdot, \cdot)$ is the classical value function.

Proof:

See [12, Ch.4]. ■

It can be seen that some regularity of the function $(t, x) \rightarrow G(t, x)$, being the solution to the PDEDP, is required, i.e. it must be at least a Lipschitz function (see, Th. 1).

Definition 1:

Let us define a set M as follows :

$$M = \left\{ \begin{array}{l} m(t, x) \text{ is Lipschitz for } t, x; t \in [0, b], \\ (t, x) \in T; m(b, x) \leq \ell(x(b)) \quad \forall (b, x) \in T; \text{ and} \\ m_t + \min_{u \in U(t)} \{m_x f(t, x, u) + L(t, x, u)\} \geq 0 \text{ a. e. } (t, x) \in T. \end{array} \right\}$$

Define on the set M the following partial ordering:

$$m \leq \hat{m} \Leftrightarrow m(t, x) \leq \hat{m}(t, x), \forall (t, x) \in T, t \in [0, b]; \forall m, \hat{m} \in M.$$

Note 1. From the definition of the classical value function $S(t, x), (t, x) \in T$ in (5) and Th.1, we observe that the function $S(t, x)$ belongs to the set M of all Lipschitz solutions to the PDIDP above, when there exists x a multiplied solution for the problem (1)-(4).

Theorem 3:

The function $S(t, x)$ (5), $(t, x) \in T, t \in [0, b]$ for the Bolza problem (1)-(4), is a maximum element of the set M , i.e., $m \leq S, \forall m \in M$.

Proof:

See [10]. ■

3. The Existence of a Minimum Solution to the Dual Partial Differential Equation of Dynamic Programming

Let $T \subset \mathbb{R}^{n+1}$ be as defined in section 2. We shall assume in this section, taking if necessary a smaller set T , that $S(t, x)$ defined in T by (5) does not take the value $\pm \infty$. Now T is, in general, larger than that defined in section 1. Let $P \subset \mathbb{R}^{n+2}$ be a set of variables $(t, y^0, y) = (t, p), t \in [a, b]$, with $y^0 \leq 0$ and a nonempty interior.

We take a function $x(t, p)$ defined on P such that $(t, x(t, p)) \in T, (t, p) \in P$, and we assume that it is a Borel measurable, locally bounded, and that for each admissible trajectory $x(t)$ lying in T there exists a function of bounded variation $p(t) = (y^0, y(t))$ lying in P such that $x(t) = x(t, p(t))$ and if all trajectories $x(t)$ start at the same (t_0, x_0) , then all the corresponding $p(t)$ have the same first coordinate y^0 .

Further, let $S_D(t, p)$ be as in (8) but with T and $x(t, p)$ defined here. We see that

$$S_D(t, p) = -y^0 S(t, x(t, p)), (t, p) \in P.$$

Note 2. The dual value function (8), has properties analogous to the classical value function (5) (see, [2]). Thus we get a modification of the known properties, i.e., Th.1,2 [2]. Also we can get a modification of Th.3 [see, 8].

The following theorem will be established when does a solution $V(t, p)$ (9) to the DPDEDP (11) satisfy the sufficient conditions for optimality of the problem (1)-(4).

Theorem 4:

Let $V(t, p), (t, p) \in P, t \in [a, b]$, be a Lipschitz solution of DPDEDP (11). Let E denote a subset of $[a, b]$ such that if $t_0 \in E$, then for $(t_0, p) \in P, V_p(t, p)$ exists. We assume that $\text{meas } E = b - a, b \in E$, and that $V(t, p)$ satisfies the boundary condition $y^0 V_{y^0}(b, p) = y^0 \lambda(-V_y(b, p)), (b, p) \in P$, and the relation:

$$V(t, p) = V_p(t, p)p, t \in E, (t, p) \in P \dots\dots\dots (12)$$

Let $x(t), u(t)$ be an admissible pair whose graph of the trajectory $x(t)$ is contained in the closure \bar{T} of $T = \{(t, x): x = -V_y(t, p), t \in E, (t, p) \in P\}$ and such that there is an absolutely continuous function $p(t) = (y^0, y(t))$ lying in P and satisfying $x(t) = -V_y(t, p(t))$ for $t \in E$. Assume further that then $V_t(t, p(t))$ exists for almost every t . Then

$$\bar{W}(t, p(t)) = -y^0 V_{y^0}(t, p(t)) + y^0 \int_t^b L(s, x(s), u(s)) ds$$

is a nondecreasing function of t . Let $\bar{x}(t), \bar{u}(t), t \in [a, b], \bar{x}(a) = c$, be an admissible pair with $\bar{x}(t)$ lying in \bar{T} and let $\bar{p}(t) = (\bar{y}^0, \bar{y}(t)), t \in [a, b]$, be a function lying in P such that $\bar{x}(t) = -V_y(t, \bar{p}(t))$ for all $t \in E$. Suppose that for almost all t in $[a, b]$,

$$V_t(t, \bar{p}(t)) + \bar{y}(t) f(t, -V_y(t, \bar{p}(t)), \bar{u}(t)) + \bar{y}^0 L(t, -V_y(t, \bar{p}(t)), \bar{u}(t)) = 0.$$

Then $\bar{x}(t), \bar{u}(t), t \in [a, b]$ is an optimal pair for the problem of Bolza (1)-(4) relative to all admissible pairs $x(t), u(t), t \in [a, b], x(a) = c$, whose graphs of trajectories are contained in \bar{T} and where the corresponding function $p(t) = (\bar{y}^0, \bar{y}(t)) (x(t) = -V_y(t, p(t)), t \in E)$ is an absolutely continuous. Moreover, $-\bar{y}^0 S(t, x(t, \bar{p}(t))) = -\bar{y}^0 V_{y^0}(t, \bar{p}(t))$ with $x(t, p) = -V_y(t, p)$ is the dual value function along $\bar{p}(t)$.

Proof:

See [2, Th.3.1]. ■

Remarks to Theorem 4.1

If we also assume that $V_{yy} \neq 0$ exists and is continuous, then for each admissible trajectory $x(t)$ whose graph is contained in T , we get that the corresponding $p(t) = (y^0, y(t))$ is really of bounded variation. 2- If V is a Lipschitz solution of (11) then it will be of form (12) when $V_{pp}(t, p)p = 0, (t, p) \in P$. 3- Put $x(t, p) = -V_y(t, p), S(t, x(t, p)) = V_{y^0}(t, p)$. Then (12) means that $(y^0, -y)$ is a normal to the epigraph of $S(t, x)$ defined in T at the point $(x(t, p), S(t, x(t, p)))$. This generalizes the classical results (if $S(t, x)$ is smooth and $y^0 = -1$) that $-y = S_x$ (see, [6],[12]). 4- We would like to stress that the $x(t, p)$ of section 2, which appeared in an artificial way, in practice, is calculated from (11) and (12) by putting $x(t, p) = -V_y(t, p)$.

Now let us in addition to the hypothesis (A), $U(t) = \{u(t) \text{ measurables; such that for } t \in [a, b], u(t) \in K, \text{ where } K \text{ is a compact subset of } \square^m\}$. $f(., \dots)$ and $L(., \dots)$ are continuous in $[a, b] \times \square^n \times K$.

The response of the problem (1)-(4) shall be denoted by $x(t)$ at instant t , for the control $u(.)$ on the interval $[a, b]$ and with the initial condition $x(a) = c$.

Definition 2:

Let W be a set of all Lipschitz solutions $H(t, p), (t, p) \in P, t \in [0, b]$, to the DPDEDP (13), when there exists $x(t) = -H_y(t, p(t))$, lying in T , as a multiplied solution for the Bolza problem (1)-(4):

$$H_t(t, p) + \max_{u \in K} \{y f(t, -H_y(t, p), u) + y^0 L(t, -H_y(t, p), u)\} \leq 0, \text{ a.e., } (t, p) \in P, t \in [0, b]$$

..... (13)

where K is a compact subset of \mathbb{R}^m ; and assume that $H(t, p)$ satisfies the boundary condition

$$y^0 H_{y^0}(b, p) \geq y^0 \ell(-H_y(b, p)), \forall (b, p) \in P;$$

and the relation :

$$H(t, p) = y^0 H_{y^0}(t, p) + y H_y(t, p) = H_p(t, p)p.$$

And define on the set W the following partial ordering:

$$H \leq \hat{H} \Leftrightarrow H(t, p) \leq \hat{H}(t, p), (t, p) \in P, t \in [0, b], \forall H, \hat{H} \in W.$$

Note 3. From the definition of the function $(t, p) \rightarrow V(t, p), (t, p) \in P$ in (9), and Th.4, we observe that the function $(t, p) \rightarrow V(t, p), (t, p) \in P$ belongs to the set W .

Now, let us formulate and prove some lemmas, which will simplify and shorten the proof of the main theorem in this paper that the function $(t, p) \rightarrow V(t, p), (t, p) \in P$ defined in (9) is a minimum element of the set W . To formulate these lemmas, let us assume that $t_0 < b$ and consider $\delta > 0$ such that the interval $[t_0 + \delta, b - \delta]$ has a nonempty interior. Now let $x_0(t_0) = x_0(t_0, p(t_0))$ be arbitrary and let it belong to $T, u(\cdot) \in U(t)$. Since $x(t, p), (t, p) \in P$ is locally bounded and $(t, x(t, p)) \in T, (t, p) \in P$, then the response of the system $t \rightarrow x(t) = x(t, p(t)) = -H_y(t, p(t)), t \in [t_0, b]$ with $x_0(t_0) = x_0(t_0, p(t_0))$, lying in T is bounded, i.e.,

$$x(t, p(t)) \in Q, \text{ for all } (t, p(t)) \in \hat{Q}, t \in [t_0, b],$$

where Q and \hat{Q} are compact subsets of T and P respectively. Here, define a set \bar{Q} as follows:

$\bar{Q} = \hat{Q} + B_1(\mathbb{R}^{n+2})$, where $B_1(\mathbb{R}^{n+2})$ is the sphere centered at the origin having a radius of 1.

Note 4. We need in the proof of the main theorem of this paper to construct a new function $(t, p) \rightarrow H_2^\varepsilon(t, p), (t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$ which is sufficiently regular and satisfies the inequality (13). So an arbitrary function $(t, p) \rightarrow H(t, p)$ of the set W

can be chosen and modified in a few steps of construction until the resulting function $(t, p) \rightarrow H_2^\varepsilon(t, p)$ satisfies the inequality (13).

Thus, for note 4 we suppose that the function $(t, p) \rightarrow H(t, p)$ be any function in the set W . We may construct a new function $(t, p) \rightarrow H_1(t, p)$ by shifting the function $(t, p) \rightarrow H(t, p)$, as follows:

$$H_1(t, p) = H(t, p) + \alpha(b - t), \text{ (14)}$$

where α is a positive real number which close to zero.

For a shorter and simpler definitions, we propose the following notations:

$$f_1(t, p, u) = f(t, -H_{1y}, u),$$

$$L_1(t, p, u) = L(t, -H_{1y}, u). \text{ (15)}$$

Since the function $(t, p) \rightarrow H(t, p), (t, p) \in P, t \in [0, b]$ belongs to the set W , and since, $H_{1t}(t, p) = H_t(t, p) - \alpha, H_{1x}(t, p) = H_x(t, p), (t, p) \in P, t \in [0, b]$, then we see that the function $(t, p) \rightarrow H_1(t, p)$ is a Lipschitz function and satisfies the following:

$$H_{1t}(t, p) + \max_{u \in K} \{y f_1(t, p, u) + y^0 L_1(t, p, u)\} \leq -\alpha < 0 \text{ a.e., } (t, p) \in P, t \in (0, b). \text{ (16)}$$

In order to define a new function $(t, p) \rightarrow H_2^\varepsilon$

$(t, p), (t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$, for arbitrary and fixed $\varepsilon < \min(1, \delta)$, such that

$$H_2^\varepsilon(t, p) \in C_0^\infty(\hat{Q}, t \in [t_0 + \delta, b - \delta])$$

and it satisfies the inequality (16), we have to define a new function $(t, p) \rightarrow H_2^\varepsilon(t, p)$ by using the convolution of the function $(t, p) \rightarrow H_1(t, p)$ with a function of class $C_0^\infty(\mathbb{R}^{n+2})$ having a compact support. So we will define a function $(t, p) \rightarrow H_2^\varepsilon(t, p), (t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$, for arbitrary and fixed $\varepsilon < \min(1, \delta)$ by using the convolution of the function $(t, p) \rightarrow H_1(t, p)$ with a function $(t, p) \rightarrow \rho_\varepsilon(t, p)$ of class $C_0^\infty(\mathbb{R}^{n+2})$ having a compact support as follows:

$$H_2^\varepsilon(t, p) = (H_1 * \rho_\varepsilon)(t, p), \text{ (17)}$$

where the function $(t, p) \rightarrow H_1(t, p)$ as defined in (14), $\rho_1: \mathbb{R}^n \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a function of class $C_0^\infty(\mathbb{R}^{n+2})$ having a compact support, and satisfies:

$$\int_{\square^{n+2}} \rho_1(t, p) dt dp = 1,$$

$$\rho_\varepsilon(t, p) = \left(\frac{1}{\varepsilon^{n+2}}\right) \rho_1\left(\frac{t}{\varepsilon}, \frac{p}{\varepsilon}\right) \in C_0^\infty(\square^{n+2}),$$

$$\text{supp } \rho_1 \subset B_1(\square^{n+2})$$

where $B_1(\square^{n+2})$ is a sphere centered at the origin having a radius of 1.

Clearly, this function $(t, p) \rightarrow H_2^\varepsilon(t, p)$ will also be a Lipschitz function, because the function $(H_1 * \rho_\varepsilon)(t, p)$ is a Lipschitz for t, p . For shorter and simpler definitions, we propose the following notations:

$$f_2(t, p, u) = f(t, -H_{2y}^\varepsilon, u),$$

$$L_2(t, p, u) = L(t, -H_{2y}^\varepsilon, u). \tag{18}$$

In order to show that the function $(t, p) \rightarrow H_2^\varepsilon(t, p)$, $(t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$ satisfies the inequality (16), i.e.,

$$\exists \varepsilon' > 0 \mid \forall \varepsilon \leq \varepsilon', H_{2y}^\varepsilon(t, p) + \max \{ y f_2(t, p, u) + y^0 L_2(t, p, u) : u \in K \} \leq \frac{-\alpha}{2} < 0,$$

we need to prove some lemmas, so that the proof of the above fact (19) becomes shorter and simpler.

Lemma 1:

Let $H_1(\cdot, \cdot)$, $H_2^\varepsilon(\cdot, \cdot)$ and $\rho_\varepsilon(\cdot, \cdot)$ be functions defined in \hat{Q} (see (17)), then for all $(t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$, we have:

$$\lim_{\varepsilon \rightarrow 0} H_{2y}^\varepsilon(t, p) = H_{1y}(t, p),$$

and this convergence is uniform.

Proof : By definition of uniformly convergent sequence of functions to prove that this lemma holds, it is sufficient to show that for arbitrary $\gamma > 0$ a $\varepsilon' > 0$ exists such that for every $\varepsilon \leq \varepsilon'$ and for all $(t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$ the following holds: $|H_{2y}^\varepsilon(t, p) - H_{1y}(t, p)| < \gamma$.

Now by using the definitions of the function $H_2^\varepsilon(\cdot, \cdot)$ and the convolution, for all $(t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$, the following holds:

$$|H_{2y}^\varepsilon(t, p) - H_{1y}(t, p)|$$

$$= |(H_{1y} * \rho_\varepsilon)(t, p) - H_{1y}(t, p)|$$

$$= \left| \int_{B_\varepsilon(\square^{n+2})} [H_{1y}(t-s, p-p') - H_{1y}(t, p)] \rho_\varepsilon(s, p') ds dp' \right|$$

$$\leq \sup_{\substack{(t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} |H_{1y}(t-s, p-p') - H_{1y}(t, p)|$$

Since the function $H_{1y}(\cdot, \cdot)$ is uniformly continuous on the compact set \bar{Q} , then we have

$$\sup_{\substack{(t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} |H_{1y}(t-s, p-p') - H_{1y}(t, p)| \rightarrow 0 \text{ when } \varepsilon \rightarrow 0.$$

Consequently, $|H_{2y}^\varepsilon(t, p) - H_{1y}(t, p)| \rightarrow 0$ when $\varepsilon \rightarrow 0$.

Therefore, for an arbitrary $\gamma > 0$, a $\varepsilon' > 0$ exists such that for all $\varepsilon \leq \varepsilon'$ and for all $(t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$ the following holds: $|H_{2y}^\varepsilon(t, p) - H_{1y}(t, p)| < \gamma$. ■ ζ

Because in the proof of Th. 5 we will need the fact that the function $L_2(\cdot, \cdot, \cdot)$ and $(L_1 * \rho_\varepsilon)(\cdot, \cdot, \cdot)$ have values arbitrary closed to each other. So lemma 2 should be given an estimate of the difference between the values of these two functions by arbitrary real number, close to zero.

Lemma 2:

Let $H_2^\varepsilon(\cdot, \cdot)$, $L_1(\cdot, \cdot, \cdot)$ and $L_2(\cdot, \cdot, \cdot)$ be functions defined in (17), (15) and (18) respectively, and $\rho_\varepsilon(\cdot, \cdot)$ be the function of class $C_0^\infty(\square^{n+2})$ defined earlier. Then for arbitrary positive real number α , described during the definition of the function $H_1(\cdot, \cdot)$ there exists $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$ and for all $(t, p, u) \in \hat{Q} \times K, t \in [t_0 + \delta, b - \delta]$ the following inequality holds :

$$|y^0 L_2(t, p, u) - y^0 (L_1 * \rho_\varepsilon)(t, p, u)| < (\alpha/4).$$

Proof : For $(t, p, u) \in \hat{Q} \times K$, the following estimation holds :

$$|y^0 L_2(t, p, u) - y^0 (L_1 * \rho_\varepsilon)(t, p, u)|$$

$$= |y^0 [L_2(t, p, u) - (L_1 * \rho_\varepsilon)(t, p, u)]|$$

$$= |y^0 \left| \int_{B_\varepsilon(\square^{n+2})} [L_2(t, p, u) - L_1(t-s, p-p', u)] \rho_\varepsilon(s, p') ds dp' \right|$$

$$\begin{aligned} &\leq |y^0| \int_{B_\varepsilon(\square^{n+2})} | [L_2(t,p,u) - L_1(t-s,p-p',u)] \\ &\quad \rho_\varepsilon(s,p') | dsdp' \\ &\leq |y^0| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} | L_2(t,p,u) - \\ &\quad L_1(t-s,p-p',u) | \\ &= |y^0| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} | L(t, -H_{2y}^\varepsilon(t,p), u) - \\ &\quad L(t, -H_{1y}(t-s,p-p'), u) |, \end{aligned}$$

since $L(\dots)$ is uniformly continuous on the compact set $\bar{Q} \times K, t \in [0, b]$, then by lemma 1, we get that

$$|y^0| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} | L(t, -H_{2y}^\varepsilon(t,p), u) - \\ L(t, -H_{1y}(t-s,p-p'), u) | \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and consequently,

$$| y^0 L_2(t,p,u) - y^0 (L_1 * \rho_\varepsilon)(t,p,u) | \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Hence, for an arbitrary positive number α , there exists $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$ and for all $(t,p,u) \in \hat{Q} \times K, t \in [t_0+\delta, b-\delta]$ the following holds :

$$| y^0 L_2(t,p,u) - y^0 (L_1 * \rho_\varepsilon)(t,p,u) | < \frac{\alpha}{4}. \blacksquare$$

Since in the proof of the main theorem we will use the fact that the value of the functions $f_2(\dots)$ and $(f_1 * \rho_\varepsilon)(\dots)$ have value arbitrarily close to each other. So lemma 3 will estimate the difference between these two functions by a positive number arbitrary close to zero.

Lemma 3:

Let $H_2^\varepsilon(\dots), f_1(\dots)$ and $f_2(\dots)$ be functions defined in (17), (15) and (18) respectively and $\rho_\varepsilon(\dots)$ be the function of class $C_0^\infty(\square^{n+2})$ defined above. Then for arbitrary positive real number α , described during the definition of function $H_1(\dots)$ there exists $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$ and for all

$(t,p,u) \in \hat{Q} \times K, t \in [t_0 + \delta, b - \delta]$ the following hold :

$$| y f_2(t,p,u) - y (f_1 * \rho_\varepsilon)(t,p,u) | < (\alpha/4).$$

Proof : For $(t,p,u) \in \hat{Q} \times K$, the following estimation holds:

$$\begin{aligned} &| y f_2(t,p,u) - y (f_1 * \rho_\varepsilon)(t,p,u) | \\ &= | y | | f_2(t,p,u) - (f_1 * \rho_\varepsilon)(t,p,u) | \\ &= | y | \left| \int_{B_\varepsilon(\square^{n+2})} [f_2(t,p,u) - f_1(t-s,p-p',u)] \right. \\ &\quad \left. \rho_\varepsilon(s,p') dsdp' \right| \\ &\leq |y| \int_{B_\varepsilon(\square^{n+2})} | [f_2(t,p,u) - f_1(t-s,p-p',u)] \\ &\quad \rho_\varepsilon(s,p') dsdp' | \\ &\leq |y| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} | f_2(t,p,u) - \\ &\quad f_1(t-s,p-p',u) | \\ &= |y| \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} | f(t, -H_{2y}^\varepsilon(t,p), u) - \\ &\quad f(t, -H_{1y}(t-s,p-p'), u) |. \end{aligned}$$

Since the function $f(\dots)$ is uniformly continuous on the compact set $\bar{Q} \times K, t \in [0, b]$, then by lemma 1, we get that

$$| y | \sup_{\substack{u \in K \\ (t,p) \in \hat{Q}, t \in [t_0+\delta, b-\delta] \\ (s,p') \in B_\varepsilon(\square^{n+2})}} | f(t, -H_{2y}^\varepsilon(t,p), u) - \\ f(t, -H_{1y}(t-s,p-p'), u) | \rightarrow 0 \text{ as } \varepsilon \rightarrow 0,$$

and consequently,

$$| y f_2(t,p,u) - y (f_1 * \rho_\varepsilon)(t,p,u) | \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Hence, for an arbitrary positive number α , there exists $\varepsilon' > 0$ such that for all $\varepsilon \leq \varepsilon'$ and for all $(t,p,u) \in \hat{Q} \times K, t \in [t_0 + \delta, b - \delta]$ the following holds:

$$| y f_2(t,p,u) - y (f_1 * \rho_\varepsilon)(t,p,u) | < (\alpha/4). \blacksquare$$

3.1 The Main Theorem

The main result of this work is formulated in the following theorem, which ensures that the function $(t,p) \rightarrow V(t,p), (t,p) \in P$ defined in (9) is a minimum element of the set W .

Theorem 5:

The function $(t, p) \rightarrow V(t, p), (t, p) \in P, t \in [0, b]$ defined in (9) for the problem of Bolza (1)-(4), is the minimum element of the set W (def. 2), that is

$$V(t, p) \leq H(t, p), \text{ for all } H(t, p) \in W, (t, p) \in P.$$

Proof : Suppose that the function $(t, p) \rightarrow H(t, p), (t, p) \in P$ is any function in the set W , then by using (9) with def. 2 of W , we get that

$$\begin{aligned} V(b, p(b)) &= -S_D(b, p(b)) - x(b, p(b))y(b) \\ &= y^0 V_y^0(b, p(b)) - x(b, p(b))y(b) \\ &= y^0 \ell(-V_y(b, p(b)) - x(b, p(b))y(b)) \\ &= y^0 \ell(x(b, p(b)) - x(b, p(b))y(b)) \\ &\leq y^0 H_{y^0}(b, p(b)) - x(b, p(b))y(b) \\ &= H(b, p(b)) - H_y(b, p(b))y(b) - x(b, p(b))y(b) \\ &= H(b, p(b)) + x(b, p(b))y(b) - x(b, p(b))y(b) \\ &= H(b, p(b)). \end{aligned}$$

Thus we obtain $V(b, p(b)) \leq H(b, p(b))$ for all $(t, p) \in P, t \in [0, b]$.

Now, let $t_0 < b$ and consider $\delta > 0$ such that the interval $[t_0 + \delta, b - \delta]$ has a non-empty interior and let $x_0(t_0) = x_0(t_0, p_0(t_0))$ be an arbitrary belonging to T , and $u(\cdot) \in U(t)$, and let the functions $(t, p, u) \rightarrow f_2(t, p, u)$ and $(t, p, u) \rightarrow L_2(t, x, u)$ and $(t, p) \rightarrow H_2^\varepsilon(t, p)$ be as defined in (17) and (18) respectively.

We need here to show that the function $(t, p) \rightarrow H_2^\varepsilon(t, p)$ satisfies the inequality (19), i.e.,

$$\exists \varepsilon' > 0 \mid \forall \varepsilon \leq \varepsilon', H_{2t}^\varepsilon(t, p) + \max\{yf_2(t, p, u) + y^0 L_2(t, p, u)\} \leq \frac{-\alpha}{2} < 0,$$

and this fact implies that the function $(t, p) \rightarrow H_2^\varepsilon(t, p)$ also belongs to the set W .

Now, to prove that the above inequality (19) is hold, we have

$$\begin{aligned} H_{2t}^\varepsilon(t, p) + yf_2(t, p, u) + y^0 L_2(t, p, u) \\ = y^0 L_2(t, p, u) - y^0 (L_1 * \rho_\varepsilon)(t, p, u) + \\ [(H_{1t} + yf_1(\dots, u) + y^0 L_1(\dots, u)) * \rho_\varepsilon](t, p) + \\ yf_2(t, p, u) - y(f_1 * \rho_\varepsilon)(t, p, u) \dots \dots \dots (20) \end{aligned}$$

In order to find the values of the left side of (20), it is sufficient to find the values of each term of the right side in (20).

From lemma 2 we know that for an arbitrary positive real number α which is close to zero, there exists $\varepsilon' > 0$ such that for all $(t, p, u) \in \hat{Q} \times K, t \in [t_0 + \delta, b - \delta]$, we get:

$$|y^0 L_2(t, p, u) - y^0 (L_1 * \rho_\varepsilon)(t, p, u)| < (\alpha/4).$$

Moreover, lemma 3 gives: for an arbitrary positive real number α which is close to zero, there exists $\varepsilon' > 0$, such that for all $\varepsilon \leq \varepsilon'$ and for all $(t, p, u) \in \hat{Q} \times K, t \in [t_0 + \delta, b - \delta]$, we have

$$|yf_2(t, p, u) - y(f_1 * \rho_\varepsilon)(t, p, u)| < (\alpha/4).$$

Therefore, by using the values of all terms in the inequality (16), lemmas 2 and 3, we see that, it is possible to estimate the values of the left side in (20) for all $(t, p) \in \hat{Q}, t \in [t_0 + \delta, b - \delta]$ as follows:

$$\begin{aligned} H_{2t}^\varepsilon(t, p) + yf_2(t, p, u) + y^0 L_2(t, p, u) \\ \leq (-\alpha * \rho_\varepsilon)(t, p) + \frac{\alpha}{4} + \frac{\alpha}{4} = \frac{-\alpha}{2} < 0, \end{aligned}$$

thus we have,

$$H_{2t}^\varepsilon(t, p) + yf_2(t, p, u) + y^0 L_2(t, p, u) < \frac{-\alpha}{2} < 0 \dots \dots \dots (21)$$

Since the right hand side of the above inequality is independent of $u(\cdot)$, then we see that the inequality (19) is satisfied. Therefore we obtain that

$$\begin{aligned} H_{2t}^\varepsilon(t, p) + yf(t, -H_{2y}^\varepsilon(t, p), u) + \\ y^0 L(t, -H_{2y}^\varepsilon(t, p), u) \leq 0 \end{aligned}$$

for all $u \in U(t), t \in [t_0 + \delta, b - \delta]$.

Now, by taking the integration of the last inequality, we get that

$$\begin{aligned} \int_{t_0+\delta}^{b-\delta} H_{2t}^\varepsilon(t, p) dt + \int_{t_0+\delta}^{b-\delta} y(t)f(t, -H_{2y}^\varepsilon(t, p(t)), u) dt \\ + y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{2y}^\varepsilon(t, p(t)), u) dt \leq 0 \dots \dots \dots (22) \end{aligned}$$

and by using the integration by parts for

$$\int_{t_0+\delta}^{b-\delta} y(t) f(t, -H_{2y}^\varepsilon(t, p(t)), u) dt,$$

where $\dot{x}(t) = f(t, -H_{2y}^\varepsilon(t, p(t)), u)$ a.e., on $[a, b]$, we get that :

$$\int_{t_0+\delta}^{b-\delta} y(t) f(t, -H_{2y}^\epsilon(t, p(t)), u) dt$$

$$= -y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta))$$

$$+ y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta))$$

$$+ \int_{t_0+\delta}^{b-\delta} H_{2y}^\epsilon(t, p(t))\dot{y}(t) dt. \dots\dots\dots (23)$$

Hence, by substituting the above equation (23) in the inequality (22), we get that

$$\int_{t_0+\delta}^{b-\delta} H_{2t}^\epsilon(t, p(t))dt - y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta))$$

$$+ y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta))$$

$$+ \int_{t_0+\delta}^{b-\delta} H_{2y}^\epsilon(t, p(t))\dot{y}(t) dt$$

$$+ y^0 \int_{t_0+\delta}^{b-\delta} L(t, H_{2y}^\epsilon(t, p(t)), u) dt \leq 0,$$

and next

$$\int_{t_0+\delta}^{b-\delta} (H_{2t}^\epsilon(t, p(t)) + H_{2y}^\epsilon(t, p(t))\dot{y}(t))dt$$

$$- y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta))$$

$$+ y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta))$$

$$+ y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{2y}^\epsilon(t, p(t)), u) dt \leq 0.$$

Since,

$$\left(\frac{d}{dt}\right)H_2^\epsilon(t, p(t)) = H_{2t}^\epsilon(t, p(t)) + H_{2p}^\epsilon(t, p(t))\dot{p}(t)$$

$$= H_{2t}^\epsilon(t, p(t)) + H_{2y}^\epsilon(t, p(t))\dot{y}(t),$$

Then,

$$\int_{t_0+\delta}^{b-\delta} \left(\frac{d}{dt}\right)H_2^\epsilon(t, p(t))dt - y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta))$$

$$+ y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta))$$

$$+ y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{2y}^\epsilon(t, p(t)), u) dt \leq 0.$$

Thus, we have

$$H_2^\epsilon(b-\delta, p(b-\delta)) - H_2^\epsilon(t_0+\delta, p(t_0+\delta))$$

$$- y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta))$$

$$+ y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta))$$

$$+ y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{2y}^\epsilon(t, p(t)), u) dt \leq 0.$$

Therefore,

$$y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{2y}^\epsilon(t, p(t)), u) dt +$$

$$y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta)) \leq -H_2^\epsilon(b-\delta, p(b-\delta))$$

$$+ y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta)) + H_2^\epsilon(t_0+\delta, p(t_0+\delta)).$$

Now since,

$$-H_2^\epsilon(b-\delta, p(b-\delta)) + y(b-\delta)H_{2y}^\epsilon(b-\delta, p(b-\delta))$$

$$= -y^0 H_{2y^0}^\epsilon(b-\delta, p(b-\delta)) \text{ (see, def. 2 of } W),$$

$$\text{and } -y^0 H_{2y^0}^\epsilon(b-\delta, p(b-\delta))$$

$$\leq -y^0 \ell(-H_{2y}^\epsilon(b-\delta, p(b-\delta))),$$

then we get

$$y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{2y}^\epsilon(t, p(t)), u) dt +$$

$$y(t_0+\delta)H_{2y}^\epsilon(t_0+\delta, p(t_0+\delta)) \leq$$

$$H_2^\epsilon(t_0+\delta, p(t_0+\delta)) - y^0 \ell(-H_{2y}^\epsilon(b-\delta, p(b-\delta)))$$

By the properties of convolution [3, p.58] and

(17) we see that, $H_2^\epsilon(t, p)$ converge to $H_1(t, p)$ unif-ormly in

$\hat{Q}, t \in [t_0+\delta, b-\delta]$. Therefore,

$$y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_{1y}(t, p(t)), u) dt +$$

$$y(t_0+\delta)H_{1y}(t_0+\delta, p(t_0+\delta)) \leq$$

$$H_1(t_0+\delta, p(t_0+\delta)) - y^0 \ell(-H_{1y}(b-\delta, p(b-\delta))),$$

and by definition of $H_1(.,.)$ in (14) we get that

$$y^0 \int_{t_0+\delta}^{b-\delta} L(t, -H_y(t, p(t)), u) dt +$$

$$y(t_0+\delta)H_y(t_0+\delta, p(t_0+\delta)) \leq H(t_0+\delta, p(t_0+\delta))$$

$$- y^0 \ell(-H_y(b-\delta, p(b-\delta))) + \alpha(b-2\delta-t_0).$$

Now, putting $\alpha \rightarrow 0$, and taking the limit $\delta \rightarrow 0$ then we have

$$y^0 \int_{t_0}^b L(t, -H_y(t, p(t)), u) dt + y^0 \ell(-H_y(b, p(b))) \leq H(t_0, p(t_0)) - y(t_0)H_y(t_0, p(t_0)).$$

Since the right hand side of the above inequality is independent of $u(\cdot)$, we observe that

$$\inf_{u(\cdot) \in K} \{y^0 \int_{t_0}^b L(t, -H_y(t, p(t)), u) dt + y^0 \ell(-H_y(b, p(b)))\} \leq H(t_0, p(t_0)) - y(t_0)H_y(t_0, p(t_0)).$$

Since $x(t, p) = -H_y(t, p)$, then we get that

$$\inf_{u(\cdot) \in K} \{y^0 \int_{t_0}^b L(t, x(t, p(t)), u) dt + y^0 \ell(x(b))\} - y(t_0) x(t_0, p(t_0)) \leq H(t_0, p(t_0)).$$

Thus, by definition of $S_D(\cdot, \cdot)$ in (8) we obtain that,

$$-S_D(t_0, p(t_0)) - x(t_0, p(t_0))y(t_0) \leq H(t_0, p(t_0)).$$

Now, since t_0 and $x(t_0, p(t_0))$ with a suitable function $p(t_0) = p_0$ are arbitrary, we see that

$$-S_D(t, p) - x(t, p)y(t) \leq H(t, p) \text{ for all } (t, p) \in P, t \in [0, b]$$

Thus, by definition of the function $(t, p) \rightarrow V(t, p)$ (9), we get that $V(t, p) \leq H(t, p)$, for all $(t, p) \in P, t \in [0, b]$. ■

Example 1: To illustrate the importance of Th. 5, we present the following example. Consider the optimal control problem:

minimize

$$\int_{-1}^{\pi/4} (a(t)x^2(t) + b(t)u^2(t)) dt - (1/2)[x(\pi/4)]^2,$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)) = B(t)u(t) \text{ a.e., in } [-1, \pi/4] \quad u(t) \in U(t) = [-1, 1], t \in [-1, \pi/4], \quad x(-1) = 0, \text{ where}$$

$$a(t) = \begin{cases} -1/2, & 0 \leq t \leq \pi/4, \\ 0, & -1 \leq t < 0. \end{cases}$$

$$b(t) = \begin{cases} 1/2, & 0 \leq t \leq \pi/4, \\ 1, & -1 \leq t < 0. \end{cases}$$

$$B(t) = \begin{cases} 1, & 0 < t \leq \pi/4, \\ -1, & t \in I_{k_1} \cup I_{k_3}, \\ 0, & t \in I_{k_2} \cup \{-1\}, \end{cases}$$

$$I_{k_j} = (-1 + (\frac{1}{2})^{3k+j}, -1 + (\frac{1}{2})^{3k+j-1}),$$

$$j = 1, 2, 3, k = 0, 1, \dots, \bigcup_{k=0}^{\infty} \bigcup_{j=1}^3 I_{k_j} = (-1, 0].$$

To study the existence of a solution to the DPDEDP of the above problem and obtain the optimal pair for the problem by using Th. 5, we help ourselves by resolving the maximum principle (the necessary optimality conditions) for the above problem, that is, $x(t), u(t), y(t)$ and $y^0 < 0$ satisfy the following conditions :

$$dy(t)/dt = -2y^0 a(t) x(t) \text{ a.e., } t \in [-1, \pi/4]$$

$$\max \{y^0 b(t)u^2(t) + y(t)B(t)u(t) + y^0 a(t)x^2(t) | u \in U(t)\}$$

$$= y^0 b(t)u^2(t) + y(t)B(t)u(t) + y^0 a(t)x^2(t) \text{ a.e.,}$$

$$t \in [-1, \pi/4], -y^0 \in [0, \infty), |y(\pi/4)| + y^0 \neq 0,$$

$$\text{and } y(\pi/4) = y^0 \ell_x(x(\pi/4)) = x(\pi/4), y^0 = -1.$$

Then the following triplets $x(t), u(t)$ and $p(t) = (y^0, y(t))$ can be calculated as follows :

$$y^0 = -e, x(t, c_1) = c_1 \sin t, y(t, ec_1) = ec_1 \cos t,$$

$$u(t, c_1) = c_1 \cos t, \text{ where } t \in [0, \pi/4], c_1 \in (-1, 1),$$

$$e \in (1/2, 3/2),$$

$$y^0 = -e, x(t, e) = 0, y(t, e) = 0, u(t, e) = 0,$$

$$t \in [-1, \frac{\pi}{4}],$$

$$y^0 = -e, x(t, c_2)$$

$$= -(c_2/2) \int_t^0 B^2(s) ds, y(t, ec_2) = ec_2,$$

$$u(t, c_2) = (c_2/2) B(t), t \in [-1, 0], c_2 \in (-1, 1).$$

For $p(t) = (y^0, y(t)), t \in [-1, \pi/4]$, we define $u(t, y^0, y)$ and $x(t, y^0, y)$ as follows:

$$u(t, y^0, y) = \begin{cases} \frac{-y}{2y^0} B(t), & t \in [-1, 0], y^0 \in (-3/2, -1/2), y \in (-3/2, 3/2), \\ 0, & t \in [-1, \pi/4], y^0 \in (-3/2, -1/2), y = 0, \\ \frac{-y}{y^0}, & t \in [0, \pi/4], y^0 \in (-3/2, -1/2), |y| < (3/2)|\cos t|. \end{cases}$$

$$x(t, y^0, y) = \begin{cases} \frac{y}{2y^0} \int_t^0 B^2(s) ds, & t \in [-1, 0], y^0 \in (-3/2, -1/2), y \in (-3/2, 3/2), \\ 0, & t \in [-1, \pi/4], y^0 \in (-3/2, -1/2), y = 0, \\ \frac{-y}{y^0} t g t, & t \in [0, \pi/4], y^0 \in (-3/2, -1/2), |y| < (3/2)|\cos t|. \end{cases}$$

$$H(t, y^0, y) = \begin{cases} \frac{-y^2}{4y^0} \int_t^0 B^2(s) ds, & t \in [-1, 0], y^0 \in (-3/2, -1/2), y \in (-3/2, 3/2), \\ 0, & t \in [-1, \pi/4], y^0 \in (-3/2, -1/2), y = 0, \\ \frac{y^2}{2y^0} t g t, & t \in [0, \pi/4], y^0 \in (-3/2, -1/2), |y| < (3/2)|\cos t|. \end{cases}$$

Therefore

$$H_{y^0}(t, y^0, y) = \begin{cases} \frac{y^2}{4y^0} \int_t^0 B^2(s) ds, & t \in [-1, 0], y^0 \in (-3/2, -1/2), y \in (-3/2, 3/2), \\ 0, & t \in [-1, \pi/4], y^0 \in (-3/2, -1/2), y = 0, \\ \frac{-y^2}{2y^0} t g t, & t \in [0, \pi/4], y^0 \in (-3/2, -1/2), |y| < (3/2)|\cos t|. \end{cases}$$

$$H_y(t, y^0, y) = \begin{cases} \frac{-y}{2y^0} \int_t^0 B^2(s) ds, & t \in [-1, 0], y^0 \in (-3/2, -1/2), y \in (-3/2, 3/2), \\ 0, & t \in [-1, \pi/4], y^0 \in (-3/2, -1/2), y = 0, \\ \frac{y}{y^0} t g t, & t \in [0, \pi/4], y^0 \in (-3/2, -1/2), |y| < (3/2)|\cos t|. \end{cases}$$

Now, define $S_D(t, y^0, y)$ (see, (8)) as follows:

$$S_D(t, y^0, y) = \begin{cases} \frac{-y^2}{4y^0} \int_t^0 B^2(s) ds, & t \in [-1, 0], y^0 \in (-3/2, -1/2), y \in (-3/2, 3/2), \\ 0, & t \in [-1, \pi/4], y^0 \in (-3/2, -1/2), y = 0, \\ \frac{y^2}{2y^0} t g t, & t \in [0, \pi/4], y^0 \in (-3/2, -1/2), |y| < (3/2)|\cos t|. \end{cases}$$

It is simple to verify that the functions $(t, y^0, y) \rightarrow H(t, y^0, y)$ as described above, are

Lipschitz functions in the sets of t and (y^0, y) and they satisfy all conditions in the set W

(def. 2). Thus, the functions $(t, y^0, y) \rightarrow H(t, y^0, y)$ in the sets of t and (y^0, y) described above, are belongs to set W . And we see that for $p(t) = (y^0, y(t))$, when $t \in [-1, \pi/4]$, y^0 is any given number in the interval $(-3/2, -1/2)$, and $y = 0$, it is not difficult to check that the function (9) for the above problem: $V(t, y^0, y) = -S_D(t, y^0, y) - x(t, y^0, y)y$, which is equal to zero, satisfy (11) and the boundary condition of Th.4, thus it belongs to the set W .

Therefore, from all above and Th.5, we observe that the function $(t, y^0, y) \rightarrow V(t, y^0, y)$, $t \in [-1, \pi/4]$ which is equal to zero is a minimum element of the set W , and by using Th.4, we find that $x(t) = 0$ and $u(t) = 0$, $t \in [-1, \pi/4]$ is an optimal pair for the above problem.

Remark 1.

As shown before in section 1, the Lagrange problem can be obtained from Bolza problem (1)-(4) by taking $x \rightarrow \ell(x)$ equal to zero. And since these two problems have the same PDED(7) and DPDED(11) [6], then it is not difficult to check that the main Th. 5 is satisfied for the Lagrange problem. Now, if we take the same example 1 for the Bolza problem (1)-(4), but with $b = \pi$ and $\ell(x(\pi)) = 0$, we get that the Lagrange problem. And by using the same manner in the above example 1, we find that $x(t) = 0$ and $u(t) = 0$, $t \in [-1, \pi]$ is an optimal pair for the Lagrange problem.

4. Conclusion:

According to the properties of the dual value function $S_D(., .)$ (8), and the function $V(., .)$ (9), for the problem of Bolza (1)-(4), we observe that, the main Th. 5 of this paper, identifies that (in the case where there is not a unique solution for the problem (1)-(4)) the minimum element of the set W which satisfies the Lipschitz condition is an approximate of optimal, i.e., if the function $V(., .)$ (9) is evaluated along any admissible trajectory such that it is a Lipschitz and satisfying the solution to the DPDED (11), (12) and the boundary condition of Th.4, then that trajectory is an optimal.

Remark 2:

It is clear in the control theory that there many not exist a minimizer for the Bolza

problem (1)-(4) [4]. Therefore it is natural to look for an approximate minimum for this problem. Thus, for the future study, we can use the method in section 3 for studying the existence of an approximate solution for the Bolza problem.

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الخلاصة

تم إثبات النظرية الأساسية التي تتعامل مع وجود حل اصغر للمعادلة التفاضلية الجزئية الموجهة للبرمجة الديناميكية لمشاكل السيطرة المثلى لبولزا ولاكرانج. كما تم إعطاء مثال يوضح قيمة هذه النظرية. تم إعطاء وصف لخواص دالة القيمة ودالة القيمة الموجهة لكل من مشكلة بولزا ولاكرانج. علاوة على ذلك، تم تقديم النظرية التي تتعامل مع وجود حل اعظم للمعادلة التفاضلية الجزئية للبرمجة الديناميكية لكل من مشكلة بولزا ولاكرانج والذي يحقق شرط ليبشترز والذي هو ايضاً يمثل دالة القيمة لهذه المشكلة.