∈- NIELSEN COINCIDENCE POINT THEORY

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Abstract

Let $f, g : X \to X$ be maps of a compact connected Riemannian manifold, with or without boundary. For $\epsilon > 0$ sufficiently small, we introduce an ϵ – Nielsen coincidence number $N^{\epsilon}(f,g)$ that is a lower bound for the number of coincidence points of all self – maps that are ϵ - homotopic to *f* and *g*. We prove that there is always maps $f_1, g_1 : X \to X$ that is ϵ – homotopic to *f* and *g* such that f_1 and g_1 have exactly $N^{\epsilon}(f,g)$ coincidence points.

Introduction

The Nielsen coincidence point theory applied to study of the calculation by computer of multiple solutions of systems of polynomials equations,using a Nielsen coincidence number to obtain a lower bound for the number of distinct solution [5].

Because machine accuracy is finite, the solution procedure requires approximations, but the information is still applicable to the original problem. The reason is that sufficiently close functions on well - behaved spaces are homotopic and Nielsen coincidence number is a homotopy invariant. Although the homotopy between two sufficiently close maps are through maps that are close to both, no limitation on the homotopies employs. The purpose of this paper is to introduce a type of Nielsen coincidence point theory that does assume that a specified to-lerance for error must be respected.

If distortion is limited to a pre–assigned amount, then it may not bepossible, without exceeding the limit, to deform maps *f* and *g* so that it has exactly $N(f,g)$ coincidence points. For very simple example, consider the maps $f, g : I \to I = [0,1]$ such that $f(0) = f(2/3) = 1$, $g(0) = g(2/3) = 0, f(1/3) = f(1) = 0$ and $g(1/3) = g(1) = 1$. If the maps f_1 and g_1 have $N(f,g)$=1 coincidence point, then there must be some $t \epsilon I$ such that $|f(t) - f_1(t)| < 1/3$ and $|g(t) - g_1(t)| < 1/3$.

This example suggests a concept of the geometric minimum (coinciden-ce point) number of maps $f, g : X \to X$ different from the one, *MF[f,g]*, that is the focus of Nielsen coincidence point theory, namely,

$$MF[f,g]=\min\{\#coin(f,g) : f_1, g_1 \text{ homotopic to } f, g \text{ respectively}\}, \quad \text{.................(1)}$$

where #coin(*f*, *g*) denotes the cardinality of the coincidence point set.The distance *d*(*f*, *g*) between maps $f, g : Z \to X$, where Z is compact and *X* is a metric space with distance function *d*, is defined by

$$d(f,g) = max\{d(f(z), g(z)) : z \epsilon Z\}. \quad \text{.........(2)}$$

Given $\epsilon > 0$, a homotopy $\{h_t\} : Z \to X$ is an ϵ – homotopy if $d(h_t, h_{t'}) < \epsilon$ for all $t, t' \epsilon I$. For a given $\epsilon > 0$, we define the ϵ - minimum (coincidence point)number $MF^{\epsilon}(f,g)$ of maps $f, g : X \to X$ of a compact metric space by

$$MF^{\epsilon}(f,g) = \min\{\#(coin(f_1, g_1)) : f_1, g_1 \text{ is } \epsilon - \text{homotopic to } f, g \text{ respectively}\}. \quad \textbf{..............................(3)}$$

Note that the concept of ϵ–homotopic maps does not give an equivalence relation

The notation *MF[f,g]* for the minimum number incorporates the symbol [*f*, *g*], generally used to denote the homotopy class of *f* and *g*, because *MF*[*f*, *g*] is a homotopy invariant. We do note use the corresponding notation for the ϵ -minimum number because it is not invariant on the homotopy class of *f* and *g*. For instance, although constants maps K_1 and K_2 of I are homotopic to maps *f* and *g* of the example,for which $MF^{\epsilon}(f,g) = 3$ for any $\epsilon \leq 1/3$,obviously $MF^{\epsilon}(K_1, K_2) = 1$ for any choice of ϵ.

Let $f, g : X \to X$ be maps of acompact manifold.Just as the Nielsen coincidence $N(f,g)$ has property $N(f,g) \leq MF[f,g]$, in the next section we will introduce the ϵ – Nielsen coincidence number $N^{\epsilon}(f,g)$, for ϵ sufficiently small,that has the property $N^{\epsilon}(f,g) \leq MF^{\epsilon}(f,g)$. My main result proven in section 3, is a " minimum coincidence theorem " : give $f, g : X \to X$, there exists f_1 and g_1 with $d(f, f_1) < \epsilon$ and $d(g, g_1) < \epsilon$ such that f_1 and g_1 have exactly $N^{\epsilon}(f,g)$ coincidence points.

The ∊ – Nielsen coincidence number

Let X be a compact, connected differentiable manifold, possibly with boundary. We introducea Riemannian metric on X and denote the associated distance function by *d.* If the boundary of X is nonempty, we choose a product metric on a tubular neighborhood of the boundary and then use a partition of unity to extend to a metric for X. There is an $\epsilon > 0$ small enough so that,if *p*,*q* ϵ X with d(*p*,*q*) < ϵ, then there is a unique geodesic c_{pq} connected them. This choice of ϵ is possible even through the manifold may have a nonempty boundary because the metric is a product on a neighborhood of the boundary for the rest of this paper, ϵ >0 will always be small enough so that points within a distance of ϵ are connected by a unique geodesic. We view the geodesic between *p* and *q* as a path $c_{pq}(t)$ in X such that $c_{pq}(0) = p$ and $c_{pq}(1) = q$. The function that takes the pair (*p, q)* to c_{pq} is continuous. If $x \epsilon c_{pq}$ then $d(p,x) \leq d(p,q)$ because c_{pq} is the shortest path from *p* to *q* (see [7, corollary 10.8 on page 62]).

If $f, g, f_1, g_1 : Z \to X$ are maps with $d(f, f_1) < \epsilon$ and $d(g, g_1) < \epsilon$, then setting $h_t(z) = c_{f(z), f_1(z)}(t)$ and $h'_t(z) = c_{g(z), g_1(z)}(t)$ defines an ϵ –homotopy between f and f_1, g and g_1 respectively. Thus an equivalent definition of the ϵ –minimum coinci-dence number of $f, g: X \to X$ is

$$MF^{\epsilon}(f,g) = min\{\#(coin(f_1, g_1)): d(f, f_1) < \epsilon \; and \; d(g, g_1) < \epsilon\} \quad \text{.....................} (4)$$

For maps $f, g: X \to X$, let

$$\Delta^{\epsilon}(f,g) = \{x \, \epsilon \, X : d(f(x), g(x)) < \epsilon\} \quad \text{.................................} (5)$$

Theorem (1) :

The set $\Delta^{\epsilon}(f,g)$ is open in X.

Proof:

Let $\mathbb{R}^{+}$denote the subspace of $\mathbb{R}$ of non– negative real numbers.Define $D_{f,g}: X \to \mathbb{R}^{+}$by $D_{f,g}(x) = d(f(x), g(x))$. Since $[0, \epsilon)$ is open in $\mathbb{R}^{+}$, it follows that $\Delta^{\epsilon}(f,g) = D_{f,g}^{-1}([0,\epsilon))$ is an open subset of X. For maps $f, g: X \to X$,define an equivalence relation on Coin(*f, g*) as follows :

$x, y \, \epsilon \, Coin(f,g)$ are ϵ - equivalent, if there is a path $w : I \to X$ from x to y such that $d(f \circ w, g \circ w) < \epsilon$.

The equivalence classes will be called the ϵ-coincidence point classes or, more briefly, the ϵ-cpc of *f* and *g*.

Theorem (2):

Coincidence points x, y of $f, g: X \to X$ are ϵ–equivalent if and only if there is a component of $\Delta^{\epsilon}(f,g)$ that contains both of them.

Proof :

Suppose $x, y \, \epsilon \, Coin(f,g)$ are ϵ – equivalence and let w be a path in X from x to y such that $d(f \circ w, g \circ w) < \epsilon$. Thus for each $s \, \epsilon \, I$ we have $d(f(w(s)), g(w(s))) < \epsilon$, so $w(I) \subset \Delta^{\epsilon}(f,g)$. Since $w(I)$ is connected it is contained in some component of $\Delta^{\epsilon}(f,g)$. Conversely, suppose $x, y \, \epsilon \, Coin(f, g)$ are in the same component of $\Delta^{\epsilon}(f,g)$.The components of $\Delta^{\epsilon}(f,g)$ are pathwise connected so there is a path w in it from x to y. Since w is in $\Delta^{\epsilon}(f,g)$ that means $d(f \circ w, g \circ w) < \epsilon$ and thus x and y are ϵ – equivalent. □

Theorems (1) and (2) imply that the ϵ– cpc are open in Coin (*f, g*), so there are finitely many of them F_1^ϵ , $F_2^\epsilon, \dots$, F_r^ϵ . We denote the component of $\Delta^\epsilon(f,g)$ that contains F_j^ϵ by $\Delta_j^\epsilon(f,g)$. An ϵ–cpc, $F_j^\epsilon = Coin(f,g) \cap \Delta_j^\epsilon(f,g)$ is essential if the coincidence point index $i((f,g),\Delta_j^\epsilon(f,g)) \neq 0$. The ϵ–Nielsen coincidence number of f and g, denoted by $N^\epsilon(f,g)$, the number of essential ϵ – cpc.

Theorem (3) :

If the coincidence points x and y of $f,g : X \to X$ are ϵ – equivalent, then x and y are in the same (Nielsen) coincidence point class. Therefore each coincidence point class is a union of ϵ – cpc and $N^\epsilon(f,g) \geq N(f,g)$.

Proof :

If x and y are ϵ – equivalent by means of a path w between them such that $d(f \circ w, g \circ w) < \epsilon$ then $h_t(s) = c_{f(w(s)),g(w(s))}(t)$ defines a homotopy, relative to the endpoints, between $f \circ w$ and $g \circ w$ so x and y are in the same coincidence point class. Therefore a coincidence point class F of f and g is the union of ϵ –cpc.If F is essential, the additively property of coincidence point index implies that at least one of the ϵ – cpc it contains must be an essential ϵ – cpc, thus $N^\epsilon(f,g) \geq N(f,g)$. □

The ϵ – Nielsen coincidence number is a local Nielsen coincidence number in the sense of [4], [1], spe-cifically $N^\epsilon(f,g) = n((f,g),\Delta^\epsilon(f,g))$.

However, in the local Nielsen coinci-dence theory, the domain U of the local Nielsen $n((f,g),U)$ is the same for all the maps considered whereas $\Delta^\epsilon(f,g)$ depends on f and g.

Theorem (4):

Let $f,g: X \to X$ be maps then $N^\epsilon(f,g) \leq MF^\epsilon(f,g)$.

Proof :

Given maps $f_1, g_1 : X \to X$ with $d(f,f_1) < \epsilon$ and $d(g,g_1) < \epsilon$, let $\{h_t\},\{h_t'\}: X \to X$ be the ϵ -homotopy with $h_0 = f$ and $h_1 = f_1$, $h_0' = g$ and $h_1' = g_1$ defined by $h_t(x) = c_{f(x),f_1(x)}(t)$ and $h_t'(x) = c_{g(x),g_1(x)}(t)$ respectively.Theorem (1) implies that $d(f(x),g(x)) \geq \epsilon$ for all x in the Boundary of ,$\Delta_j^\epsilon(f,g)$. Thus for x in the boundary of, $\Delta_j^\epsilon(f,g)$ and $t \in I$ we have

$$d(f(x),h_t(x)) + d(h_t(x),h_t'(x)) + d(h_t'(x),g(x)) \geq d(f(x),g(x)) \geq \epsilon. \quad \dots\dots\dots (6)$$

Since $\{h_t\}$ and $\{h_t'\}$ are an ϵ-homotopy, $d(h_t(x),f(x)) = d(h_t(x),h_0(x)) < \epsilon$ and $d(h_t'(x),g(x)) = d(h_t'(x),h_0'(x)) < \epsilon$ so $d(h_t(x),h_t'(x)) > 0$, that is h_t and h_t' have no coincidence points on the boundary of $\Delta_j^\epsilon(f,g)$.

Therefore, the homotopy property of coincidence point index implies that

$$i((f,g),\Delta_j^\epsilon(f,g)) = i((f_1,g_1),\Delta_j^\epsilon(f,g)). \quad \dots\dots\dots (7)$$

Consequently, if $F_j^\epsilon = Coin(f,g) \cap \Delta_j^\epsilon(f,g)$ is an essential ϵ–cpc, then $i((f_1,g_1),\Delta_j^\epsilon(f,g)) \neq 0$ so f_1 and g_1 have a coincidence point in $\Delta_j^\epsilon(f,g)$.We conclude that f_1 and g_1 have at least $N^\epsilon(f,g)$ coincidence points. □

The minimum coincidence theorem :

The main result in this section is to prove the minimum coincidence theorem, but before that we need the following theorem.

Lemma (5) :

Let F be a closed subset of a compact manifold X and let U be an open, connected subset of X that contains F, then there is an open, connected subset V of X containing F such that the closure of V is contained in U.

Proof :

Since F and $X - U$ are disjoint compact sets, there is an open set W containing F such that the closure of W is contained in U.There are finitely many components $W_1, \dots, W_r$ of W that contain points of the compact set F. Let α_1 be a path in U from $x_1 \epsilon\ W_1 \cap F$ to $x_2 \epsilon\ W_2 \cap F$ and let A_1 be an open subset of U containing α_1 such that the closure of A_1 is in U.

Since α_1 is connected,we may assume A_1 is also connected. Continuing in this manner, we let

$\boldsymbol{V = W_1 \cup A_1 \cup W_2 \cup A_2 \cup \ldots \cup W_r \cup A_r}, \ldots$. (8)
Which is connected. The closures of each of the W_i and A_i are in U so the closure of V is also in U. □

Let $F_j^\epsilon = coin(f,g) \cap \Delta_j^\epsilon(f,g)$ be an ϵ – cpc. By lemma 5, there is an open, connected subset V_j of $\Delta_j^\epsilon(f,g)$ containing F_j^ϵ whose closure cl(V_j) is in $\Delta_j^\epsilon(f,g)$. For the map $D_{f,g}: X \to \mathbb{R}^+$ defined by $D_{f,g}(x) = d(f(x), g(x))$, we see that $D_{f,g}\left(cl\left(V_j\right)\right) = [0, \delta_j]$ where $\delta_j < \epsilon$. Choose $\alpha_j > 0$ small enough so that $\delta_j + 2\alpha_j < \epsilon$.

<u>Theorem(6)</u> (Minimum Coincidence Theorem):

Given $\boldsymbol{f, g: X \to X}$, there exists $\boldsymbol{f_1, g_1: X \to X}$ with $\boldsymbol{d(f, f_1) < \epsilon}$ and $\boldsymbol{d(g, g_1) < \epsilon}$ such that $\boldsymbol{f_1}$ and $\boldsymbol{g_1}$ have exactly $\boldsymbol{N^\epsilon(f, g)}$ coincidence points.

<u>Proof:</u>

We will define f_1 and g_1 outside $\Delta^\epsilon(f,g)$ to be a simplicial approximation of f and g respectively such that $d(f, f_1) < \alpha$ and $d(g, g_1) < \alpha$, where α denotes the minimum of the α_j . The proof then consists of describing f_1 and g_1 on each $\Delta_j^\epsilon(f,g)$ so, to simplify notation, we will assume for now that $\Delta^\epsilon(f,g)$ is connected and thus we are able to suppress the subscript j. Triangulate X and take a subdivision of such small mesh that if u and v are a simplicial approximation to f and g respectively with respect to that triangulation, then $d(u,f) < \alpha/2$ and $d(v,g) < \alpha/2$ and, for σ a simplex that intersect X –int(V), we have $u(\sigma) \cap \sigma = \phi$ and $v(\sigma) \cap \sigma = \phi$. By the Hopf construction, we may modify u and v, moving no point more than $\alpha/2$, so that it has finitely many coincidence points, each of which lies in a maximal simplex in V and therefore in the interior of X (see [2,Theorem 2 on page 118]), [9]. We will still call the modified maps u and v, so we now have maps u and v with finitely many coincidence points and it has the property that $d(u,f) < \alpha$ and $d(v,g) < \alpha$. Refine the triangulation of X so that the coincidence points of u and v are vertices.Since V is a connected n–manifold, we may connect the coincidence points of u and v by paths in V, let P be the union of all these paths. With respect to a sufficiently fine subdivision of the triangulation of X, the star neighbor-hood $S(P)$ of P, which is a finite, connected polyhedron, has the property that the derived neighborhood of $S(p)$ lies in V. Let T be a spanning tree for the finite connected graph that is the 1–skeleton of $S(P)$, then T contains coin(u,v). Let $R(T)$ be a regular neighborhood of T in $V \cap int(X)$ then, since T is collapsible, $R(T)$ is the n – ball by [8, Corollary 3.27 on page 41].

Thus we have a subset W = int($R(T)$) of V containing coin (u,v) and a homeomorphism $\varphi: W \to \mathbb{R}^n$.We may assume that φ (coin(u,v)) lies in the interior of the unit ball in $\mathbb{R}^n$,which we denote B_1. Set $\varphi^{-1}(B_1) = B_1^*$. If $x \epsilon B_1^*$, then

$$\boldsymbol{d(u(x), v(x)) \le d(u(x), f(x)) + d(f(x), g(x)) + d(g(x), v(x)) < \delta + \alpha + \delta < 2\delta + \alpha < \epsilon}, \quad \textbf{(9)}$$

So there is a unique geodesic $C_{u(x)v(x)}$ connected $u(x)$ to v(x). Consider the map $H: B_1^* \times I \to X$ defined by $H(x,t) = C_{u(x),v(x)}(t)$, then $H^{-1}(W)$ is an open subset of $B_1^* \times I$ containing $B_1^* \times \{0\}$. Therefore there exists $t_0 > 0$ such that $H(B_1^* \times [0,t_0]) \subset W$.

Denote the origin in $\mathbb{R}^n$ by 0 and let $0^* = \varphi^{-1}(0)$. Define a retraction $\rho: B_1^* - 0^* \to \partial B_1^*$, the boundary of

B_1^*, by

$$\boldsymbol{\rho(x) = \varphi^{-1}\left(\frac{1}{|\varphi(x)|}\varphi(x)\right)} \ldots\ldots\ldots\ldots\ldots\ldots \text{(10)}$$

Define $K: B_1^* \times [0,t_0] \to W$ by setting $K(0^*,t) = 0^*$ for all t and, otherwise let

$$\boldsymbol{k(x,t) = \varphi^{-1}(\varphi|(x)|(H(\rho(x),t)))} \ldots\ldots\ldots\ldots \text{(11)}$$

The function K is continuous because $\varphi(H(\partial B_1^* \times I))$ is a bounded subset of $\mathbb{R}^n$. Now define $D_K: B_1^* \times [0,t_0] \to \mathbb{R}^+$ by $D_K(x,t) = d(x, K(x,t))$. Since $D_K^{-1}([0,\eta))$ is an open subset of $B_1^* \times [0,t_0]$ containing $B_1^* \times \{0\}$, there exists $0 < t_1 < t_2 < t_0$ such that $d(x, K(x,t_1)) < \alpha$ and $d(x, K(x,t_2)) < \alpha$. Define

$h_1, h_2: B_1^* \to X$ by $h_1(x) = K(x, t_1)$ and $h_2(x) = K(x, t_2)$ respectively.

Next we extend h_1 and h_2 to the set B_2^* consisting of $x \epsilon W$ such that $0 \le |\varphi(x)| \le 2$ by letting

$$h_1(x) = C_{u(x),v(x)}((1-t_1)|\varphi(x)| + 2t_1 - 1),$$
$$h_2(x) = C_{u(x),v(x)}((1-t_2)|\varphi(x)| + 2t_2 - 1) \quad ,\ldots\ldots(12)$$

where $1 \le |\varphi(x)| \le 2$. Noting that $h_1(x) = u(x)$ and $h_2(x) = v(x)$ if $\varphi\ (x) = 2$, we extend h_1 and h_2 to all X by setting $h_1 = u$ and $h_2 = v$ outside B_2^*.

The maps h_1 and h_2 have a single coincidence point at 0^*. If $i\left((f,g), \Delta_j^\epsilon(f,g)\right) \neq 0$, we let $f_1 = h_1$, $g_1 = h_2 : X \to X$. If $i\left((f,g), \Delta_j^\epsilon(f,g)\right) = 0$, by [2, Theorem 4 on page123], there is maps $f_1, g_1 : X \to X$, identical to *u* and *v* respectively outside of B_1^*, such that f_1 and g_1 have no coincidence point in B_1^* and $d(f_1, u) < \alpha$ and $d(g_1, v) < \alpha$.We claim that$d(f, f_1) < \epsilon$ and $d(g, g_1) < \epsilon$. For $x \notin B_2^*$, we defined $f_1(x) = u(x)$ and $g_1(x) = v(x)$ where $d(u,f) < \alpha < \epsilon$ and $d(v, g) < \alpha < \epsilon$.

If $x \epsilon B_2^* - B_1^*$, then $f_1(x) = h_1(x) \epsilon C_{u(x),v(x)}$ and $g_1(x) = h_2(x) \epsilon C_{u(x),v(x)}$ so $d(h_1(x), u(x)) \le d(v(x), u(x))$ and $d(h_2(x), v(x)) \le d(u(x), v(x))$.

Therefore,

$$d(f_1(x), f(x)) = d(h_1(x), f(x))$$
$$\le d(h_1(x), u(x)) + d(u(x), f(x))$$
$$\le d(v(x), u(x)) + d(u(x), f(x))$$
$$\le \left(d(v(x), f(x)) + d(f(x), u(x))\right)$$
$$+ d(u(x), f(x))$$
$$< \delta + 2\alpha < \epsilon.$$
$$d(g_1(x), g(x)) = d(h_2(x), g(x))$$
$$\le d(h_2(x), v(x)) + d(v(x), g(x))$$
$$\le d(u(x), v(x)) + d(v(x), g(x))$$
$$\le \left(d(u(x), g(x)) + d(g(x), v(x))\right)$$
$$+ d(v(x), g(x))$$
$$< \delta + 2\alpha < \epsilon. \ldots\ldots(13)$$

Now suppose $x \epsilon B_1^*$. If $i\left((f, g), \Delta_j^\epsilon(f, g)\right) \neq 0$, then $f_1(x) = h_1(x) = K(x, t_1)$ and $g_1(x) = h_2(x) = K(x, t_2)$ so

$$d(f_1(x), f(x)) = d(K(x,t_1), f(x))$$
$$\le d(K(x,t_1), u(x)) + d(u(x), f(x))$$
$$< \alpha + \delta < \epsilon.$$
$$d(g_1(x), g(x)) = d(K(x,t_2), g(x))$$
$$\le d(K(x,t_2), v(x)) + d(v(x), g(x))$$
$$< \alpha + \delta < \epsilon \ \ldots\ldots(14)$$

If $i\left((f,g), \Delta_j^\epsilon(f,g)\right) = 0$ then

$$d(f_1(x), f(x)) \le d(f_1(x), h_1(x))$$
$$+ d(h_1(x), f(x))$$
$$= d(f_1(x), h_1(x)) + d(K(x, t_1), f(x))$$
$$< \alpha + (\alpha + \delta) < \epsilon .$$
$$d(g_1(x), g(x)) \le d(g_1(x), h_2(x))$$
$$+ d(h_2(x), g(x))$$
$$= d(g_1(x), h_2(x)) + d(K(x, t_2), g(x))$$
$$< \alpha + (\alpha + \delta) < \epsilon. \ldots\ldots(15)$$

Which completes the proof that $d(f, f_1) < \epsilon$ and $d(g, g_1) < \epsilon$.

We return now to the general case, in which $\Delta^\epsilon(f,g)$ may be not connected.Applying the construction above to each $\Delta_j^\epsilon(f,g)$ gives us maps $f_1, g_1: X \to X$ with exactly $N^\epsilon(f,g)$ coincidence points. For $x \notin \Delta^\epsilon(f,g)$ we defined $f_1$and $g_1$to be a simplicial approximation with $d(f, f_1) < \alpha < \epsilon$ and $d(g, g_1) < \alpha < \epsilon$.For $x \epsilon \Delta_j^\epsilon(f,g)$, the argument just concluded proves that

$$d(f, f_1) \le 2\alpha_j + \delta_j < \epsilon$$
$$d(g, g_1) \le 2\alpha_j + \delta_j < \epsilon, \ldots\ldots(16)$$

because α is the minimum of the α_j, so we know that $d(f, f_1) < \epsilon$ and $d(g, g_1) < \epsilon$. □

Theorem 6 throws some light on the failure of the Wecken property for surfaces [3]. For instance,consider the celebrated example of

Jiang [6], of maps f and g of the paths surface with $N(f\ ,\ g)\ =\ 0$ but $MF[f,g]=2$.The coincidence point set of f and g consists of three points, one of them of index zero. The other two coincidence points, y_1 and y_2 are of index +1 and -1 respectively and Jiang described a path, call it σ from y_1 to y_2 such that $g\circ\sigma$ is homotopic to $f\circ\sigma$ relative to the endpoints. Suppose $\epsilon>0$ is small enough so that points in the pants surface that are within ϵ of each other are connected by a unique geodesic. If there were a path τ from y_1 to y_2 such that $g\circ\tau$ and $f\circ\tau$ were ϵ - homotopic, then $N^{\epsilon}(f,g)=0$ and therefore, by theorem 6, there would be a coincidence point free maps homotopic to f and g. Since Jiang proved that no maps homotopic to f and g can be coincidence point free, we conclude that no such path τ exists. In other words,for any paths τ from y_1 to y_2 that is homotopic to $g\circ\tau$ and $f\circ\tau$ relative to the endpoints, it must be that $d(g\circ\tau,f\circ\tau)>\epsilon$.

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الخلاصة

لتكن $f,g\colon X\to X$ دوال لتشعب ريمان المتصل المتراص، مع أو بدون حد لتكن **0**<ϵ صغيرة بشكل كافي سوف نقدم ϵ – عدد نيلسن المتطابق $N^{\epsilon}(f,g)$ أي الحد الأدنى لعدد النقاط المتطابقة لكل الدوال التي تكون ϵ – هوموتوبيا ل f و g . سوف نبرهن في هذا البحث أن هناك دوال f_1 , g_1 : $X\to X$ التي تكون ϵ – هوموتوبيا لـ f و g وعدد النقاط المتطابقة لها هي $N^{\epsilon}(f,g)$ بالضبط .