STABILITY ANALYSIS OF PERIODIC SOLUTIONS TO THE NON-STANDARD DISCRETIZATION MODEL OF THE LOTKA-VOLTERRA PREDATOR-PREY SYSTEM

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Abstract

The standard classical discretization methods of differential equations often produce difference equations that do not share dynamics with their continuous counterparts; Recently,[4] has developed successful non-standard discretization schemes that produce dynamical consistency, which numerical analysis value highly. Many authors have adapted these methods to various biological models. We reviewed a non-standard discretized biological model of a Lotka-Volterrs Predator-Prey system in a general form and discussed the stability analysis of its periodic solutions. We also discussed a numerical example of this analysis using the non-standard discretized Predator-Prey model the name of executed program for drawing and calculation is “MATLAB 7.0”.

Introduction

A wide variety of numerical schemes are available to solve the dynamical systems that cannot be solved analytically. The standard classical discretization methods involved in these numerical schemes often produce systems of difference equations that do not inherit the dynamical properties of their continuous counterparts. When they exist, stability of fixed points and periodic solutions are the most important properties of continuous dynamical systems and discretized model. Thus, a discretization methods involved in numerical scheme is useful if the solution of that scheme is exact for at least a subclass of original system, if it preserves the dynamics, and if it conserves energy like its continuous analogue.

Mickens developed non - standard discretization methods that have proved to be very fruitful, producing numerical schemes that are highly desirable because they meet the criteria above. These methods are relatively easy to implement and have much greater computational efficiency than standard numerical methods. The relative importance of advection and biological and chemical reaction is directly incorporated into the corresponding numerical scheme, large time steps can be taken without affecting the accuracy of the numerical solutions. Generally, non-standard methods can be used in numerical schemes to construct highly accurate algorithms for solving a varity of stiff dynamical systems,[8]. Many researchers [Dohtani, 1992; Gopalsamy & Liu, 1999; Jian & Rogers, 1987], applyied these techniques to obtain numerical solutions to the various differential equations that rise in interesting problems in the natural and engineering sciences. [Al-Kahby al, 2000] and his Co-workers have used non-standard discretization methods with some biological models, they applied this approach to discretize the competitive and cooperative models of predator-prey. In that work, they consider the simple predator-prey model:

$$\left.\begin{aligned} \frac{du}{dt} &= u(b_1 + b_2 v), u(0) = u_0 \\ \frac{dv}{dt} &= v(a_1 + a_2 u), v(0) = v_0 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots (1)$$

If $\beta_1 < 0$, $\beta_2 > 0$, $\alpha_1 > 0$ and $\alpha_2 < 0$, then $(u^*, v^*) = (-\frac{a_1}{a_2}, -\frac{b_1}{b_2})$ is the only positive equilibrium point of the system and all other solutions are periodic and lie on closed curves surrounding the equilibrium point (u^*,v^*). They showed that if applying Mickens' discretization method to above predator-prey model, the resulting difference equations possess solutions that either spiral in toward the positive equilibrium point (u^*,v^*) or spiral out of it. As defined elsewhere [Al-Kahby al, 2000] we say that a difference equation is dynamically consistent with counterpart continuous dynamical system if they both posses the same dynamic with respect to stability. Using this discretized system, can demonstrate the stability of periodic solutions around the positive equilibrium points (such as $(-\frac{a_1}{a_2},-\frac{b_1}{b_2})$ in (1)) that exist in both the continuous and difference systems. We proved this by preserving area in non-canonical Hamiltonian systems. Finally, based upon our analytical results, we discuss numerical example to demonstrate the stability of periodic solution.

Elementary Bifurcation of Non-standard Discretization Models

In this section we construct non-standard discretization models for some elementary example to demonstrate dynamic consistency between the discretized models and the original systems.

Transcritical Bifurcation

Our first example is the famous one–parameter logistic differential equation

$$\frac{dx}{dt} = x(m - bx) \quad \dots\dots\dots\dots\dots\dots\dots\dots (2)$$

,[1]

It is clear that $x^* = 0$ and $x^* = \frac{m}{b}$ are the two fixed points for (2). Taking $f'_x = m - 2bx$, we have $f'_x = m$ for $x^* = 0$ and $f'_x = -m$ for $x^* = \frac{m}{b}$. These two fixed points have different stabilities regardless of the value of b ≠ 0 and they exchange stability at the bifurcation point. There for, eq.(2) has a transcritical bifurcation for the value of μ = 0 .

Applying Mickens' non-standard discretization method [Mickens, 2000] to eq.(2) we obtain

$$x_{n+1} - x_n = j\,(h)(m\,x_n - b\,x_n x_{n+1})$$

or

The h in Euler's method is replared here by Φ (h), a function of h, for details on special techniques that produce an appropriate Φ(h) refer to, [3].

$$x_{n+1} = \frac{(1 + m j\,(h))x_n}{1 + b j\,(h) x_n}, n \in Z^+ \quad \dots\dots\dots\dots (3)$$

$1 + b\,\Phi(h)x_n \neq 0$

For details on special techniques that produce an approximate φ(h) , refer to [Liu & Elaydi,2001]. Clearly eq.(3) has the same fixed points as eq.(2). Let

$$H(m,x) = \frac{(1+mj\,(h))x}{1+bj\,(h)x}.$$

Then $H'_x(m,x) = \frac{(1+mj\,(h))}{(1+bj\,(h)x)^2}$. Thus for the first fixed point $x^* = 0$, we have $H'_x = (m,0) = 1 + mj\,(h)$ and hence, $H'_x = (0,0) = 1$, $H'_x(m,0) < 1$, for μ < 0, and $H'_x(m,0) > 1$ for μ > 0 (note that φ(h) > 0). For the second fixed point $x^* = \frac{m}{b}$, we have $H'_x(m,\frac{m}{b}) = \frac{1}{1+mj\,(h)}$, $H'_x(0,0) = 1$, $H'_x(m,\frac{m}{b}) < 1$ for μ >0, and $H'_x(m,\frac{m}{b}) > 1$ for μ < 0. This means that μ = 0 is the bifurcation value. The exchange of stability leads to Transcritical bifurcation and the bifurcation diagram is the same as eq.(2) see Fig.(1). Therefore, eq.(2) and eq.(3) concide in bifurcation value and type.

Saddle-Node Bifurcation

Consider

$$\frac{dx}{dt} = bx^2 + m \quad \dots\dots\dots\dots\dots\dots\dots\dots (4)$$

and it non-standard discretization

$$x_{n+1} = \frac{mj\,(h) + x_n}{1 - bj\,(h)x_n}, n \in Z^+ \qquad (5)$$

Equations (4) and (5) have the same fixed points $x^* = 0$ for μ = 0 and $x^* = \pm\sqrt{\frac{-M}{b}}$ for different signs of μ (≠ 0) and b, and hence they have the same saddle–node bifurcation diagrams for μ = 0. see Fig.(2). Although we could present more such examples.

Numerical Example:-

Consider the predator-prey model (1)

$$\left.\begin{aligned}\frac{du}{dt} &= b_1 u + b_2 uv \quad ,u(0)=u_0\\ \frac{dv}{dt} &= a_1 v + a_2 uv \quad ,v(0)=v_0\end{aligned}\right\}, [1]$$

Hence, *u(t)* represent the density of the predator at time *t* and *v(t)* represent the density of prey at time *t* , nothing that in the Lofka - Volterra predator - prey system $\beta_1<0,\beta_2>0$, and $\alpha_2 < 0$, it follows that the only positive equilibrium point $w^* = (-\frac{a_1}{a_2}, -\frac{b_1}{b_2})$ is non-hyperbolic with eigenvalues $l_1 = 0,\ l_2 = -\frac{a_2 b_1}{b_2}$. This equilibrium point is known to be stable, [3] and surrounded by nested closed curves (periodic orbits). This model can be written as:

$$\frac{du}{dt} = uv\frac{\partial H}{\partial v} \quad ,u_0$$
$$\frac{dv}{dt} = -uv\frac{\partial H}{\partial u} \quad ,v_0$$

Where the Hamiltonian H, described by :-

$$\mathrm{H} = -\int \frac{\mathrm{f}(\mathrm{u},a)}{\mathrm{u}}\,\mathrm{du} + \int \frac{\mathrm{g}(v,a)}{v}\,\mathrm{d}v \ ,$$

Now consider the non-standard discretized model of (1). The equation corresponding to equation

$$u_{n+1} = \frac{u_n}{1 - j_1 g(v_n + b)}$$
$$v_{n+1} = \frac{v_n}{1 - j_2 \mathrm{f}(u_{n+1} + a)}$$

for this model are:

$$u_{n+1} = \frac{u_n}{1 - j_1(h)(b_1 + b_2 v_n)}$$
$$v_{n+1} = \frac{v_n}{1 - j_2(h)(a_1 + a_2 u_{n+1})}.$$

For simplicity we take β_1=-1, β_2=1,α_1= 1 and α_2= -1, so the positive fixed point w^* is (1,1), so we may choose $\varphi_1(h) = 1- e^{-h}$ and $\varphi_2(h) = e^h-1$. Figs. (3-b to 3-e) show periodic solutions for different values of *h* with the same initial condition (1.2,1.2). Note that starting from (1.2,1.2), the amplitudes are the same, but different values of *h* yield different time periods. The time series solutions, in these figures show that, for the same initial conditions, *h* and the larger time period are inversely correlated. For example, in Fig (3-b) with *h*=0.1 and initial condition (1.2,1.2), the time period is almost 60. With the same starting point (1.2,1.2) for *h*= 0.5 this time period is 15 and for *h*=1 it is 5 (see Figs. (3-c and 3-d)). As we can see in Fig. (3-f), these periodic solutions break down. Indeed all periodic solutions in the first quadrant are dependent upon the two elements *h* (the variable in functions φ_1 and φ_2) and the initial condition. The variable *h* may range through the interval (0, $\ln(3+2\sqrt{2}$)) and for these values of *h*, the initial condition for $u = v$ ranges through the interval (1,7). If we fix *h*, then Figs. (4-b to 4-e) illustrate different amplitudes for different values of initial conditions. For example, in Fig.(4-c) with *h*=0.1 and starting point (1.5,1.5), the amplitude is 1.4 while it is 3.5 in Fig (4-d), for the same *h* and starting point (2.5,2.5). In Fig. 4-b we note that even in small neighborhoods of the fixed point (1,1), the periodic solution can be predicted. Here, the starting point is (1.05,1.05), which is close to the fixed point. As shown in Fig.(4-a), there is no periodic solution for starting point (1,1) with values of $h \in (0,2)$. Finally, the relation between *h* and the initial condition necessary to preserve the periodic solutions in the neighborhood of (1,1) is shown in Fig.(5). In this figure the area between the curve and the line x=1 is the region on which the periodic solutions are preserved using the non-standard discretization model, for the predator-prey model (1). These results are not only consistent with other similar results [Gander](see2), but also produce a larger region on which periodic solutions exist by using our discretized model.

Conclusion

In our model problem, we used Micken's method to discretized the general form of the Lotka–Volterra predator–prey system. This system was written as a canonical Hamiltonian system.

The stability of the periodic solutions of our model problems in both the continuous system and its discretized counterparts. This stability analysis completes the work of other authors, [1]. All of these results show that Mickens' non-standard discretization methods produce discretized systems that inherit the corresponding dynamical properties of the original continuous systems. The phase-portraits in the (u,v) plane, as illustrated in Figs. (6-a and 6-b), show additional periodic solutions to system for fixed value of h=0.1 with different initial conditions. These figures show that the first quadrant periodic solutions determined by using the non-standard discretization model are smooth curves, are more accurate and exist in a large region than the similar ones found by other discretization methods, [2].

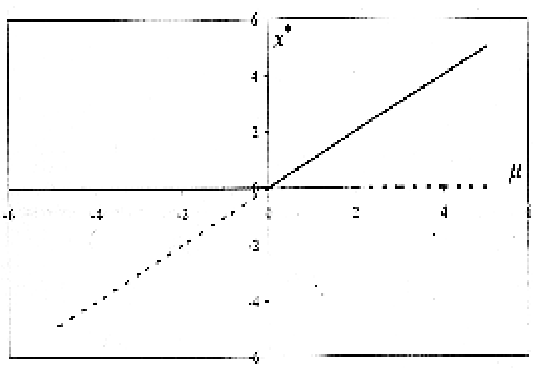

Fig. (1): Transcritical bifurcation in systems (2) for different values of μ with b = 1.

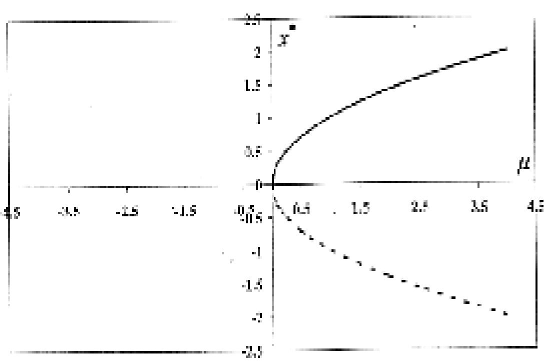

Fig.(2): Saddle–node bifurcation in system(4)or(5) for different value μ with b = 1.

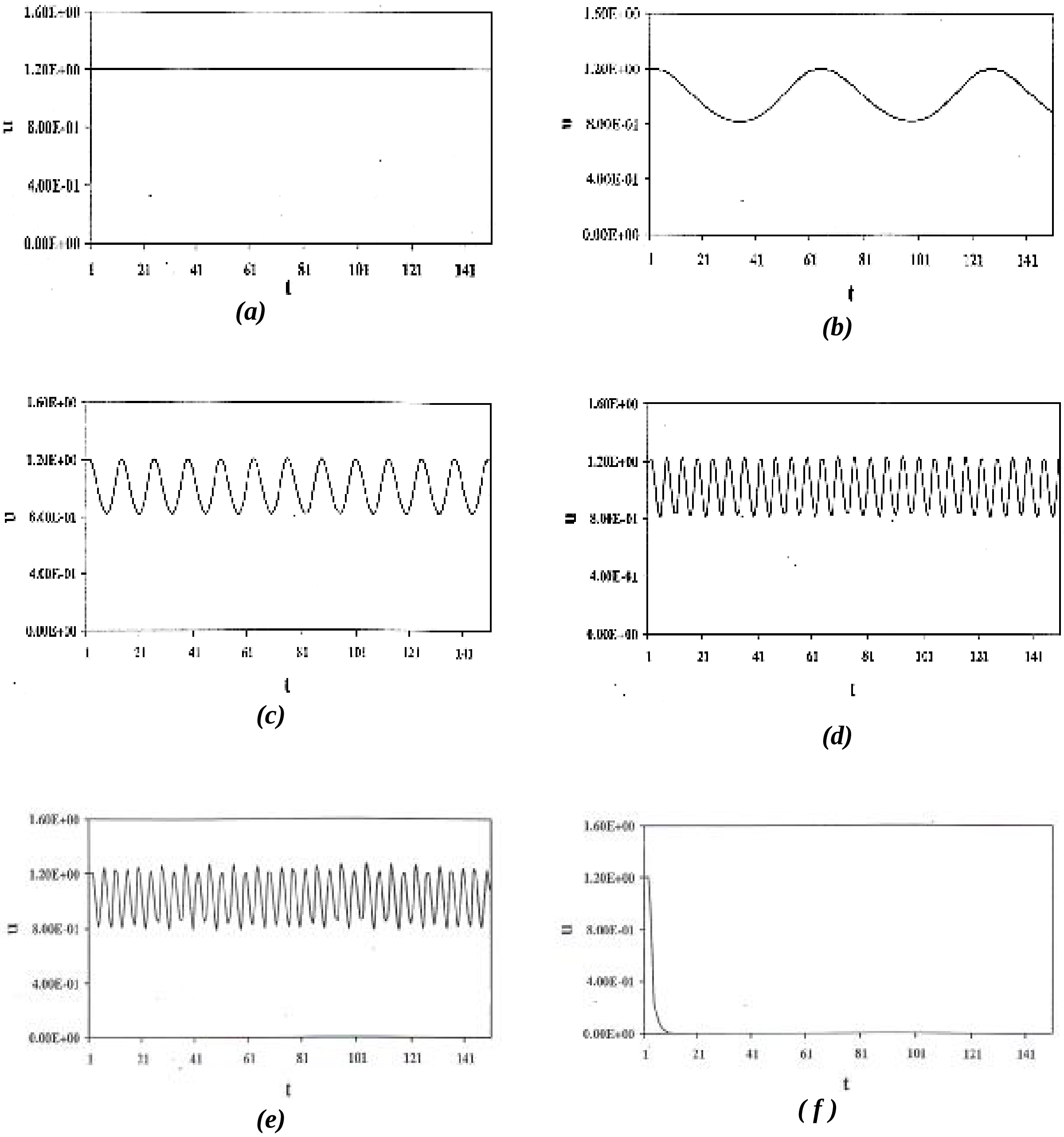

Fig. (3): Time series solutions for different values of h. In all these figures we began with the initial condition (1.2,1.2). The values of h and the larger time period are inversely correlated. (a) study state solution for h=0. The value of h is 0.1, 0.5, 1 and 1.2 in figures (b) to (e), respectively. In (f), h = 2.

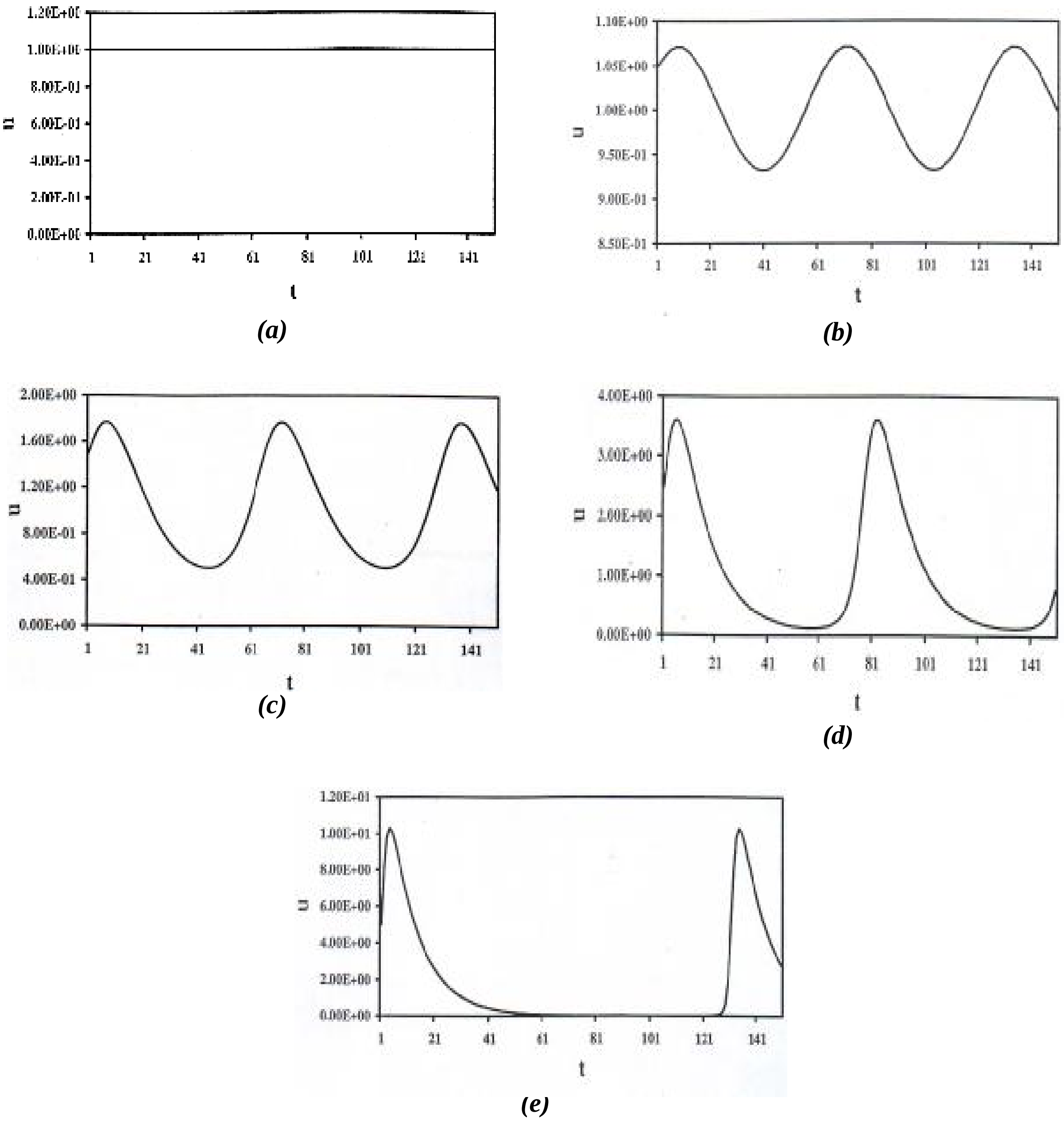

Fig. (4): Time series solutions for h = 0.1 with different initial conditions. Differing amplitudes are shown in figures (b) to (e) for initial conditions (1.05,1.05), (1.5,1.5), (2.5,2.5) and (5,5) respectively. The starting point in (a) is (1,1).

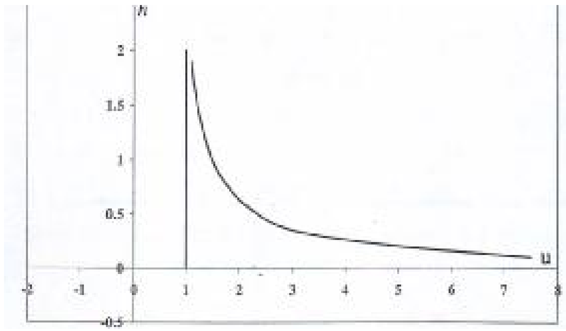

Fig.(5): The area between the curve and the line x = 1 is the region on which periodic solutions are preserved using the non-standard discretization model for predator-prey model (1).

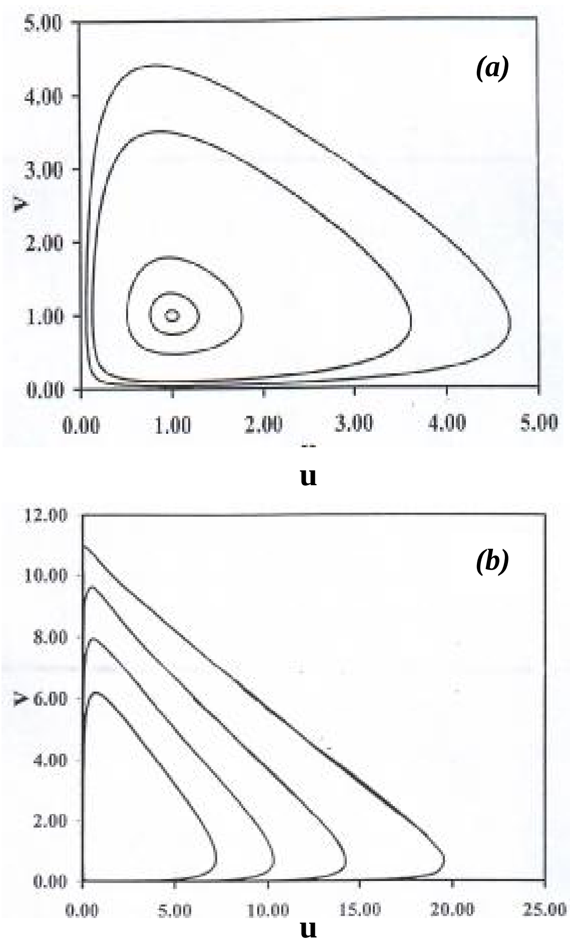

Fig.(6): Phase-portraits in the (u,v) plane of the discretized system for varying initial conditions and fixed h=0.1. Smooth periodic solutions illustrate the accuracy of the solutions of this discretized system for system(1).

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الخلاصة

أن المعادلات التفاضلية غالباً ما تنتج معادلات مختلفة عند استخدام إحدى الطرائق المتقطعة القياسية الكلاسيكية التي لا تتشارك ديناميكيا مع نظيراتها ذات استمرارية الحركة. وحديثاً طور بنجاح أسلوب (schemesnon-standard discretization)والذي يَكون حركة متوافقة (Dynamical Consistency) تكون فيها دقة التحليلات العددية كبيرة.

لقد وظف العديد من المؤلفين هذه الطرائق بوصفها موديلات لتطبيقات حياتية مختلفة. وفي هذا البحث قام

الباحث بمراجعة البحوث السابقة في مجال (non-standard discretizatized biological model) والخاصة بنظام(Lotkca-Volterra predator-prey system) بشكل عام ومناقشة تحليل الاستقرار(Stability analysis) للحلول الدورية. وكذلك تم تطبيق عدد من الأمثلة لاستخدام (non-standard discretized predator–prey model) ومناقشة النتائج المستحصلة، أما اسم البرنامج الذي استخدم في الرسم و الحساب فهو"Matlab 7.0".