TOTALLY M*-PARANOORMAL OPERATORS

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Abstract

In this paper we introduce a class of operators on a Hilbert space, we call the elements of this class totally M*-paranormal operator.We study some basic properties of these operators and we give some sufficient conditions for a totally M*-paranormal operator to be normal operator.

Introduction

Let H be a separable complex Hilbert space and B(H) be the Banach algebra of all bounded linear operators on H . An operator T is called *-paranormal if $\|T^*x\|^2 \le \|T^2x\|$ for every unit vector x in H [3], T is called dominant operator if for each λ∈ ¢ there exists a number $M_\lambda > 0$ such that $\|(T-\lambda)^*x\| \le M_\lambda \|(T-\lambda)x\|$ for all x∈ H. Furthermore, if the constants M_λ are bounded By a positive number M, then T is called M-hyponormal operator [2]. Let $\sigma(T), \sigma_p(T), \sigma_{ap}(T), \rho(T)$ denote the spectrum, the point spectrum, the approximate point spectrum of T and the resolvent of T respectively and let $E_T(\mu)$ be the μ-space of T that is $E_T(\mu) = \{x \in H : Tx = \mu x\}$.

Totally M*-paranormal operator, definitions and some basic properties .

Definition 1.1 Let T∈B(H),we call T a totally M*-paranormal operator if for each $\lambda \in \sigma(T)$ there exists a number $M_\lambda > 0$ such that $\|(T-\lambda)^*x\|^2 \le M_\lambda \|(T-\lambda)^2x\|$ for all x ∈H ,with $\|x\|=1$.

Example 1.2

1-Let T:H→H ,defined as follows T=4iI then T is totally M*-paranormal operator .

2- Let H = ℓ_2(¢) ={x:x=(x_1,x_2, x_3…..x_n) $\sum_{i=l}^{\infty} |x_i|^2 < \infty$ } define T:H→H as follows T(x_1, x_2, x_3,…) =(0,x_1,0,0,…..).It is easy to check that T^*(x_1, x_2, x_3,…)=(x_2,0, 0,0,……) , and $\sigma(T)=\{0\}$, if we take x=(0, x_2, x_3, x_4,……). such that $x_2 \neq 0$ then Tx=0 and T^2x=0 but T^*(x_1, x_2, x_3,) =(x_2,0,0,0,…) ≠ 0. Therefore T is not M*-paranormal operator.

Proposition 1.3

Let T∈B(H),T is totally M*-paranormal operator if and only if for each λ∈¢, there exists a number $M_\lambda > 0$ such that $\|(T-\lambda)^*x\|^2 \le M_\lambda \|(T-\lambda)^2x\|$, for all x∈H.

Proof :

Suppose that T is a totally M*-paranormal operator and let $\lambda \in \rho(T)$ then T-λI is invertible therefore

$\|(T-\lambda)^*x\|^2 \le \|(T-\lambda)^*\|\|x\|^2 \le$

$\|(T-\lambda)^*\|^2 \|(T-\lambda)^{-2}\|\|(T-\lambda)^2x\|$

Let $M_\lambda = \|(T-\lambda)^*\|^2 \|(T-\lambda)^{-2}\|$, then T is a totally M*-paranormal operator.

The other direction is clear.

Proposition 1.4

Let T∈B(H), T a totally M*-paranormal operator if and only if

$M_\lambda^2 (T^* - \bar{\lambda})^2 (T-\lambda)^2 - 2k(T-\lambda)(T-\lambda)^* + k^2 \ge 0$

for each k > 0.

Proof :

We know that for each positive b_λ and c_λ, $c_\lambda - 2b_\lambda k + k^2 \ge 0$ for all k >0 if and only if $b_\lambda^2 \le c_\lambda$. Let $b_\lambda = \|(T-\lambda)^*x\|^2$ and $c_\lambda = M_\lambda^2 \|(T-\lambda)^2x\|^2, \|x\|=1$. Then T is a totally M*-paranormal operator if and only if $b_\lambda^2 \le c_\lambda$ this means that T is a totally M*-paranormal operator if and only if

$M_\lambda^2 \|(T-\lambda)^2x\|^2 - 2k\|(T-\lambda)^*x\|^2 + k^2 \ge 0$.

Proposition 1.5

Let T be a totally M*-paranormal operator, then: 1-T-λI and λT are totally M*- paranormal operators.

2- if T is invertible operator then T^{-1} is totally M*-paranormal operator

Proof :(1)

$\left\|[(T-\lambda I)-\alpha I]^{*}x\right\|^{2}=\left\|[T-(\lambda+\alpha)I]^{*}\right\|^{2}$

$\leq M_{\lambda}\left\|[T-(\lambda+\alpha)I]^{2}x\right\|\leq M_{\lambda}\left\|[(T-\lambda)-\alpha]^{2}x\right\|$

If λ= 0 , then λT is totally M*-paranormal operator .

Suppose that λ≠0 then

$\left\|(\lambda T-MI)^{*}x\right\|^{2}=|\lambda|^{2}\left\|[T-(M/\lambda)I]^{*}x\right\|^{2}$

$\leq M_{\lambda}|\lambda|^{2}\left\|[T-(M/\lambda)I]^{2}x\right\|\leq$

$M_{\lambda}\left\|[(\lambda T-MI)^{2}x\right\|$ Hence λT is totally M*-paranormal operator.

(2) if λ=0 then $\left\|T^{*-1}x\right\|^{2}\leq$

$\left\|T^{*-1}\right\|^{2}\|x\|\leq\left\|T^{*-1}\right\|^{2}\left\|T^{2}\right\|\left\|T^{-2}x\right\|$, take

$M_{0}=\left\|T^{*-1}\right\|^{2}\left\|T^{2}\right\|$.

let $\lambda\neq0$ then $\left\|(T^{*-1}-\overline{\lambda})x\right\|^{2}=$

$\left\|-\overline{\lambda}T^{*-1}[T^{*}-(1/\overline{\lambda})I]x\right\|^{2}\|x\|=$

$|\lambda|^{2}\left\|T^{*-1}\right\|^{2}\left\|[T^{*}-(1/\overline{\lambda})I]x\right\|^{2}\leq$

$M_{1/\lambda}|\lambda|^{2}\left\|T^{*-1}\right\|^{2}\left\|[T-(1/\lambda)I]^{2}x\right\|$

$\leq M_{1/\lambda}\left\|T^{*-1}\right\|^{2}\left\|T^{2}\right\|\left\|(T^{-1}-\lambda I)^{2}x\right\|$, take

$M_{\lambda}=\left\|T^{*-1}\right\|^{2}\left\|T^{2}\right\|M_{1/\lambda}$.

Hence T^{-1} is totally M*- paranormal operator.

By using definition it is easy to prove the following Proposition

Proposition 1.6

Let T a totally M*-paranormal operator then :

1-$E_T(\lambda)\subseteq E_{T^*}(\overline{\lambda})$ for all λ∈¢, in fact if x is an eigenvector for T with eigenvalue λ then x is an eigenvector for T* with eigenvalue $\overline{\lambda}$.

2- For a fixed scalar λ, $E_T(\lambda)$ reduces T

3- $E_T(\lambda)\perp E_T(\mu)$ whenever $\lambda\neq\mu$.

For any operator T∈B(H), we set Re(T)=(1/2)[$T+T^{*}$], and ImT=(1/2i)[$T-T^{*}$], [1.p148].

Proposition 1.7

Let T a totally M*-paranormal operator then :

1- if $\sigma(T)\cap$ R ≠ϕ then $0\in\sigma_{ap}(\text{Im}T)$.

2- if $\sigma(T)\cap$ iR ≠ϕ then $0\in\sigma_{ap}(\text{Im}T)$.

3- $\text{Re}\,\sigma(T)\subset\sigma_{ap}(\text{Re}T)$.

4- $\text{Im}\,\sigma(T)\subset\sigma_{ap}(\text{Im}T)$.

Proof :

1-Since $\sigma(T)\cap$ R ≠ϕ then there exists a real number r such that r ∈ $\sigma(T)$.Thus the line L={z∈¢ | Imz=Imr=0} intersect $\sigma(T)$ at a boundary point c. Therefore $c\in\sigma_{ap}(T)$.Then there exist a sequence of unit vectors $\{x_n\}$ in H. Such that (T-cI)$x_n\to0$ and $(T-cI)^{2}x_n\to0$ then $(T-cI)^{*}x_n\to0$ when n→ ∞. Thus (ImT - ImcI)x_n = (1/2i)[($T-T^{*}$)-($c-\overline{c}$)I] $x_n\to$ 0 .

Hence $\text{Im}c\in\sigma_{ap}(\text{Im}T)$.Thus $0\in\sigma_{ap}(\text{Im}T)$

By the same way we can prove (2),(3), and (4).

It easy to prove the following theorem.

Theorem 1.8

1-Let (T-λ I) be a *-paranormal for each λ ∈ $\sigma(T)$ then T is a totally M*-paranormal operator.

2-Let T be a dominant operator and (T-λI) is an idempotent operator for each λ ∈ $\sigma(T)$ then T is totally M*-paranormal operator.

Corollary1.9

Every M-hyponormal operator in particular, every hyponormal operator , normal operator , selfadjoint operator and (T-λI) is idempotent operator for each λ ∈ $\sigma(T)$ then T is totally M*-paranormal operator.

M*-paranormal operator and normal operators

The following theorems give conditions under which a totally M*-paranormal operator is normal operator.

Theorem 2.1

If T is a totally M*-paranormal operator then T can be expressed uniquely as the direct sum $T = T_1 \oplus T_2$ defined on the space $H=H_1 \oplus H_2$ with the following properties:

1- H_1 is the closure of the space spanned by the eigenvectors of T
2- T_1 is normal
3- $\sigma_p(T_2) = \phi$
4-T is normal if and only if T_2 is normal

Proof:

1. Let $H_1 = \sum_{\lambda \in \sigma_p(T)} \oplus E_T(\lambda)$. Since $H_1^\perp$ is a closed linear subspace, then $H= H_1 \oplus H_2$ where $H_2 = H_1^\perp$ Let $T_1 = T|_{H1}$ and $T_2 = T|_{H_2}$ then $T= T_1 \oplus T_2$ uniquely

2- Let $H_1 = E_T(\lambda_1) \oplus E_T(\lambda_2) \oplus E_T(\lambda_3) \oplus ....$ and $x \in H_1$. Then $x = x_{\lambda_1} + x_{\lambda_2} + x_{\lambda_3} + ...$ for any $x_{\lambda i} \in E_T(\lambda i)$, i. e., $x = \sum_{\lambda \in \sigma_p(T_1)} x_{\lambda i}$ for each i , and $\sum |x|^2_{\lambda i} < \infty$

$$T_1^* T_1 \left[\sum_{\lambda_i \in \sigma(T_1)} x_{\lambda_i} \right] = T_1^* \left[\sum_{\lambda_i \in \sigma_p(T_1)} T_1 x_{\lambda i} \right] =$$

$$\lambda_1 T_1 * x_{\lambda_1} + \lambda_2 T_2 * x_{\lambda_2} + \lambda_3 T_3 * x_{\lambda_3} + ...$$

$$= |\lambda_1|^2 x_{\lambda_1} + |\lambda_2|^2 x_{\lambda_2} + |\lambda_3|^2 x_{\lambda_3} + ...$$ and

$$T_1 T_1 * x = T_1 T_1 * (x_{\lambda_1} + x_{\lambda_2} + x_{\lambda_3} + ..)$$

$$= \overline{\lambda_1}\, Tx_{\lambda_1} + \overline{\lambda_2}\, Tx_{\lambda_2} + \overline{\lambda_3}\, Tx_{\lambda_3} + ...$$

$$= |\lambda_1|^2 x_{\lambda_1} + |\lambda_2|^2 x_{\lambda_2} + |\lambda_3|^2 x_{\lambda_3} + ... \quad .$$

Hence T_1 is normal .

3- Suppose that $\sigma_p(T_2) \neq \phi$ and let $M \in \sigma_p(T_2)$ Then there exists $x \neq 0 \in H_2$ such that $(T_2-M)x = 0$ .Since $T(0 + x) = T_2 x = Mx = M(0 + x)$, hence x in H_1.
This is contradiction to $H_2 = H_1^\perp$ and $x \neq 0$, therefore $\sigma_p(T_2)= \phi$

4- Since

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix}, \text{and } T^* = \begin{bmatrix} T_1^* & 0 \\ 0 & T_2^* \end{bmatrix} \in B(H_1, H_2)$$

Then $T^*T = \begin{bmatrix} T_1^*T_1 & 0 \\ 0 & T_2^*T_2 \end{bmatrix}$ and $TT^* = \begin{bmatrix} T_1T_1^* & 0 \\ 0 & T_2T_2^* \end{bmatrix}$

Thus $T^*T = TT^*$ if and only if $T_2^*T_2 = T_2T_2^*$.

Theorem 2.2

Let $T \in B(H)$,be a totally M*-paranormal operator .If the eigenspaces $E_T(\lambda)$ of T Form a total family , then T is normal operator .

Proof :

Let H_0 be the null space of TT*-T*T, the problem is to show that $H_0=H$ or equivalently $H_0^\perp = \{\theta\}$, claim $E_T(\mu) \subseteq H_0$ for all μ. Let $x \in E_T(\mu)$, thus $T^* x \in E_T(\mu)$ Therefore $TT^*x = \mu (T^*x)=T^*(\mu x)=T^*Tx$. Thus $(TT^*-T^*T)x=0$ and $x \in H_o$.
It follows that if $x \perp H_0$, then $x \perp E_T(\mu)$ for all M .But the eigenspaces form total family. Hence x=0 then $(TT^*-T^*T)x=0 \ \forall \ x \in H$ .Thus T*T=TT* and T is normal operator

Collarary 2.3

Let $T - \lambda I$ be a *-paranormal for each $\lambda \in \sigma(T)$. If the eigenspaces $E_T(\lambda)$of T from a total family then T is normal operator.

References

[1] S.K. Berberrian, "Introduction to Hilbert space", Second edition. Chelsea Publising Company. New York.n .Y., 1976.
[2] Hou., jinchaan ,"Some Results on M-hyponormal operators",Journal of math. Research and Exposition vol.4, No.2, 1984,pp.101-103.
[3] C.S.Rtoo,"some class of operators" Math. J, Toyamess Univ,Vol. 21,1998,pp147-152.

الخلاصة

فـي هذا البحـث نقدم صـنفا مـن المـؤثرات علـى فضـاء هلبرت. يطلـق علـى هذا الصـنف من المؤثرات بـ (المؤثرات الموازيـة للسـوية مـن الـنمط – *M) سـوف نـدرس بعـض الخواص الاساسية لهذه المؤثرات .كما نعطي بعض الشروط عليها للحصول على المؤثر السوي.