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NUMERICAL METHODS FOR SOLVING LINEAR FREDHOLM-VOLTERRA INTEGRAL EQUATIONS

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Abstract

The aim of this work is to use some numerical methods to solve linear Fredholm-Volterra integral equations of the first and second kinds. These methods namely are the repeated Trapezoidal method and the repeated Simpson's 1/3 method. Numerical examples are presented to show the efficiency and accuracy of the presented work.

1-Introduction

Integral equations have received considerable interest in the mathematical literatures, because of their many fields of application in different areas of sciences (see, for example [1-4]). Many authors give numerical solutions for different types of Fredholm integral equations and Volterra integral equations (see, for example [3-8]).

In this paper, we show how the numerical methods which are based on the repeated Trapezoidal quadrature formula and repeated Simpson's 1/3 quadrature formula can be used to solve the linear Fredholm-Volterra integral equation of the second kind

$$u(x) = g(x) + \lambda\int_a^b L(x,y)u(y)dy + \mu\int_a^x K(x,y)u(y)dy \text{ .... } (1.1)$$

and the linear (FVIE) of the first kind:

$$g(x) = \lambda\int_a^b L(x,y)u(y)dy + \mu\int_a^x K(x,y)u(y)dy \text{ .... } (1.2)$$

where $a \le x \le b$, λ and μ are real numbers, g(x), L(x, y) and K(x, y), are given continuous functions and u(x) is the unknown function to be determined. If μ = 0 then equation (1.1) is called linear Fredholm integral equation. Also, if λ = 0 then equation (1.1) is called linear Volterra integral equation. It is known that if λ = 0 then an analytic solution exists for special types of equation (1.1), (see, [3, 4, 6]), otherwise the solution is difficult.

2-The Repeated Trapezoidal Method:

Consider the linear Fredholm-Volterra integral equation of second kind given by equation (1.1). To solve this equation on the finite interval [a, b], we divide it into n smaller intervals of width h, where h = (b − a)/n. The i-th point of subdivision is denoted by x_i, such that $x_i = a + ih$, i = 0, 1, …, n. The approximate solution will be defined at the mesh point x_i is denoted by $u(x_i)$ and is given by.

$$u(x_i) = g(x_i) + \lambda\int_a^b L(x_i,y)u(y) + \mu\int_a^{x_i} K(x_i,y)u(y)dy,\ i = 0,1,\ldots,n \text{ ... . } (2.1)$$

If we approximation the integrals that appeared in equation (2.1) by the repeated Trapezoid formula which will yield the following system of equations:

$$u_0 = g_0 + \frac{\lambda h}{2}\left(L_{0,0}u_0 + 2\sum_{j=1}^{n-1} L_{0,j}u_j + L_{0,n}u_n\right),$$

$$u_i = g_i + \frac{h}{2}\left(\left(\lambda L_{i,0} + \mu K_{i,0}\right)u_0 + 2\sum_{j=1}^{i-1}\left(\lambda L_{i,j} + \mu K_{i,j}\right)u_j + \left(2\lambda L_{i,j} + \mu K_{i,j}\right)u_i + 2\lambda\sum_{j=i+1}^{n-1} L_{i,j}u_j + \lambda L_{i,n}u_n\right)$$

$$i = 1,2,\ldots,n-1$$

$$u_n = g_n + \frac{h}{2}\Big(\big(\lambda L_{n,0} + \mu K_{n,0}\big)u_0 + 2\sum_{j=1}^{n-1}\big(\lambda L_{n,j} + \mu K_{n,j}\big)u_j + \big(\lambda L_{n,n} + \mu K_{n,n}\big)u_n \qquad \text{.................. (2.2)}$$

where:

$K_{i,j} = K(x_i, x_j)\,(j = 0,1,\ldots,i)$,

$L_{i,k} = L(x_i, x_k)\,(k = 0,1,\ldots,n)$,

$g_i = g(x_i)$,

and u_i is the approximate value of the unknown function u at the node x_i, i = 0, 1, …, n.

By solving the system given by equation (2.2) which consists of (n+1) equations and (n+1) unknowns, the approximate solution of (1.1), is obtained.

3-The Repeated Simpson's 1/3 Method:

Consider the linear Fredholm-Volterra integral equation of second kind given by equation (1.1). Here we use Simpson's 1/3 method to find the solution of equation (1.1).To do this, we divide the finite interval [a, b] into 2n smaller interval of width h, where h = (b − a)/2n. The approximate solution of (1.1) in the even nods (x_{2i}) is given by

$$u(x_{2i}) = g(x_{2i}) + \lambda\int_a^b L(x_{2i}, y)u(y) + \mu\int_a^{x_{2i}} K(x_{2i}, y)u(y)dy,\ i = 0,1,\ldots,n \qquad \text{.................. (3.1)}$$

and in the odd nods (x_{2i+1}) is given by:

$$u(x_{2i+1}) = g(x_{2i+1}) + \lambda\int_a^b F(x_{2i+1}, y)u(y) + \mu\int_a^{x_{2i+1}} K(x_{2i+1}, y)u(y)dy,\ i = 0,1,\ldots,n \qquad \text{.................. (3.2)}$$

By using the repeated Simpson's 1/3 formula to approximate the integrals that appeared in equations (3.1) - (3.2) one can get the following system of equations:

$$u_0 = g_0 + \frac{\lambda h}{3}\Big(L_{0,0}u_0 + 4\sum_{j=1}^{n} L_{0,2j-1}u_{2j-1} + 2\sum_{j=1}^{n-1} L_{0,2j}u_{2j} + F_{0,2n}u_{2n}$$

$$u_{2i} = g_{2i} + \frac{h}{3}\Big(\big(\lambda L_{2i,0} + \mu K_{2i,0}\big)u_0 + 4\sum_{j=1}^{i}\big(\lambda L_{2i,2j-1} + \mu K_{2i,2j-1}\big)u_{2j-1} + 2\sum_{j=1}^{i-1}\big(\lambda L_{2i,2j} + \mu K_{2i,2j}\big)u_{2j} + \big(2\lambda L_{2i,2i} + \mu K_{2i,2i}\big)u_{2i} + 4\lambda\sum_{j=i+1}^{n} L_{2i,2j-1}u_{2j-1} + 2\lambda\sum_{j=i+1}^{n-1} L_{2i,2j}u_{2j} + \lambda L_{2i,2n}u_{2n}\Big)$$

$i = 1, 2, \ldots, n-1$

$$u_{2i+1} = g_{2i+1} + \frac{h}{3}\Big(\big(\lambda L_{2i+1,0} + \mu K_{2i+1,0}\big)u_0 + 4\sum_{j=1}^{i}\big(\lambda L_{2i+1,2j-1} + \mu K_{2i+1,2j-1}\big)u_{2j-1} + 2\sum_{j=1}^{i-1}\big(\lambda L_{2i+1,2j} + \mu K_{2i+1,2j}\big)u_{2j} + \left(2\lambda L_{2i+1,2i} + \frac{5}{2}\mu K_{2i+1,2i}\right)u_{2i} + \left(4\lambda L_{2i+1,2i+1} + \frac{3}{2}\mu K_{2i+1,2i+1}\right)u_{2i+1} + 4\lambda\sum_{j=i+2}^{n} L_{2i+1,2j-1}u_{2j-1} + 2\lambda\sum_{j=i+1}^{n-1} L_{2i+1,2j}u_{2j} + \lambda L_{2i+1,2n}u_{2n}\Big)$$

$i = 0, 1, \ldots, n-1$

$$u_{2n} = g_{2n} + \frac{h}{3}\Big(\big(\lambda L_{2n,0} + \mu K_{2n,0}\big)u_0 + 4\sum_{j=1}^{n}\big(\lambda L_{2n,2j-1} + \mu K_{2n,2j-1}\big)u_{2j-1} + 2\sum_{j=1}^{n-1}\big(\lambda L_{2n,2j} + \mu K_{2n,2j}\big)u_{2j} + \big(\lambda L_{2n,2n} + \mu K_{2n,2n}\big)u_{2n}\Big) \qquad \text{.................. (3.3)}$$

This system solves equation (1.1) more accurately than system (2.2). Because in this system we use the repeated Simpson's 1/3 method for solving (1.1) instead repeated Trapezoid method.

Remark:

The repeated Trapezoidal method and the repeated Simpson's 1/3 method can be also used to solve the linear (FVIE) of first kind given by equation (1.2).To do this suppose that the kernels and the left-hand side in equation (1.2) have continuous derivatives with respect to x and that the condition K(x, x) ≠ 0 holds. In this case, after differentiating equation (1.2) and dividing the resulting expression by K(x, x) we arrive at the following Linear Fredholm-Volterra integral equation of the second kind:

$$u(x) = G(x,y) + \lambda\int_a^b H(x,y)u(y) + \mu\int_a^x J(x,y)u(y)dy \qquad \text{…………(3.4)}$$

Where:

$$G(x) = \frac{-g'_x(x)}{K(x,x)},\ H(x,y) = \frac{\lambda L'_x(x,y)}{K(x,x)} \text{ and}$$

$$J(x,y) = \frac{\mu K'_x(x,y)}{K(x,x)}.$$

for which the condition K(x, x) ≠ 0 must hold.

4-Numerical Examples:

In this section we give some of the numerical examples to illustrate the above methods for solving the linear Fredholm-Volterra integral equations of the first and second kinds. In all case we chose g(x) in such a way that we know the exact solution. This exact solution is used only to show that the numerical solution obtained with our method is correct. Then, in such example, we calculate the absolute errors at some points. We solve these examples by using MATLAB v7.

Example 1:

Consider the FVIE of the second kind:

$$u(x) = 2\cos x - x\cos 2 - 2x\sin 2 + x - 1 + \int_0^2 xyu(y)dy + \int_0^x (x-y)u(y)dy,\ 0 \le x \le 2$$

for which the exact solution is u(x) = cos(x). Table 1 and 2 show that the absolute errors at some mesh points obtained by using the repeated Trapezoid method and the repeated Simpson's 1/3 method respectively for h = 0.2, 0.1, 0.05. Therefore, Table (1 and 2) show that the repeated Simpson's 1/3 method more gave accurate results than the repeated Trapezoid method.

Table (1)
The Absolute Errors at some mesh points of Example 1obtained by using the repeat Trapezoidal formula.

Points	h=0.2	h=0.1	h=0.05
x=0	0	0	0
x=0.2	3.36697×10^{-4}	8.52287×10^{-5}	2.13739×10^{-5}
x=0.4	8.17363×10^{-4}	2.06533×10^{-4}	5.17720×10^{-5}
x=0.6	1.45337×10^{-3}	3.66811×10^{-4}	9.19223×10^{-5}
x=0.8	2.25741×10^{-3}	5.69301×10^{-4}	1.42638×10^{-4}
x=1	3.24451×10^{-3}	8.17835×10^{-4}	2.04883×10^{-4}
x=1.2	4.43334×10^{-3}	1.11717×10^{-3}	2.79851×10^{-4}
x=1.4	5.84776×10^{-3}	1.47338×10^{-3}	3.69067×10^{-4}
x=1.6	7.51871×10^{-3}	1.89433×10^{-3}	4.74509×10^{-4}
x=1.8	9.48653×10^{-3}	2.39027×10^{-3}	5.98747×10^{-4}
x=2	1.18036×10^{-2}	2.97449×10^{-3}	7.45117×10^{-4}

Table (2)
The Absolute Errors at some mesh points of Example 1obtained by using the repeat Simpson's 1/3 formula.

Points	h=0.2	h=0.1	h=0.05
x=0	0	0	0
x=0.2	8.35541×10^{-5}	2.62342×10^{-6}	2.62342×10^{-6}
x=0.4	3.06842×10^{-4}	5.26641×10^{-6}	5.26641×10^{-6}
x=0.6	1.08547×10^{-4}	7.92778×10^{-6}	7.92778×10^{-6}
x=0.8	6.32625×10^{-4}	1.06148×10^{-5}	1.06148×10^{-5}
x=1	1.91537×10^{-4}	1.33473×10^{-5}	1.33473×10^{-5}
x=1.2	9.79100×10^{-4}	1.61622×10^{-5}	1.61622×10^{-5}
x=1.4	9.06532×10^{-5}	1.91179×10^{-5}	1.91179×10^{-5}
x=1.6	1.36428×10^{-3}	2.22986×10^{-5}	2.22986×10^{-5}
x=1.8	2.64724×10^{-4}	2.58190×10^{-5}	2.58190×10^{-5}
x=2	1.83138×10^{-3}	2.98298×10^{-5}	2.98298×10^{-5}

Example 2:

Consider the FVIE of the first kind

$$u(x) = -\frac{2}{5}x^7 - \frac{5}{4}x^4 - \frac{59}{20}x + \int_0^1 x(y+1)u(y)dy + \int_0^x (2x^2y+1)u(y)dy, 0 \le x \le 1$$

for which the exact solution is:

$$u(x) = x^3 + 1.$$

Table (3 and 4) show that the absolute errors at some mesh points obtained by using the repeated trapezoid method and the repeated Simpson’s 1/3 method respectively for h =0.2, 0.1, 0.05. Therefore, Table 3 and 4 show that the repeated Simpson’s 1/3 method gave more accurate results than the repeated Trapezoid method.

Table (3)
The Absolute Errors at some mesh points of Example 2obtained by using the repeat Trapezoidal formula.

Points	h=0.2	h=0.1	h=0.05
x=0	2.21141×10^{-3}	5.53482×10^{-4}	1.38410×10^{-4}
x=0.2	2.20379×10^{-3}	5.51599×10^{-4}	1.37940×10^{-4}
x=0.4	2.16267×10^{-3}	5.41356×10^{-4}	1.35382×10^{-4}
x=0.6	2.08942×10^{-3}	5.23067×10^{-4}	1.30811×10^{-4}
x=0.8	2.02228×10^{-3}	5.06320×10^{-4}	1.26627×10^{-4}
x=1	2.01821×10^{-3}	5.05415×10^{-4}	1.26408×10^{-4}
x=1.2	2.12177×10^{-3}	5.31501×10^{-4}	1.32941×10^{-4}
x=1.4	2.34438×10^{-3}	5.87367×10^{-4}	1.46921×10^{-4}
x=1.6	2.66747×10^{-3}	6.68290×10^{-4}	1.67161×10^{-4}
x=1.8	3.05958×10^{-3}	7.66372×10^{-4}	1.91685×10^{-4}
x=2	3.49113×10^{-3}	8.74235×10^{-4}	2.18649×10^{-4}

Table (4)
The Absolute Errors at some mesh points of Example 2obtained by using the repeat Simpson’s 1/3 formula.

Points	h=0.2	h=0.1	h=0.05
x=0	2.26374×10^{-4}	3.05501×10^{-5}	3.94906×10^{-6}
x=0.2	2.24277×10^{-4}	3.03950×10^{-5}	3.93125×10^{-6}
x=0.4	2.17214×10^{-4}	2.94728×10^{-5}	3.81899×10^{-6}
x=0.6	1.30262×10^{-4}	2.73344×10^{-5}	3.55475×10^{-6}
x=0.8	1.74088×10^{-4}	2.41930×10^{-5}	3.16565×10^{-6}
x=1	1.72379×10^{-4}	2.10750×10^{-5}	2.78130×10^{-6}
x=1.2	1.33094×10^{-4}	1.93274×10^{-5}	2.57047×10^{-6}
x=1.4	5.66677×10^{-4}	1.98240×10^{-5}	2.64108×10^{-6}
x=1.6	1.56855×10^{-4}	2.25897×10^{-5}	2.99422×10^{-6}
x=1.8	8.76536×10^{-4}	2.70251×10^{-5}	3.55353×10^{-6}
x=2	2.34124×10^{-4}	3.23478×10^{-5}	4.22071×10^{-6}

5-Conclusions and Recommendations

The Linear Fredholm-Volterra integral equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions, for this purpose the presented methods can be proposed. From numerical examples it can be seen that the proposed numerical methods are efficient and accurate to estimate the solution of these equations, also, we show that when the values of h decreases, the absolute errors decrease to small values. We will use this method to study linear Fredholm–Volterra integro-differential-difference equation of the first and second kinds in our future work.

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الخلاصة

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