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ON MARCINKIEWICZ-ZYGMUND INEQUALITIES IN $L_{P,\mu}$-SPACES, $0 < p \leq 1$

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Abstract

We found the relationships between Marcinkiewicz-inequalities and linear trigonometric operators in the quasi-normed spaces $L_{p,\mu}$, $0 < p \leq 1$.

Introduction

Here, we assert the Marcinkiewz-Zygmund inequalities which are given by the following theorem. [1]. If $n \geq 1$ is an integer, $1 < p < \infty$, and S is a trigonometric polynomial of order at most n, then:

$$\int_{-\pi}^{\pi} |S(t)|^p \, dt \leq \frac{c_1}{2n+1} \sum_{j=1}^{2n+1} \left| S\left(\frac{2\pi j}{2n+1}\right) \right|^p \leq c_2 \int_{-\pi}^{\pi} |S(t)|^p \, dt \quad \text{.........(1)}$$

Where c_1 and c_2 are positive constants depending only on p, but not on S or n. These inequalities have an important application which deduce the mean boundedness of the trigonometric interpolation polynomial that of the n-th partial sum operator of the trigonometric.

These inequalities have been studied by many authors in connection with orthogonal polynomials both in the case of Lagrange interpolation and Jacobi weights on [–1, 1] ([2], [3]).

Many authors studied the convergence of the linear operators in different spaces like ([4], [5]), [6], [7].

In this paper, we shall investigate the relation ship between these operators and inequalities similar to (1) but in the quasi-norm $L_{p,\mu}$-space, $0 < p \leq 1$.

Definitions and Notations

Let f be a bounded function defined on $[-\pi,\pi]$, and μ be a positive measure on a measure space Σ. For a μ-measurable function f, we write:

$$\|f\|_{p,\mu} = \left(\int |f(x)|^p \, d\mu(x) \right)^{1/p}, \; 0 < p < 1 \text{.. (2)}$$

The space $L_{p,\mu}$ consists of all functions f for which $\|f\|_{p,\mu} < \infty$.

Mhaskar H. N. and Prcstin J. proved the following theorem [8].

Theorem

For a trigonometric polynomial S on $[-\pi, \pi]$,
$1 < p < \infty$, we get:

$$\|S\|_{p,\mu} \leq c_p \|s\|_{p,\upsilon} \leq c_2 \|S\|_{p,\mu} \quad \text{.............(3)}$$

μ is the Lebesgue measure and υ is a discrete measure.

Now, let us consider two σ-finite measures μ and υ on a measure space Σ and corresponding operators (when defined) ($f \in L_{p,\upsilon}$, $0 < p \leq 1$).

$$T(\tau, f, x) = \int f(t) k(x,t) \, d\tau(t), \; \tau = \mu, \upsilon,$$

where k is a symmetric function. We assume that k is essentially bounded as well as integrable with respect to all the product measures $\mu\times\mu$, $v\times v$ and $\mu\times v$. We also consider two weight functions w and W which are both bounded measurable and positive almost everywhere with respect to both μ and υ.

For $0 < p \leq 1$, by condition (4(p)) we mean that:

$$\|w(wW)^{-1/p} T(\upsilon, f)\|_{p,\mu} \leq c \|W(wW)^{-1/p} f\|_{p,\upsilon} \quad \text{............ (4(p))}$$

By (5(p)) we also mean that:

$$\|w(wW)^{-1/p} T(\mu, f)\|_{p,\upsilon} \leq c \|W(wW)^{-1/p} f\|_{p,\mu} \quad \text{.................(5p))}$$

where we note that the constants in both inequalities depend on μ and not on υ.

We also denote by (6(p)):

$$\| w(wW)^{-1/p} T(\mu, f) \|_{p,\mu} \le c \| W(wW)^{-1/p} f \|_{p,\mu} \qquad ....(6(p))$$

Main Results

Now, we prove the following theorem which shows a close connection between these conditions in the quasi normed space $L_{p,\mu}$, $0 < p \le 1$.

Theorem (1):

Let μ and υ be σ-finite measures on a measure space Σ, $0 < p \le q = p' \le 1$ and p' be such that $p'=q=1-p$. Further suppose that w and W^{-1} are measurable bounded weight functions both in $L_{p,\mu}$ and $L_{p,\upsilon}$.

(a) Let f, g : $\Sigma \longrightarrow R$, and f(x)g(x)k(x, t) be integrable with respect to the product measures $\mu(t)\times\upsilon(x)$. Then the following reciprocity law holds:

$$\int fT(\mu; g)\, d\upsilon = \int T(\upsilon; f)\, g\, d\mu$$

(b) The conditions (4(p)) and (5(q)) (0<p<q<=1)are equivalent. The conditions (6(p)) and (6(q)) are equivalent.

Proof:

By Fubini's theorem and the fact that k(x, t) = k(t, x), we have:

$$\int f(x) T(\mu; g, x)\, d\upsilon(x)$$
$$= \int f(x) \int g(t) k(x,t)\, d\mu(t) d\upsilon(x)$$
$$= \int g(t) \int f(x) k(x,t)\, d\upsilon(x) d\mu(t)$$
$$= \int g(t) T(\upsilon; f, t)\, d\mu(t)$$

This proves part (a)

To prove (b), we first prove that (5(q)) implies (4(p)).

Let g : $\Sigma \longrightarrow R$, be an arbitrary μ-measurable, simple function such that $\|g\|_{p',\mu} \le 1$. Then:

$$\left\| w(wW)^{-\frac{1}{p}} \right\|_{1,\mu}$$
$$= \left[\int \left(w(wW)^{-\frac{1}{p}} \right)^p w(wW)^{1-p} \right]^{\frac{1}{p}}$$
$$\left[\int \left(w(wW)^{-\frac{1}{p}} \right)^p w(wW)^{1-p} \right]^{1-\frac{1}{p}}$$
$$= c_p \left\| W^{-\frac{1}{p}} \right\|_{p,\mu} \left\| W^{-\frac{1}{p'}} \right\|_{p',\mu}$$
$$= c_p \|w\|_{1,\mu}^{\frac{1}{p'}} \|W^{-1}\|_{1,\mu}^{\frac{1}{p}}$$

Which shows that $w(wW)^{-\frac{1}{p}} \in L_{p,\mu}$.

Since g is μ-essentially bounded, the function $T\left(\mu; w(Ww)^{-\frac{1}{p}} g\right)$ is defined and the conditions of part (a) are also satisfied. Then using the reciprocity law and the fact that:

$$W^{-1}(wW)^{\frac{1}{p}} \le w(wW)^{-\frac{1}{p'}},\ O(p'),$$
$$w(wW)^{-\frac{1}{p}} \le W^{-1}(wW)^{\frac{1}{p'}}$$

and as in the lines above, we obtain:

$$\left| \int w(wW)^{-\frac{1}{p}} T(\upsilon; f) g\, d\mu \right|$$
$$= \left| \int fT(\mu; w(wW)^{-\frac{1}{p}}) g\, d\upsilon \right|$$

$$\leq \left\| W(wW)^{-\frac{1}{p}} f \right\|_{p,\mu}$$

$$\left\| w(wW)^{-\frac{1}{p'}} T(\mu; w(wW)^{-\frac{1}{p}})g \right\|_{p',\upsilon}$$

$$\leq c \left\| W(wW)^{-\frac{1}{p}} f \right\|_{p,\upsilon}$$

$$\left\| W(wW)^{-\frac{1}{p'}} w(wW)^{-\frac{1}{p}} g \right\|_{p',\mu}$$

$$\leq c \left\| W(wW)^{-\frac{1}{p}} f \right\|_{p,\upsilon}$$

Since:

$$\left\| w(wW)^{-\frac{1}{p}} T(\upsilon; f) \right\|_{p,\mu}$$

$$\leq \sup \left| \int w(wW)^{-\frac{1}{p}} T(\upsilon; f) g \, d\mu \right|$$

Where the supremum is taken over all μ-measurable simple function g with $\|g\|_{p',\mu} \leq 1$, we have proved (4(p)) similarly the condition (4(p)) implies (5(q)).

Finally, we observe that when $\mu = \upsilon$, the conditions (4(p)), (6(p)) and (5(p)) are the same. This completes the proof of part (b). <

In order to rate theorem (1) with Marcinkiewiz-Zygmund type (M-Z) inequalities, we restrict ourselves to the case w = W. We consider an increasing sequence of sets $\{\pi_k\}$,
$\pi_k \subseteq \pi_{k+1}$, k = 0, 1, …, which may be thought of as subsets of $L_{p,\mu}$ and $L_{p,\upsilon}$. And consider a sequence of symmetric kernel function k_k and operators T_k defined by (if possible):

$$T_k(\tau; f, x) = \int f(t) k_k(x,t) \, d\tau(t);$$

$$\tau = \mu, \ \upsilon, k = 1, 2, \ldots$$

Assume that there exist integers a ≥ 1 and b such that $T_k(\tau; f) \in \pi_{ak+b}$, $\tau = \mu, \upsilon$; k = 1, 2,

Assume that the property (7n) is such that:

$$T_k(\tau, p) = p, \ p \in \pi_k, \ \tau = \mu, \upsilon, \ k = 1, 2, \ldots, n \qquad \ldots\ldots (7n)$$

where υ will be chosen depending on n, so that, the condition (7n) will be satisfied.

The conditions (4n(p)), (5n(p)) and (6n(p)) denote the fact that each of the operators T_k, $1 \leq k \leq n$ satisfies the condition (4(p)) (respectively (5(p)), (6(p)).

Clearly, the condition (5n(p)) (with w = W) and (7n) imply the simpiler M-Z inequality:

$$\left\| w^{\frac{p-2}{p}} p \right\|_{p,\upsilon} \leq c \left\| w^{\frac{p-2}{p}} p \right\|_{p,\mu} \quad , p \in \pi_n \ldots\ldots (8)$$

The condition will be referred as:

$$\left\| w^{\frac{p-2}{p}} p \right\|_{p,\mu} \leq c_1 \left\| w^{\frac{p-2}{p}} p \right\|_{p,\upsilon}$$

$$\leq c_2 \left\| w^{\frac{p-2}{p}} p \right\|_{p,\mu} \quad , p \in \pi_n \qquad \ldots\ldots (9)$$

The inequality (8) will be called $SMZ_n(p)$ and (9) will be called $MZ_n(p)$ or full M-Z inequality.

Theorem (2):

Let μ, υ be as in theorem (1), n ≥ 1 be an integer, w = W and w , w^{-1} be in both $L_{p,\mu}$ and $L_{p,\upsilon}$, then if $0 < p \leq 1$ and $SM_{an+b}(q)$ holds. The condition (6n(p)) implies (4n(p)).

Proof:

Let $1 \leq k \leq n$ be an integer and (6n(p)) hold.
By theorem (1), we get (6n(q)). Now, let $w_{p'} = w^{\frac{p'-2}{p'}}$. Since $T_k(\mu; f) \in \pi_{an+b}$, for all f for which it is defined, the condition $SM_{an+b}(q)$ and (6n(q)) together imply condition $SM_{an+b}(q)$ and (6n(q)) together imply:

$$\left\| w_{p'} T_k(\mu; f) \right\|_{p',\upsilon} \le c \left\| w_{p'} T_k(\mu; f) \right\|_{p',\mu} \le c \left\| w_{p'} f \right\|_{p',\mu}$$

Which is the condition (5n(q)) which implies (4n(p)).

Conclusion

We are using two σ-finite measures μ and υ on a measure space ∑ and define a new trigonometric operators

$$T(\tau, f, x) = \int f(t) K(x,t) d\tau(t), \tau = \mu, \upsilon.$$

and K is a symetric function then we investigate the relation ship between these operators and the inequalities similar to (1) in the quasi-norm

$L_{p,\mu}$ - space, o < p £ 1,

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الخلاصة

في هذا البحث وجدنا العلاقة بين متراجحات مارسينز - زيكموند والمؤثرات المثلثية الخطية في الفضاء الشبه قياسي $L_{p,\mu}$، $0 < p \le 1$.