NON-STANDARD DISCRETIZATION METHODS FOR SOME BIOLOGICAL MODELS

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Abstract

It has been observed for some time that the standard (classical) discretization methods of differential equations often produce difference equations that do not share their dynamics Mickens[21]. An illustrative example is the logistic difference equations $\frac{dx}{dt} = \beta x(1-x)$.

Where x(t) represent the density of species A at time t, β is positive number, Euler's discretization scheme produces the logistic difference equation

$x(n+1) = \mu x(n)(1-x(n))$,

Which possesses a remarkably different dynamics such as period-doubling bifurcation route to chaos. A more popular discretization method is to modify the given differential equation to another with piecewise-constant arguments and then to integrate the modified equation. In some instance, this produces a different equation whose dynamics is closed to its original differential equation. However, oftentimes this is not the case. Nevertheless, many authors [1, 3, 7, 8, 9, 10] find it interesting to study the resulting difference equations. This is not a criticism of these author's research, since the study of nonlinear difference equations is of paramount importance regardless of whether or not they have connections with differential equations. But what we are actually saying is that from the point of view of numerical analysis such study is of less importance. This paper itself with those numerical schemes that produce difference equations whose dynamics resembles that of their continuous counter-parts. The most fruitful methods are those of Mickens[14] (for asymptotically stable systems) and of Kahan[16] (for periodic systems).

The paper is organized as follows. Section 2 establishes the basic stability results for Lotka-Volterra differential systems. Section 3 surveys some classical discretization methods that are widely used and show their shortcomings. Section 4 provides the reader with essential intgredients of Mickens nonstandard discretization scheme. In section 5, we discretize a periodic Lotka-Volterra differential system using Kahan's scheme[16]. It is shown that the solutions of the resulting difference equation lie on closed curves surrounding the positive equilibrium point. In section 6, we consider a Kolmogrove continuous model of cooperative system[13]. This model was discretized in[7] using the method of piecewise-constant argument. Surprisingly, the resulting difference equation is dynamically consistent with its continuous counterpart.

Stability of Lotka - Volterra Differential Equations

Consider two species A and B. Then we have three types of relationship between them. The first type is competition, in which A and B competes for common resources such as food, living space, etc. Here the presence of B adversely affects the growth of A and vice versa. The second type of relationship between A and B is cooperation, where the presence of B produces positive effects on the growth of A and vice versa. The third and last type is predator-prey or host-parasite relationship. If A is the prey (host) and B is the predator (parasite), then the presence of B produces a negative effect on the growth of A, while the presence of A produces positive effects on the growth of B K. Gopalsamy[5].

Now, let x(t) be the density of species A at time t , and y(t) be the density of species B at time t K. Gopalsamy[6]. Then the growth rate of species A and B can be modeled by the following system of differential equations:

$$\left.\begin{aligned}\frac{dx}{dt} &= x(t)[r_1 - a_{11}x(t) - a_{12}y(t)]\\ \frac{dy}{dt} &= y(t)[r_2 - a_{21}x(t) - a_{22}y(t)]\end{aligned}\right\} \quad \text{........(2.1)}$$

Equation (2.1) is often referred to as a Lotka-Volterra system, was introduced by Volterra in 1931[21] and by Lotka in 1920 [12]. Here, r_1, and r_2 represent intrinsic growth (or decay) rate for species A, and B respectively, while a_{11}, a_{22} represents the negative effects of squabbles among members of the same species A, and B, respectively. Finally, a_{12} represents the effect on the growth of species A from species B, and a_{21} represents the effect on the growth of species B from species A. It is now evident that $a_{11} \geq 0$ and $a_{22} \geq 0$. However, for the signs of a_{12} and a_{21}, we have three cases:

Case I: Competitive species: $a_{12} \geq 0$, $a_{21} \geq 0$.
Case II: Cooperative species: $a_{12} \leq 0$, $a_{21} \leq 0$.
Case III: Predator-prey species: $a_{12} > 0$, $a_{21} < 0$ or $a_{12} < 0$, $a_{21} > 0$

$-a_{12}\, x(t)\, y(t) < 0$ The presence of B produces a negative effect on the growth of A.

$-a_{21}\, x(t)\, y(t) > 0$ The presence of A produces a positive effect on the growth of B.

If we write eq.(2.1) in the form

$$\frac{dx}{dt} = f(x,y), \qquad \frac{dy}{dt} = g(x,y)$$

Then we say that (x^*, y^*) is an equilibrium point if $f(x^*, y^*) = g(x^*, y^*) = 0$. The equilibrium point (x^*, y^*) is said to be stable if for any open neighborhood U of (x^*, y^*) there exists an open neighborhood V of (x^*, y^*) such that if $(x_0, y_0) \in$ V then:

$(x(t, x_0), y(t, y_0)) \in U$ for all $t \geq 0$. If in addition, $\lim\limits_{t\to\infty}(x(t,x_0), y(t,y_0)) = (x^*, y^*)$ for all (x_0, y_0) in an open neighborhood W of (x^*,y^*), then (x^*,y^*) is said to be asymptotically stable if $W = R^2$, then (x^*, y^*) is said to be globally asymptotically stable.

Observe that if system (2.1) possess a positive equilibrium point (x^*, y^*) then it must satisfy the equations;

$$a_{11}x^* + a_{12}y^* = r_1 \quad \text{..................................(2.2)}$$
$$a_{21}x^* + a_{22}y^* = r_2 \quad \text{..................................(2.3)}$$

Hence Cramer's Role

$$x^* = \frac{r_1 a_{22} - r_2 a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \; y^* = \frac{r_2 a_{11} - r_1 a_{21}}{a_{11}a_{22} - a_{12}a_{21}} \quad (2.4)$$

We are in a position to state the main stability result for system (2.1).

Theorem [2]

Suppose that system (2.1) has an asymptotically stable positive equilibrium point (x^*, y^*) then (x^*, y^*) is globally asymptotically stable if $a_{11} > 0$, $a_{22} > 0$.

Classical Discretization

There are numerous discretization schemes in numerical analysis literature. The simplest numerical scheme is the forward Euler in which dx/dt is replaced $\dfrac{x(t+h) - x(t)}{h}$ and dx/dt is replaced by $\dfrac{y(t+h) - y(t)}{h}$, where h is the step size of the numerical method. Making this replacement in eq.(2.1) and letting $t = nh$, $x(t) = x(nh) = x(n)$, and $y(t) = y(nh) = y(n)$ yield the difference system;

$$\left.\begin{aligned}x(n+1) &= x(n)[1 + r_1 h - a_{11}hx(n) - a_{12}hy(n)]\\ y(n+1) &= y(n)[1 + r_2 h - a_{21}hx(n) - a_{22}hy(n)]\end{aligned}\right\}$$
$$\text{..................................(3.1)}$$

Observed that the dynamics of Eq. (3.1) differs form that of Eq. (2.1) and for some parameter values may exhibit chaotic behavior. Hence, the search for a better numerical scheme continues. Another popular method is to consider Eq. (2.1) with a piecewise constant arguments [22] as follows:

$$\left.\begin{aligned}\frac{dx}{dt} &= x(t)[r_1 - a_{11}x(\lfloor t \rfloor) - a_{12}y(\lfloor t \rfloor)] \\ \frac{dy}{dt} &= y(t)[r_2 - a_{21}x(\lfloor t \rfloor) - a_{22}y(\lfloor t \rfloor)]\end{aligned}\right\} \quad \ldots\ldots\ldots (3.2)$$

Where $0 \le n \le t < n+1$, and $\lfloor t \rfloor$ is the greatest integer in t. Integrating both sides of Eq. (3.2) yields;

$$\left.\begin{aligned}x(t) &= x(n)\exp[r_1 - a_{11}x(n) - a_{12}y(n)] \\ y(t) &= y(n)\exp[r_2 - a_{21}x(n) - a_{22}y(n)]\end{aligned}\right\} \ldots. (3.3)$$
$$t \in [n, n+1)$$

If we let $t \to n+1$ in the preceding system, we obtain the following system of difference equations.

$$\left.\begin{aligned}x(n+1) &= x(n)\exp[r_1 - a_{11}x(n) - a_{12}y(n)] \\ y(n+1) &= y(n)\exp[r_2 - a_{21}x(n) - a_{22}y(n)]\end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3.4)$$

System (3.4) has been investigated by Krawcewicz and Rogers [10] for the case of cooperative systems ($a_{12} \le 0$, $a_{21} < 0$) and by Jiang and Rogers[9] for competitive systems ($a_{12} \ge 0$, $a_{21} \ge 0$). In both cases, it was shown that system (3.4) may exhibit a dynamical behavior quite different from its continuing counterpart (2.1).

In spite of its deficiency, system (3.3) has been given a lot of attention by several authors including Hofbauer[8], Dohtani[3].

Nonstandard Discretization Schemes

One of the main aims of numerical analysis is to find a numerical scheme that produces difference equations that exhibits the same qualitative behavior as its continuous counterpart (differential equations). We say that a difference equation is dynamically consistent with its differential equation if they both posses the same dynamics such as stability, Bifurcation and chaos. In [14], Mickens developed successful nonstandard discretization schemes that produce what every numerical analyst dreams about, namely dynamically consistency.

Here pargraf we adapt Mickens method to the setting of biological models mainly of Lotka-Volterra type. To illustrate Mickens general scheme, we start with a very simple example, the logistics differential equation:

$$\frac{dx}{dt} = x(t)(a - bx(t)) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.1)$$

Integrating Eq. (4.1) from t to t+h yields

$$x(t+h) = \frac{e^{ah}x(t)}{1 + (\frac{e^{ah}-1}{a})bx(t)} \ldots\ldots\ldots\ldots\ldots (4.2)$$

We now let t = nh, and x(nh) = x(n) in (4.2). Hence, we have

$$x(n+1) = \frac{e^{ah}x(n)}{1 + (\frac{e^{ah}-1}{a})bx(n)}, n \in Z^+ \ldots\ldots (4.3)$$

Observe that the solutions of Eq. (4.1) and Eq. (4.3) are equal on Z^+, regardless of the value of the step size h setting $\varphi(h) = \frac{e^{ah}-1}{a}$ Eq. (4.2) may be written in the form:

$$\frac{x(t+h) - x(t)}{\varphi(h)} = ax(t) - bx(t)x(t+h) \ldots (4.4)$$

Notice that (4.4) is similar to the difference equation obtained by forward Euler's method with two major differences: (i) h in the denominator of the left hand side is now replaced by a function of h, $\varphi(h)$, (ii) the term $x^2(t)$ is now replaced by x(t) x(t+h). The resulting equation is given by;

$$x(n+1) = \frac{1 + a\varphi(h)x(n)}{1 + b\varphi(h)x(n-1)}$$

We now formulize the above steps for the general differential equation:

$$\left.\begin{aligned}\frac{dx}{dt} &= f(t, x(t), y(t)) \\ \frac{dy}{dt} &= g(t, x(t), y(t))\end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.5)$$

Step 1: Replace the derivative dx/dt by an expression of the form $\frac{x(t+h)-x(t)}{\varphi_1(h)}$, where $\varphi_1(h) = h + O(h)$, and dy/dt by the expression of the form $\frac{y(t+h)-y(t)}{\varphi_2(h)}$ where $\varphi_2(h) = h + O(h)$.

$$
\begin{array}{lll}
x^2(t) & \longrightarrow & x(t)\,x(t+h) \\
y^2(t) & \longrightarrow & y(t)\,y(t+h) \\
y(t)\,x(t) & \longrightarrow & x(t)\,y(t+h) \\
 & \searrow & x(t+h)\,y(t) \\
 & \searrow & x(t)\,y(t)
\end{array}
$$

Step 2: Vary the nonlinear terms by non-local expressions for example,

For step 1, the main question is how to chose the appropriate function $\varphi_1(h)$ and $\varphi_2(h)$. At this time, we are unable to give a general method for the selection of these "denominator" functions. However, we will demonstrate to the reader some special techniques that produce appropriate "denominator" function[11].

As for step 2, the selection of appropriate expressions provides to be simple for competitive and cooperative Lotka-Volterra systems and most challenging for predator-prey model[11]. While performing step 2, one should make sure that solutions with non- negative initial value must stay non-negative all the time, i.e., the cone;

$\Re_+^2 = \{(x,y): x \geq 0, y \geq 0\}$ Must be invariant.

Other Nonstandard Numerical Schemes

In this section, we consider the discretization of the following simple predator-prey model;

$$
\left.\begin{aligned}
\frac{dx}{dt} &= \alpha x + \beta xy \\
\frac{dy}{dt} &= \gamma x + \delta xy
\end{aligned}\right\} \quad \text{..................................} (5.1)
$$

Where x(t) represents the density of the prey at time t , and y(t) represents the density of the predator at time t.

$$\text{If } \alpha < 0,\ \beta > 0,\ \gamma > 0,\ \delta < 0 \text{ ....} (5.2)$$

Then $(-\alpha/\beta,\ -\gamma/\delta)$ is the only positive equilibrium point of the system (5.1). All other solutions are periodic and lie on closed curves surrounding the equilibrium point (x^*, y^*).

If we apply Mickens discretization scheme, the resulting difference equation possesses solutions that either spiral in towards the equilibrium point (x^*, y^*) or spiral out of the equilibrium point (x^*, y^*).

In [16], the author employed a discretization scheme attributed to W. Kahan that produces a difference equation whose solutions stay on closed curves.

He proposed the following discretization scheme:

$$
\left.\begin{aligned}
\frac{x(t+h)-x(t)}{h} &= \frac{\alpha}{2}(x(t+h)+x(t)) + \frac{\beta}{2}(x(t+h)y(t)+x(t)y(t+h)) \\
\frac{y(t+h)-y(t)}{h} &= \frac{\gamma}{2}(y(t+h)+y(t)) + \frac{\delta}{2}(x(t+h)y(t)+x(t)y(t+h))
\end{aligned}\right\}
$$

$$\text{..................................} (5.3)$$

Which yield the difference system;

$$
\left.\begin{aligned}
x(n+1) &= x(n)\frac{[1+\frac{\alpha h}{2}+\frac{\beta h}{2}y(n+1)]}{[1-\frac{\alpha h}{2}-\frac{\beta h}{2}y(n)]} \\
y(n+1) &= y(n)\frac{[1+\frac{\gamma h}{2}+\frac{\delta h}{2}x(n+1)]}{[1-\frac{\gamma h}{2}-\frac{\delta h}{2}x(n)]}
\end{aligned}\right\} \text{......} (5.4)
$$

Note that the discretization (5.3) differs from all classical discretization schemes. It replaces the nonlinear term x(t)y(t) by (1/2) (x(t+h) y(t)+x(t) y(t+h)) while it is replaced by (1/2) (x(t+h) y(t+h) + x(t)y(t)) in the standard trapezoidal rule and by:

(1/2) (x(t+h) + x(t)) (y(t+h) + y(t)) in the midpoint rule.

To explain Kahan's scheme (5.3) works while most of other numerical schemes produce spiraling solutions, the author in [16] observed that for systems of differential equations in the plane, the situation where all trajectories in the phase spaces are closed curves in nongeneric, i.e., atypical. Hence in this case any small perturbation of the right-hand side may change the closed curves into spirals. The effect of numerical integration amount to changing the system being solved into a nearby system whose solutions would typically spiral.

There is, however, a class of differential equations in the plane where closed curves are typical. This is the class of canonical Hamiltonian systems of the form;

$$\left.\begin{aligned}\frac{dx}{dt}&=-\frac{\partial H}{\partial y}\\ \frac{dy}{dt}&=\frac{\partial H}{\partial x}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5.5)$$

Where H=H(x,y) is a Hamiltonian function [17]. The most important of system (5.5) is that is trajectories are the level curves of the Hamiltonian function H. Moreover, if all trajectories of system (5.5) are closed then all nearby Hamiltonian systems also have closed trajectories. We also observe that the flow φ_h induced by system (5.5), where $\varphi_h(x(t),y(t)) = (x(t+h), y(t+h))$ is an area-preserving map. Hence we should look for a numerical scheme that is also area-preserving.

Such numerical schemes are called canonical. This precisely what Kahan's scheme (5.3) achieves? Now, according to KAM theory[17], a canonical numerical method applied to a canonical Hamiltonian system preserves the property of closed curves. It is straightforward to verify that the system (5.1) is not Hamiltonian since its vector field (f,g) is not divergence free, i.e., $\frac{\partial h}{\partial x}+\frac{\partial g}{\partial y}$ fails to be identically zero, where

$f(x,y) = \alpha x + \beta xy$ and $g(x,y) = \gamma y + \delta xy$. However, by the change of variable $\xi = \ln x$, $\eta = \ln y$, system (5.1) becomes;

$$\left.\begin{aligned}\frac{d\xi}{dt}&=\alpha+\beta e^{\eta}\\ \frac{d\eta}{dt}&=\gamma+\delta e^{\xi}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5.6)$$

Observe that system (5.6) is divergent free, i.e. $\frac{\partial\tilde{f}}{\partial x}+\frac{\partial\tilde{g}}{\partial y}=0$,

Where $\tilde{f}(\xi,\eta)=\alpha+\beta e^{\eta}$, $\tilde{g}(\xi,\eta)=\gamma+\delta e^{\xi}$.

The corresponding Hamiltonian function of (5.6) is given by $\tilde{H}(\xi,\eta)=N(e^{\xi})+M(e^{\eta})$, where N(z) is the antiderivative of $\frac{\gamma+\delta z}{z}$ and M(z) is the anti derivative of $\frac{-\alpha-\beta z}{z}$.

Hence $N(e^{\xi})=\gamma\xi+\delta e^{\xi}$, $M(e^{\eta})=-\alpha\eta-\beta e^{\eta}$.

Observe that the trajectories of systems (5.6) lie on the level curves of $\tilde{H}$ in the (ξ, η) plane. This implies that the trajectories of the system (5.1) lie on the level curves of the function:

$$\begin{aligned}H(x,y)&=\tilde{H}(\ln x,\ln y)\\ &=\gamma\ln x+\delta x-\alpha\ln y-\beta y\end{aligned} \quad \ldots\ldots\ldots (5.7)$$

Observe that system (5.1) can be written in the following noncanonical Hamiltonian system;

$$\left.\begin{aligned}\frac{dx}{dt}&=-\frac{1}{\sigma(x,y)}\frac{\partial H}{\partial y}\\ \frac{dy}{dt}&=\frac{t}{\sigma(x,y)}\frac{\partial H}{\partial x}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (5.8)$$

Where $\sigma(x,y) = 1/xy$.

A Kolmogorov Model of Cooperative Systems

In [13], Robert may suggested the following differential system to model a cooperative system of two species with densities x(t) and y(t):

$$\left.\begin{aligned}\frac{dx}{dt}&=r_1x(t)\left(1-\frac{x(t)}{\beta_1+\alpha_1y(t)}\right)\\ \frac{dy}{dt}&=r_2y(t)\left(1-\frac{y(t)}{\beta_2+\alpha_2x(t)}\right)\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6.1)$$

Where $r_1,r_2,\alpha_1, \beta_1, \alpha_2, \beta_2$ are positive numbers. Model (6.1) it is known that if:

$\alpha_1\alpha_2 < 1$, then system (6.1) has globally asymptotically stable positive equilibrium point (x^*,y^*) [6]. Although the discretization in (3.2) and (3.3) produced a dynamically inconsistent difference equation for Model (2.1), it has been effective in dealing with Model (6.1)[7]. As in (3.2), we consider a modification of system (6.1) to a system of picewise-constant argument

$$\left.\begin{aligned} \frac{dx}{dt} &= r_1 x(t)\left(1-\frac{x(t)}{\beta_1+\alpha_1 y(\lfloor t \rfloor)}\right) \\ \frac{dy}{dt} &= r_2 y(t)\left(1-\frac{y(t)}{\beta_2+\alpha_2 x(\lfloor t \rfloor)}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6.2)$$

Where $\lfloor t \rfloor$ denotes the greatest integer in t. Integrating both side of (6.2) on [n , n+1)

And letting $t \to n+1$ yields the difference systems:

$$\left.\begin{aligned} x(n+1) &= \frac{e^{r_1} x(n)}{1+\left(\dfrac{e^{r_1}-1}{R_1(n)}\right)x(n)} \\ y(n+1) &= \frac{e^{r_2} y(n)}{1+\left(\dfrac{e^{r_2}-1}{R_2(n)}\right)y(n)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (6.3)$$

Where $n \in Z^+$, and $R_1(n) = \beta_1 + \alpha_1\ y(n)$, $R_2(n) = \beta_2 + \alpha_2\ x(n)$. It can be shown that all positive solutions of Eq. (6.3) are bounded away from zero. Moreover, if $\alpha_1\ \alpha_2 < 1$, then all positive solutions of Eq.(6.3) are bounded above[7].

Now, if $\alpha_1\ \alpha_2 < 1$, then there exists a positive equilibrium point (x^*, y^*) which satisfies the equations:

$$\left.\begin{aligned} -x^* + \alpha_1 y^* &= -\beta_1 \\ \alpha_2 x^* - y^* &= -\beta_2 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.4)$$

The linearized system around (x^*,y^*) has the coefficient matrix:

$$B = \begin{pmatrix} e^{-r_1} & \alpha_1(1-e^{-r_1}) \\ \alpha_2(1-e^{-r_2}) & e^{-r_2} \end{pmatrix} \ldots\ldots\ldots\ldots (6.5)$$

It is easy to verify that matrix B satisfies the Schur-Cohn criterion. Hence the equilibrium point (x^*,y^*) is (**locally**) asymptotically stable.

Indeed, Gopalsamy and Liu[7] proved that (x^*,y^*) is globally asymptotically stable. Hence (6.3) is dynamically consistent with the differential system (6.1)

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الخلاصة

لقد لوحظ لمدة من الزمن أن الطرائق المتقطعة (القياسية) المعيارية للمعادلات التفاضلية غالبا ما تنتج معدلات فروق لا تتشارك معها بالديناميكية، والمثال التوضيحي هـو المعادلـة التفاضلية الشائعة:-

$$\frac{dx}{dt}=\beta x(1-x)$$

خطة اويلر(Euler's) المتقطعة تنـتج معـادلات فـروق شائعة.

$$x(n+1)=\mu x(n)(1-x(n))$$

والتي تمتلك بصورة واضحة ديناميكية مختلفة، مثلا لهـا تفرع دوري مضاعف بصورة فوضوية، طرق حل المعادلات الشائعة المتقطعة هي بتغيير المعادلات التفاضلية المعطاة إلى أخرى متقطعة بصورة ثابتة، ومـن ثـم تكامـل المعـادلات المحورة.

في بعض الأمثلة هذا ينتج معادلة فروق التي ديناميكيتهـا قريبة من المعادلات الأصلية. مع ذلك في اغلب الأحيان فان المعادلات المذكورة أنفاً ليست تكاملية دائما مع إن المـؤلفين [1، 3، 7، 8، 9، 10] يبـدون اهتمامـا بدراسـة النتـائج للمعادلات التفاضلية. ولسنا هنا فـي موضـع نقـد لبحـوث المؤلفين، منذ ذلك الحين معادلات الفروق غير الخطيـة ذات أهمية رئيسة بغض النظر إذا كانت أو لم تكـن ذات ارتبـاط بالمعادلات التفاضلية. ولكن ما نقوله حقيقة انه من وجهة نظر التحليل العددي فان تلك الدراسة ذات أهمية اقل.

هذا البحث يتضمن في ذاته المخططات العددية التي تنتج معادلات فروق التي ديناميكيتها مشابهة إلى القسـم المكمـل المستمر. أغلب الطرائق الناتجة منسوبة إلى[14] Mickens (الأنظمـة المقاربـة للاسـتقرار) والـى[16] Kahan (الأنظمة الدورية).

نظم البحث بالشكل الآتي الجزء الأول يمثل عرض موجز للبحث. الجـزء الثـاني أظهـر النتـائج الثابتـة لطريقـة (Lotka-Volterra) للأنظمة التفاضلية أما الجزء الثالث فهو نظرة عامة لبعض الطرائق القياسية التي تستعمل بشكل واسع وإظهار قصورها، أما الجزء الرابع يقدم للقـارئ الأجـزاء

المقومة الضرورية من Mickens ذات المخطط المتقطع غير القياسي. أما الجزء الخـــامس فخصـــص للنظـــام التفاضـــلي لـــــ (Lotka-Voltera) المتقطـــع الـــدوري باســـتخدام مخطط[16] Kahan’s. لقد تبين أن حل معـــادلات الفـــرق الناتجة يقع على منحني مغلق حول نقطة موجبـــة متوازنـــة. وفي الجزء الســـادس نشـــاهد الأنمـــوذج (Kolmogorov) مستمر لنظام مساعد[13]، هذا الأنموذج الذي قطَّـــع فـــي[7] مستخدمين المناقشة على طريقة القطعة الثابتـــة، للاســـتزادة ينظر[21].