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ABOUT THE DISTANCE FUNCTION BETWEEN FUZZY SETS

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Abstract

In this paper, we will study and construct the distance function between two fuzzy sets in a fuzzy metric space and give an example to illustrate this function.

1-Introduction

Since its inception 45 years ago the theory of fuzzy sets has advanced in a variety of ways and in many disciplines. Applications of this theory can be found for example in artificial intelligence, computer science, control engineering, decision theory, expert systems, logic, management, science operation research and robotics. Fuzzy sets are not "fuzzy" because of a lack of clarity, which could perhaps be remedied by focusing more clearly on the situation, and in that respect the use of the word "fuzzy" is misleading, [Zadeh, 1965].

In this paper we will define the distant between two fuzzy sets by using the level sets and prove that (I^X, D) is a fuzzy semi metric space, where I^X is the set of all fuzzy subsets of the universal set X, and D is the Hausdorff distance function defined on I^X.

2-Basic Concepts

Next, Some basic definitions and fundamental notations are given for completeness. First of all, we start with the definition of the characteristic function related to non-fuzzy sets.

Definition (1), [Dubois, 1980]:

Let X be a classical set of objects, called the universal set whose generic elements are denoted by x. The membership in a classical subset A of X is often viewed as a characteristic function μ from X onto {0, 1}, such that:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$

where {0, 1} is called a valuation set.

If the valuation set is allowed to be the real interval [0, 1], Then A is called a fuzzy set, and is denoted symbolically by $\tilde{A}$, and the characteristic function μ is denoted by $\mu_{\tilde{A}}$, where $\mu_{\tilde{A}}(x)$ is the grade of membership of x in $\tilde{A}$, [Zadeh, 1965].

The closer the value of $\mu_{\tilde{A}}(x)$ to 1, the more x belong to $\tilde{A}$. Clearly, $\tilde{A}$ is a subset of X that has no sharp boundary. $\tilde{A}$ is completely characterized by:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) \mid x \in X, 0 \le \mu_{\tilde{A}}(x) \le 1\}.$$

The basic algebraic operations and relations related to fuzzy set theory may be given. Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be fuzzy sets of an universal set X, then [Erceg, 1979], [Klir, 2000]:

1. $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}}(x) \le \mu_{\tilde{B}}(x)$, $\forall\, x \in X$.
2. $\tilde{A}^c$ is the complement of $\tilde{A}$ if and only if $\mu_{\tilde{A}^c}(x) = 1 - \mu_{\tilde{A}}(x)$.
3. $\tilde{A} \cap \tilde{B}$ is a fuzzy set with membership function:
$$\mu_{\tilde{A}\cap\tilde{B}}(x) = \text{Min}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall\, x \in X$$
4. $\tilde{A} \cup \tilde{B}$ is a fuzzy set with membership function:
$$\mu_{\tilde{A}\cup\tilde{B}}(x) = \text{Max}\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\},\ \forall x \in X.$$

The fuzziness of $\tilde{A} \cap \tilde{B}$ and $\tilde{A} \cup \tilde{B}$ follows directly from the uniqueness of the operators Max and Min.

More generally, for any index set I, we have $\bigcup_{i\in I} \tilde{A}_i$ and $\bigcap_{i\in I} \tilde{A}_i$ are fuzzy subsets of X with membership functions, respectively:

$$\mu_{\bigcup_{i\in I} \tilde{A}_i}(x) = \text{Sup}\,\{\mu_{\tilde{A}_i}(x) : \forall\, i \in I\}$$

$$\mu_{\bigcap_{i\in I} \tilde{A}_i}(x) = \text{Inf}\,\{\mu_{\tilde{A}_i}(x) : \forall\, i \in I\}$$

for all $x \in X$.

One of the important aspects of fuzzy set theory is that the law of ordinary set theory that is no longer valid here, which is so called sometimes, the excluded middle law, because $\tilde{A} \cap \tilde{A}^c \neq \tilde{\varnothing}$ and $\tilde{A} \cup \tilde{A}^c \neq X$. Since the fuzzy set $\tilde{A}$ has no definite boundary and neither $\tilde{A}^c$, i.e., for all $x \in X$, we have:

$$\text{Min}\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{A}^c}(x)\} \leq 0.5$$

and

$$\text{Max}\{\mu_{\tilde{A}}(x),\ \mu_{\tilde{A}^c}(x)\} \geq 0.5$$

and it can be seen that $\tilde{A}$ and $\tilde{A}^c$ may overlap and then $\tilde{A} \cup \tilde{A}^c$ do not cover X exactly, [Fadhel, 1998].

3- α-Level Sets

The scope of this section is to cover the basic and the most important properties of the so-called α-level sets, which corresponds to any fuzzy set $\tilde{A}$. α-level sets are the collection between fuzzy sets and ordinary sets, which may be used to prove that most of the results that are satisfied in ordinary sets are also satisfied here in fuzzy sets, which means, there is another approach in which the classical sets and fuzzy sets are connected to each other.

Definition (2), [Yan, J., 1994]:

The α-level (or α-cut) set of a fuzzy set $\tilde{A}$, labeled by A_α, is the crisp set of all x in X such that $\mu_{\tilde{A}}(x) \geq \alpha$.

One can notice that an α-level set discards the points whose membership values are lower than α, in mathematical symbols α-level sets are defined as:

$$A_\alpha = \{x \in X \mid \mu_{\tilde{A}}(x) \geq \alpha,\ \alpha \in (0, 1]\}$$

Remark (1), [Kandel, A., 1984]:

The level fuzzy sets of a fuzzy set $\tilde{A}$ are denoted as the fuzzy sets $\tilde{A}_\alpha$, $\alpha \in [0, 1]$, such that:

$$\tilde{A}_\alpha = \{(x,\ \mu_{\tilde{A}}(x)) \mid x \in A_\alpha\}$$

Remark (2), [Whaib, S. A., 2005]:

One also defines the strong α-level sets as:

$$A_\alpha^+ = \{x \in X \mid \mu_{\tilde{A}}(x) > \alpha,\ \alpha \in (0, 1]\}$$

It is easily checked that the following properties hold for all $\alpha, \beta \in (0, 1]$:

1. $(A \cup B)_\alpha = A_\alpha \cup B_\alpha$.
2. $(A \cap B)_\alpha = A_\alpha \cap B_\alpha$.
3. $\tilde{A} \subseteq \tilde{B}$ gives $A_\alpha \subseteq B_\alpha$.
4. $\tilde{A} = \tilde{B}$ equivalent to $A_\alpha = B_\alpha$.
5. $A_\alpha \cap A_\beta = A_\beta$ and $A_\alpha \cup A_\beta = A_\alpha$, if $\alpha \leq \beta$.

It is noticeable that all α-level sets corresponding to any fuzzy set form a family of nested crisp sets, as visually depicted in Fig.(1).

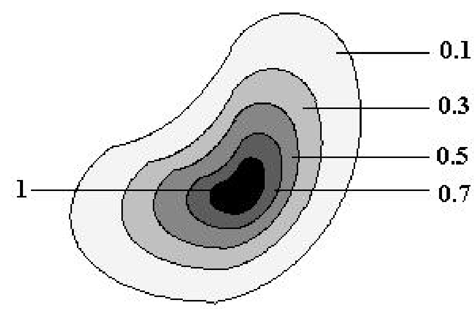

Fig.(1) : Nested α-level sets.

Sometimes, we have two different fuzzy sets with the same family of level sets, [Fadhel, 1998].

4-Fuzzy Metric Spaces

Fuzzy metric spaces have so many difficulties in their definition and construction. In fuzzy metric spaces, since such distance is defined between sets rather than elements, which require some modifications. But still there is some studies concerning this subject given by Ereeg M. A. in 1979. Since it seems to be very difficult to give a precise definition for the distance that may be used in constructing the metric spaces. Further studies of fuzzy metric spaces are given by Fadhal S.F in 1998 [Fadhel, 1998], and Marry G. in 2004 [Mary, 2004].

The next definition is introduced for Fuzzy distance function between fuzzy sets.

Definition (3):

Let (X, d) be an ordinary metric space and let $\tilde{A}, \tilde{B}$ be two fuzzy subset of X, let A_α, B_α be the α-level subset of $\tilde{A}, \tilde{B}$, respectively, for all $\alpha \in (0, 1]$, then the distance between $\tilde{A}$ and $\tilde{B}$ is defined by:

$$D(\tilde{A}, \tilde{B}) = \sup_{\alpha \in (0,1]} \text{Max}\{d(A_\alpha, B_\alpha), d(B_\alpha, A_\alpha)\}$$

Where $d(A_\alpha, B_\alpha)$ and $d(B_\alpha, A_\alpha)$ are the distance between A_α and B_α, and B_α and A_α respectively, which may not be equal.

Theorem (1):

Suppose I^X, where I = [0, 1] is the set of all fuzzy subsets of the universal set X, then (I^X, D) is a fuzzy semi metric space.

Proof:

D is a function from $I^X \times I^X$ onto R^+, then we must prove that D satisfies the condition of the metric space.

(1) Since $d(A_\alpha, B_\alpha) > 0$ and $d(B_\alpha, A_\alpha) > 0$, then it is clear that $D(\tilde{A}, \tilde{B}) > 0$. Also, since $d(A_\alpha, B_\alpha) = 0$ may implies that $A_\alpha \neq B_\alpha$, hence if $A_\alpha = B_\alpha$, $\forall \alpha$ Then $D(\tilde{A}, \tilde{B}) = 0$ but $D(\tilde{A}, \tilde{B}) = 0$ may implies that $\tilde{A} \neq \tilde{B}$.

(2)
$$D(\tilde{A}, \tilde{B}) = \sup_{\alpha} \mathrm{Max}\{d(A_\alpha, B_\alpha), d(B_\alpha, A_\alpha)\}$$
$$= \sup_{\alpha} \mathrm{Max}\{d(B_\alpha, A_\alpha), d(A_\alpha, B_\alpha)\}$$
$$= D(\tilde{B}, \tilde{A})$$

Hence:
$$D(\tilde{A}, \tilde{B}) = D(\tilde{B}, \tilde{A})$$

(3) From the properties of the distance between ordinary sets and the properties of the Housdorff distance. Then we may have:
$$D(\tilde{A}, \tilde{B}) \leq D(\tilde{A}, \tilde{C}) + D(\tilde{C}, \tilde{B})$$
where $\tilde{A}, \tilde{B}, \tilde{C}$ are fuzzy subsets of X.

Hence (I^X, D) is a fuzzy semi metric space.

The following is an illustrative example:

Example (1):

Let $\tilde{A}$ and $\tilde{B}$ be two Fuzzy subsets of X = R, such that:
$$\tilde{A} = \{(1,0.1), (-1,0.2), (2,0.3), (-2,0.4), (0,0.5)\}$$
$$\tilde{B} = \{(3,0.1), (-3,0.2), (2,0.3), (-4,0.4), (0,0.5)\}$$
and to find the distance between $\tilde{A}$ and $\tilde{B}$.

Then, the 0.1, 0.2, 0.3, 0.4, and 0.5 level subsets of $\tilde{A}$ and $\tilde{B}$ are:
$$A_{0.1} = \{1, -1, 2, -2, 0\}$$
$$A_{0.2} = \{-1, 2, -2, 0\}$$
$$A_{0.3} = \{2, -2, 0\}$$
$$A_{0.4} = \{-2, 0\}$$
$$A_{0.5} = \{0\}$$
and
$$B_{0.1} = \{3, -3, 4, -4, 0\}$$
$$B_{0.2} = \{-3, 4, -4, 0\}$$
$$B_{0.3} = \{4, -4, 0\}$$
$$B_{0.4} = \{-4, 0\}$$
$$B_{0.5} = \{0\}$$
Therefore, the distance between $A_{0.1}$ and $B_{0.1}$ is:
$$d(A_{0.1}, B_{0.1}) = \sup_{x \in A_{0.1}} \inf_{y \in B_{0.1}} d(x, y)$$
$$= \mathrm{Sup}\,\inf\{\{1, -1, 2, -2, 0\}, \{3, -3, 4, -4, 0\}\}$$
$$= \mathrm{Sup}\{1, 1, 1, 1, 0\} = 1$$
$$d(B_{0.1}, A_{0.1}) = \sup_{y \in B_{0.1}} \inf_{x \in A_{0.1}} d(y, x)$$
$$= \mathrm{Sup}\,\inf\{\{3, -3, 4, -4, 0\}, \{1, -1, 2, -2, 0\}\}$$
$$= \mathrm{Sup}\{1, 1, 2, 2, 0\} = 2$$
Then
$$\mathrm{Max}\{d(A_{0.1}, B_{0.1}), d(B_{0.1}, A_{0.1})\} = \mathrm{Max}\{1, 2\}$$
$$= 2$$
Similarly:
$$d(A_{0.2}, B_{0.2}) = \sup_{x \in A_{0.2}} \mathrm{Inf}_{y \in B_{0.2}} d(x, y)$$
$$= \mathrm{Sup}\{1, 0, 1, 0\} = 1$$
$$d(B_{0.2}, A_{0.2}) = \sup_{y \in B_{0.2}} \mathrm{Inf}_{x \in A_{0.2}} d(y, x)$$
$$= \mathrm{Sup}\{1, 2, 2, 0\} = 2$$
Hence:
$$\max\{d(A_{0.2}, B_{0.2}), d(B_{0.2}, A_{0.2})\} = 2$$
Also:
$$d(A_{0.3}, B_{0.3}) = \sup_{x \in A_{0.3}} \inf_{y \in B_{0.3}} d(x, y)$$
$$= \sup\{2, 2, 0\} = 2$$
$$d(B_{0.3}, A_{0.3}) = \sup_{x \in A_{0.3}} \inf_{y \in B_{0.3}} d(y, x)$$
$$= \sup\{2, 2, 0\} = 2$$
Therefore:
$$\max\{d(A_{0.3}, B_{0.3}), d(B_{0.3}, A_{0.3})\} = 2$$

Similarly:

$$\max\left\{d\left(A_{0.4},B_{0.4}\right),\ d\left(B_{0.4},A_{0.4}\right)\right\}=2$$

and

$$\max\left\{d\left(A_{0.5},B_{0.5}\right),d\left(B_{0.5},A_{0.5}\right)\right\}=0$$

Hence the distance between the two fuzzy sets $\tilde{A}$ and $\tilde{B}$, is given by:

$$D\left(\tilde{A},\tilde{B}\right)=\sup\left\{2,2,2,2,0\right\}=2$$

5-References

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الخلاصة

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