EXTREMITY CONCEPTS OF LIFTING MODULES

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Abstract

Recall that an R-module M is lifting if every submodule of M lies above a direct summand of M. In this paper, we introduce and study the classes of modules which are extremity of lifting modules. We call an R-module M is strongly lifting if every submodule of M lies above a stable direct summand of M. Also, we call R-module M is S-lifting if every stable submodule of M lies above a direct summand of M. In fact, the following proper hierarchy is concluded:

Strongly lifting modules $\Rightarrow$ Lifting modules $\Rightarrow$ S-lifting modules

Some counter examples are given to separate these concepts. Also, many characterizations and properties of strongly lifting (respectively, S-lifting) modules are obtain. It is shown that a module M is strongly lifting if and only if M is lifting and M is SS-modules. Moreover, we investigate whether the class of strongly lifting (respectively, S-lifting) modules are closed under particular class of submodules, direct summands and direct sums. It shown that a finite direct sum of S-lifting modules is S-lifting.

Kew words: Lifting modules, (quasi-)discrete modules, supplemented modules.

1-Introduction and Preliminaries

The dual concepts of (quasi-)continuous (respetively, extending) modules namely (quasi-) discrete (respectively, lifting) modules were studied extensively by many authors. S. Mohamed and S. Singh [7] defined discrete modules under the name dual-continuous modules, and Oshiro in [8], quasi-discrete modules were given under the name "quasi-semiperfect module". Lifting modules were studied by S. Singh [10], under the name "semi-dual-continuous modules".

Since thirty years ago, the developments of modules with lifting (extending) property have been a major area in ring and module theory. Recently, extremities concepts of extending modules introduced and studied in [2]. Motivated by these ideas, in this paper we introduce and study extremities concepts of lifting modules.

Throughout this paper all rings are associative with identity element and all modules are unitary left modules. A submodule N of a module M is called small in M (denoted by N<<M) provided $M \neq N + K$ for any proper submodule K of M. A module M is called hollow if every proper submodule of M is small in M. Let N be submodule of a module M. A submodule X is called supplement of N in M provided M =N +X and $M \neq N + Y$ for any proper submodule Y of X. It is easy to check that X is a supplement of N in M if and only if M =N + X and $N \cap X << X$. A submodule N is called supplement of M if N is a supplement of some submodule of M. If every submodule of M has a supplement in M, then M is called supplemented. A module M is called imply supplemented if for any submodules A and B of M with M =A + B, there exists a supplement X of A such that $X \subseteq B$. A submodule A of M is called coclosed in M if whenever A/B<<M/B implies that A=B for every submodule B of M with $B \subseteq A$. Clearly, every supplement submodule is coclosed but the converse is true for imply supplemented modules. A submodule U of a module M is called lies above X in M if U/X<<M/X. A module M is called lifting (or satisfies (D_1)) if every submodule of M lies above a direct summand of M. A module M is said to have the condition (D_2) in case of if A is a submodule of

M such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M. Also, a module M is said to have the condition (D_3) in case of if A_1 and A_2 are direct summands of M with $A_1 + A_2 = M$, then $A_1 \cap A_2$ is a direct summand of M. A module

M is called discrete if it has the conditions (D_1) and (D_2). Also, a module M is called quasi-discrete if it has the conditions (D_1) and (D_3). Note that all concepts which mentioned above and for more information about these concepts see ([11], [6], [5], [3]). Recall that a submodule N of an R-module M is fully invariant if f (N) $\subseteq$ N for each R-endomorphism f of M [11]. Moreover, a stronger than that of fully invariant submodules M.S.Abbas [1] introduced the concept of stable submodules. A submodule N of an R-module M is called stable, if $f(N) \subseteq N$ for each homomorphism $f: N \longrightarrow M$. An R-module M is fully stable if every submodule of M is a stable.

2-Strongly lifting Modules

Recall that an R-module M is called strongly extending if, every submodule of M lies under a stable summand of M [2]. As a stronger extremity concept than of lifting modules and dual concept of strongly extending modules, we introduce the following concept:

Definition (2.1):

An R-module M is called strongly lifting if, every submodule of M lies above a stable direct summand of M.

It follows immediately from the definitions, every strongly lifting module is lifting, but the converse is not true in general (see (Remarks (2.5) (2), (3)).

Firstly, the next result gives characterizations of strongly lifting modules. Compare this result with [11, Theorem (41.11), P. 357].

Proposition (2.2):

The following statements are equivalent for an R-module M:
(1) M is strongly lifting;
(2) For every submodule N of M there is a decomposition M $=M_1 \oplus M_2$ such that $M_1 \subseteq$ N with M_1 is a stable submodule of M and $N \cap M_2 <<$ M.
(3) Every submodule N of M can written as N $=N_1 \oplus N_2$ where N_1 is a stable direct summand of M and $N_2 <<$ M.

(4) M is amply supplement and every coclosed submodule of M is a stable direct summand of M.

Proof:

(1) $\Rightarrow$ (2). Suppose that every submodule of M lies above a stable direct summand of M. Let N be a submodule of M, thus there is a decomposition M $=M_1 \oplus M_2$ with M_1 is stable submodule of M and $N/M_1 << M/M_1$. Since $M/M_1 \cong M_2$ and $N/M_1 = (M_1 \oplus (N \cap M_2)/M_1) \cong N \cap M_2$ [11] . It follows that $N \cap M_2$ is small in M_2 and hence (by [11, (19.3) (5), P.160]) is small in M.

(2) $\Rightarrow$ (3). Let N be a submodule of M, by (2), there exists a decomposition M $=M_1 \oplus M_2$ such that $M_1 \subseteq$ N with M_1 is a stable submodule of M and $N \cap M_2 <<$ M. But $N = N \cap M = N \cap (M_1 \oplus M_2) = (N \cap M_1) \oplus (N \cap M_2) = M_1 \oplus (N \cap M_2)$.

(3) $\Rightarrow$ (4). Suppose that M $= A \oplus B$ for submodules A, B of M. To prove that B contains a supplement of A. By (3), B $= C \oplus D$ where C is a stable direct summand of M and $D << $ M. Hence, M $= A + C$ (by smallness of D). Again by (3), $A \cap C = H \oplus K$ where $K << $ M and H is a stable direct summand of M. Write M $= H \oplus N$. Thus, $K << $ C and $C = H \oplus T$ where T= $C \cap N$. Now, we claim that T is a supplemented of H +K in C. To show that consider a submodule C = E +H and so E =T since T is a supplement of H in C. Hence T is a supplement of H + K = $A \cap C$ in C. Then M $= A + C = A + (A \cap C) + T- A + T$ and moreover, $A \cap H = (A \cap C) \cap H << $ H. Thus H is a supplement of A in M. Therefore, M is supplement.

Now, let A be a coclosed submodule of M. Hence, by (3), A $= B \oplus C$ where B is a stable direct summand of M and $C << $ M. Then, A lies above B in M. Hence, A =H (i.e) A is a stable direct summand of M.

(4) $\Rightarrow$ (1). Let N be a submodule of M. If N is small submodule of M, then it lies above the zero submodule which is a stable direct summand of M. If N is not small in M, then by [5], it lies above a coclosed submodule of M and by using (4), N lies above a stable direct summand of M. Thus, M is strongly lifting. ◘

Since, by [2, lemma (2.1.6)], every fully invariant direct summands are stable, so we can rewrite all results in this paper with "stable direct summand" being replaced by "fully invariant direct summand". For example, we can restate proposition (2.2) as follows:

Proposition (2.3):

The following statements are equivalent for an R-module M:

(1) M is strongly lifting;

(2) For every submodule N of M there a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ with M_1 is a fully invariant submodule of M and $N \cap M_2 \ll M$.

(3) Every submodule N of M can written as $N = N_1 \oplus N_2$ where N_1 is a fully invariant direct summand of M and $N_2 \ll M$.

(4) M is amply supplement and every coclosed submodule of M is a fully invariant direct summand of M.

Recall that an R-module M is SS-module if, every direct summand of M is stable [2]. The following result provides us an important characterization of strongly lifting modules.

Proposition (2.4):

An R-module M is strongly lifting if and only if M is lifting and M is SS-module.

Proof:

($\Rightarrow$). Assume that M is strongly lifting, so directly by definition, M is lifting. Also, let D be a direct summand of M, hence D is coclosed submodule of M [5]. Hence, by proposition (2.2), D is a stable submodule of M, so M is SS-module.

($\Leftarrow$). Conversely, let N be a submodule of M. By lifting property of M, N lies above a direct summand D of M. But by SS-module property of M, D is stable submodule. So, M is strongly lifting. ◘

Remarks (2.5):

(1) The concepts of lifting modules and SS-modules are different. In fact, Z as Z-module is SS-module since the only direct summands of Z as Z-module are (0) and Z, so they are stable of Z_Z, while Z is not lifting Z-module [3]. In other direction, the vector space $V = F^{(2)}$ over the field F is not SS-module [2], but it is lifting F-module since it is semi-simple [6].

(2) By using the same above argument in (1), we can use the vector space $V = F^{(2)}$ over the field F as counter example which is lifting F-module, but it is not strongly lifting.

(3) Consider the Z-module $M = (Z/pZ) \oplus (Z/p^2 Z)$ where p is a prime integer. By [3, 32.20], M is lifting. But M is not strongly lifting since $N = (Z/pZ) \oplus (pZ/p^2 Z)$ is a submodule of M which is not small in M and N does not contain any nonzero stable direct summand of M.

It is known that every closed submodule (and then direct summand) of a strongly extending module is strongly extending. As dual result, we have the following:

Proposition (2.6):

Every coclosed submodule (and then direct summand) of a strongly lifting module is strongly lifting.

Proof:

Let D be a coclosed submodule of a strongly lifting module M. By [11, 41.7], D is amply supplemented. Now, let C be a coclosed submodule of D, thus C is coclosed submodule of M [11]. From preposition (2.2) (4), since M is strongly lifting C is stable direct summand of M. But $C \subseteq D \subseteq M$, thus C is a direct summand of D [9, lemma (2.4.3)]. Also, easily one can check that since $C \subseteq D \subseteq M$ and C is a stable submodule of M, then C is a stable submodule of D. Thus, by preposition (2.2) (4), D is strongly lifting. ◘

Motivated by in [11, Theorem (41.11)], we obtain further characterizations of strongly lifting modules.

Proposition (2.7):

The following statements are equivalent for an R-module M:

(1) M is strongly lifting;

(2) For each submodule U of M, there is an idempotent $f \in \text{End}(M)$ such that f(M) is a stable submodule of M with $f(M) \subseteq U$ and $(I-f)(U) \ll (I-f)(M)$, where I is the identity mapping of M.

(3) For each submodule U of M, there exists a stable direct summand X of M with $X \subseteq U$, $U = X + Y$ and $Y \ll M$.

Proof:

(1) $\Rightarrow$ (2). Assume that M is strongly lifting and let U be a submodule of M. By proposition (2.2) (2), there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq U$ with M_1 is a stable submodule of M and $U \cap M_2 \ll M$. Now, let $\pi_1 : M = M_1 \oplus M_2 \to M_1$ be a projection mapping. Thus, it easy check that π_1 is an idempotent with $\pi_1(M_1) = M_1 \subseteq U$. Also, $(I - \pi_1)(M) = M_2$. Since $U \cap M_2 \ll M$ and M_2 is a direct summand of M, then $U \cap M_2 \ll M_2$ [11]. It is enough to show that $(I - \pi_1)(U) = U \cap M_2$. To see this, let $x \in U$, then $(I - \pi_1)(x) = x \pi_1(x)$. But $\pi_1(M_1) \subseteq U$ and $(I - \pi_1)(M) = M_2$, then $(I - \pi_1)(U) \subseteq M_2$ and so $(I - \pi_1)(U) \subseteq U \cap M_2$. On other direction, let $y \in U \cap M_2$, then $y \in M_2$. But $(I - \pi_1)(M) = M_2$, so there is $m \in M$ such that $(I - \pi_1)(m) = w$. Thus, $m - \pi_1(m) = w$ and so $m = w + \pi_1(m) \in U$. Then, $w \in (I - \pi_1)(U)$. So, $(I - \pi_1)(U) = U \cap M_2$. But $U \cap M_2 \ll M_2$, hence $(I - \pi_1)(U) \ll (I - \pi_1)(M)$.

(2) $\Rightarrow$ (3). Let U be a submodule of M. Thus, by using (2), there is an idempotent $f \in End (M)$ such that $f(M) \subseteq U$ with $f(M)$ is a stable submodule of M and $(I - f)(U) \ll (I - f)(M)$. Now, since f is idempotent then $M = f(M) \oplus (I - f)(M)$ [5, (8.4)]. Put $f(M) = A$, then $U = U \cap M = U \cap (X + (I - f)(M)) = X + (U \cap (I - f)(M))$. Set $(U \cap (I - f)(M)) = Y$, thus, $U = X + Y$. Now, it is enough to show that $Y \ll M$. Since $Y = U X + Y (I - f)(M) \subseteq (I - f)(U)$ and $(I - f)(U) \ll (I - f)(M)$, then $Y \ll (I - f)(M)$ [7]. But, $(I - f)(M)$ is a direct summand of M, so $Y \ll M$ [11].

(3) $\Rightarrow$ (1) Let N be a submodule of M. By using the hypothesis of (3), there is a stable direct summand X of M such that $N = X + Y$ with $X \subseteq N$ and $Y \ll M$. Write $M = X \oplus D$, where D is a submodule of M. We claim that D is a supplement of N in M. Since $M = X + D \subseteq N + D$, then $M = N + D$. If there is $H \subseteq D$ with $M = N + H$, thus $M = X + Y + H = X + H$. But D is supplement of X in M, thus $D = H$ and so D is supplement of N in M which implies that $N \cap D \ll D$. So, $N \cap D \ll M$ since D is direct summand of M [11]. Therefore, by proposition (2.2), M is strongly lifting. ◪

3-Strongly (quasi-)discrete Modules.

In this section, classes of modules which are stronger than that of (quasi-) discrete modules are introduced. Firstly, consider the following conditions for an R-module M:

(SD$_1$) Every submodule of M lies above a stable direct summand of M.

(SD$_2$) If N is a submodule of M such that M/ N is isomorphic to a direct summand of M, then N is a stable direct summand.

(SD$_3$) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is a stable direct summand of M.

Definition (3.1):

An R-module M is called strongly discrete if it satisfies the conditions (SD$_1$) and (SD$_2$).

Definition (3.2):

An R-module *M* is called strongly quasi-discrete if it satisfies the conditions (SD$_1$) and (SD$_3$).

Remarks and Examples (3.3):

(1) It is clear that every strongly discrete (resp. strongly quasi-discrete) module is discrete (resp. quasi-discrete) while the converses are not true in general (see (8)).

(2) It is known that, every hollow module is quasi-discrete [6]. Here, we conclude that the class of hollow modules is contained in the class of strongly quasi-discrete modules. In fact0, let M be a hollow module and A be a submodule of M, then A is small in M. Let $M = (0) \oplus M$ be a decomposition of M , hence $(0) \subseteq A$ with (0) is a stable direct summand of M and $A \cap M = A \ll M$. So M has (SD$_1$) condition. Also, let A_1 and A_2 are direct summands of M such that $M = A_1 + A_2$. Thus, A_1 (resp. A_2) is either M (resp. (0)) or (0) (resp. M). If $A_1 \neq A_2$ and since A_1 is small in M, then $A_1 = (0)$ (since (0) the only small direct summand of M). Similarly for A_2. Both cases implies that $A_1 \cap A_2 = (0)$ is a stable direct summand of M. If $A_i = M$ (i=1,2), then $A_1 \cap A_2 = M$ is a stable direct summand of M. Hence, M has (SD$_3$) condition. So, M is strongly quasi-discrete.

(3) The converse of (2) is not true in general, for example, the Z-module Z_6 is strongly quasi-discrete which is not hollow.

(4) If an R-module M has (SD$_2$) condition then M has (SD$_3$). Indeed, if N and H are

direct summands of M with M =N + H. Let M= $N \oplus D$. Thus, $D \cong (N + H)/ N \cong H/(N \cap H)$. Hence, by (SD$_2$) condition $N \cap H$ is stable direct summand of M. So, M valid (SD$_3$) condition.

(5) By (4), every strongly discrete module is strongly quasi-discrete.

(6) The converse of (4) is not true in general, for example, $Z_{p\infty}$ is strongly quasi-discrete Z-module since it is hollow [6]. While, $Z_{p\infty}$ is not strongly discrete Z-module since for any non-zero submodule A of $Z_{p\infty}$, $Z_{p\infty}/A \cong Z_{p\infty}$ [7], but A is stable, not direct summand of $Z_{p\infty}$,(i.e) M has no (SD$_2$) condition.

(7) The Z-module Z is not strongly (quasi-) discrete. In fact, Z has no (SD$_1$) condition. If not, let H =3Z, then by (SD$_1$) property of Z, then there exists a=$A_1 \oplus A_2$ with $A_1 \subseteq 3Z$ and $A_2 \cap 3Z << Z$ which is a contradiction since 3Z + 2Z =Z and $2Z \neq Z$

(8) It is known that, every semi-simple module is (quasi-) discrete [6]. This results is not valid true for strongly (quasi-) discrete modules. For example, the vector space V=$F^{(2)}$ over the field F is semi-simple F-module which is not strongly (quasi-) discrete F-module (by lemma (2.4) and [2 , Examples (2.3.3) (4)]).

(9) It is clear that every fully stable semi-simple module is strongly (quasi-)discrete.

(10) It is known that, every commutative ring with unity has (D$_3$) [6]. We can generalize this result as: every SS-module has (SD$_3$) condition (and then (D$_3$) condition). Let A_1 and A_2 are direct summands of M such that M =A_1 + A_2. By SS-module property of M and [2, proposition.(2.2.6)], $A_1 \cap A_2$ is a stable direct summand of M. So, M has (SD$_3$) condition.

(11) We can easily show that, if M is an indecomposable module, then the following statements are equivalent:
    (a) M is hollow;
    (b) M has (D$_1$) condition;
    (c) M has (SD$_1$) condition.

The following lemmas give useful characterizations of strongly (quasi-)discrete modules. Firstly, we can rewrite proposition (2.4) as follows:

Lemma (3.4):
An R-module M has (SD$_1$) condition if and only if M has (D$_1$) condition and M is SS-module.

Lemma (3.5):
An R-module M has (SD$_2$) condition if and only if M has (D$_2$) condition and M is SS-module.

Proof:
Directly by definitions (SD$_2$) condition implies (D$_2$) condition. Now, let N be a direct summand of a module M. Write M =$N \oplus H$, where H is a submodule of M. Thus, M/N = $(N \oplus H)/N \cong H/(N \cap H) \cong H$. By (SD$_2$) property of M, N is a stable submodule of M. So, M is SS-module.

Conversely, let N be a submodule of M such that M/ N is isomorphic to a direct summand of M. From (D$_2$) property of M, N is a direct summand of M and by SS-module property of M, N is stable submodule of M. So M has (SD$_2$) condition. ◘

Proposition (3.6):
An R-module M is strongly (quasi-) discrete if and only if M is (quasi-) discrete and M is SS-module.

Since every indecomposable module is SS-module [2], thus the next corollaries are concluded:

Corollary (3.7):
Let M be an indecomposable R-module. Then M is strongly quasi-discrete if and only if M is quasi-discrete.

Since every module with local endomorphisms ring is indecomposable [4, Theorem (3.52)]. Hence, we have:

Corollary (3.8):
Every quasi-discrete module with local endomorphisms ring is strongly quasi-discrete.

Recall that an R-module M is directly finite if, M is not isomorphic to a proper direct summand of itself [6].On other direction, an R-module M has the cancellation property if whenever $M \oplus X \cong M \oplus Y$, then $X \cong Y$ [6, p.9]. Also, an R-module M has the internal

cancellation property whenever $M = A_1 \oplus B_1 = A_2 \oplus B_2$ with $A_1 \cong A_2$, then $B_1 \cong B_2$ [6, p.9].

It is well-known that a directly finite (quasi)-discrete module has the (internal) cancellation property [6, corollary (4.20)]. Since, every SS-module is directly finite [2, Lemma (2.3.24)]. Thus, directly we have the following results:

Proposition (3.9):
Every strongly discrete module has the cancellation property.

Proposition (3.10):
Every strongly quasi-discrete module has the internal cancellation property.

The following propositions investigate further characterizations of strongly (quasi-)discrete modules:

Proposition (3.11):
The following statements are equivalent for an *R*-module *M*:
  (1) *M* is strongly discrete;
  (2) *M* has (SD_1) condition and (D_2) condition;
  (3) *M* has (D_1) condition and (SD_2) condition.
Proof: By Lemma (3.4) and Lemma (3.5). ◘

Proposition (3.12):
The following statements are equivalent for an *R*-module *M*:
  (1) *M* is strongly (quasi-)discrete;
  (2) *M* has (SD_1) condition and (D_3) condition;
Proof:
By Lemma (3.4). ◘

The next result ensures that the strongly (quasi-) discrete property is inherited by direct summands.

Proposition (3.13):
A direct summand of strongly (quasi-) discrete module is strongly (quasi-) discrete.
Proof:
It follows immediately by using Proposition (3.6) and the fact that (quasi-)discrete property (resp. SS-module property) is inherited by direct summands [6, lemma (4.7)] (resp. [2, Proposition (2.2.25)]). ◘

4- S-lifting modules.

Recall that an R-module M is S-extending if every stable subnodule of M is essential in a direct summand of M [2]. As a dual concept of S-extending modules and as a generalization of Lifting modules we introduce the following concept:

Definition (4.1):
An R-module M is called stable lifting (shortly, S-lifting) if, every stable submodule of M lies above a direct summand of M.

It is clear that lifting modules, semi-simple modules and hollow modules are trivial examples of S-lifting modules. Moreover, S-lifting modules is proper generalization of lifting modules since, for example, *Z* as *Z*-module is S-lifting since *(0)* and *Z* are the only stable submodule of *M* and they lie above a direct summands of *Z*. In other hand, it is easy to check that *Z* is not lifting *Z*-module.

Firstly, we have the following characterizations of S-lifting modules.

Proposition (4.2):
The following statements are equivalent for an *R*-module *M*:
(1) *M* is S-lifting**;**

(2) For every stable submodule N of M there a decomposition $M = M_1 \oplus M_2$ such that $M_1 \subseteq N$ with and $N \cap M_2 << M$;
(3) For each stable submodule U of M, there is an idempotent $f \in End(M)$ such that $f(M) \subseteq U$ and $(I-f)(M) << (I-f)(M)$;
(4) Every stable submodule N of M can written as $N = N_1 \oplus N_2$ where N_1 is a direct summand of M and $N_2 << M$;
(5) For each stable submodule U of M, there exists a direct summand X of M with $X \subseteq U$, $U = X + Y$ and $Y << M$.

Proof:
By using the same argument of [11, Theorem (41.11), P.357]. ◘

Definition (4.3):
An R-module M is called stable hollow (shortly, S-hollow) if, every stable proper submodule of M is small.

It is clear that every hollow module is S-holllow while the converse is not true in general. For example Z is S-hollow Z-module which is not hollow.

The next result gives the relationship between S-lifting modules and S-hollow modules. Compare this result with [6, corollary (4.9)].

Proposition (4.4):

An indecomposable module is S-hollow if ad only if S- lifting.

Proof:

Suppose that *M* is S-hollow and let *A* be a stable submodule of *M*, thus *A* is small and hence $A = (0) \oplus A$ with *(0)* is direct summand of *M* and $A << M$. So, by proposition (4.1) (4), *M* is S-lifting.

Conversely, suppose that *M* is S-lifting and let *A* be stable proper submodule of *M*, thus by proposition (4.1) (4), $A = N \oplus D$ where *N* is a direct summand of *M* and $D << M$. Now, since *M* is indecomposable, then either $N = (0)$ or $N = M$. If $N = M$, then $M = N \subseteq A$ which implies that $A = M$ which is a contradiction. So, $N = (0)$, thus $A = D << M$. Hence, *M* is S-hollow. ◘

It is known that a direct sum of lifting modules need not be lifting [3]. In the next result, we assert that this property is partially valid for S-extending modules.

Proposition (4.5):

A finite direct sum of S-lifting modules is lifting.

Proof:

Let $M = \oplus_{i=1}^{n} M_i$, where M_i is S-lifting module for each $i = 1\ldots n$. Let A be a stable submodule of M. By [1, proposition (4.5)], $A = \oplus_{i=1}^{n} (A \cap M_i)$. Also, it is easy to check that $A \cap M_i$ is a stable submodule of M_i for each $i = 1\ldots n$. Since, M_i is S-lifting for each $i = 1\ldots n$, thus $A \cap M_i = D_i \oplus K_i$, where D_i is a direct summand of M_i and $K_i << M_i$ for each $i = 1\ldots n$. Put $D = \oplus_{i=1}^{n} D_i$ and $K = \oplus_{i=1}^{n} K_i$. Thus, $A = D \oplus K$ with D is a direct summand of M [4] and $K << M$ [11, 19.3(3)]. Hence, by proposition ((4.2) (4)), $M = \oplus_{i=1}^{n} M_i$ is S-lifting.

Corollary (4.6):

A finite direct sum of lifting (hollow) modules is S-lifting.

Example (4.7):

Consider $M = (Z/pZ) \oplus (Z/p^3 Z)$ where p is a prime integer. Since Z/pZ and $Z/p^3 Z$ are hollow Z-modules, then by corollary (4.6), M is S-lifting. While, $M = (Z/pZ) \oplus (Z/p^3 Z)$ is not lifting Z-module [5, Example (23.5)].

We do not know in general whether S-lifting property is inherited by stable submodules. The following result gives partially answer. Firstly, let X be an R-module, recall that an R-module M is called stable-injective relative to X if for each stable submodule A of X, each R-homomorphism f:A→M can be extended to an R-homomorphism g:X→M [2]. In [2, Proposition (3.2.14)], it is shown that if X a stable submodule of S-extending such that M is a stable-injective relative to X, then X is S-extending. Thus, as dual result, we have the next result.

Proposition (4.8):

Let M be a stable-injective relative to a stable submodule N. If M is S-lifting, then so is N.

Proof:

Let A be a stable submodule of N. since M is stable-injective relative to N, thus A is a stable submodule of M. But, M is S-lifting, then A lies above a direct summand D of M (i.e) A/D << M/D. Since $D \subseteq N \subseteq M$ and D is a direct summand of M, then D is direct summand of N [9, lemma (2.4.3).]. Also, by [3], A/D << N/D. So, N is S-lifting. ◘

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