THE COLLOCATION METHOD FOR SOLVING NONHOMOGENEOUS FUZZY BOUNDARY VALUE PROBLEMS

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Abstract

In this paper, the collocation method is considered to solve the nonhomogeneous fuzzy boundary value problems, in which the fuzziness appeared together in the boundary conditions and in the nonhomogeneous term of the differential equation. The method of solution depends on transforming the fuzzy problem to equivalent crisp problems using the concept of α-level sets.

Keywords: Fuzzy sets, fuzzy boundary value problems, α-level sets, the collocation method.

1-Introduction

Fuzzy set had been introduced by Zadeh in 1965, in which, Zadeh's original definition of fuzzy set> is as follows "a fuzzy set (denoted by $\tilde{A}$) is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$, where X is the universal set, which assigns to each object a grade of membership ranging between zero and one", i.e., the fuzzy set may be given by, [5], [3]:

$$\tilde{A} = \left\{(x, \mu_{\tilde{A}}(x)) : x \in X,\ 0 \le \mu_{\tilde{A}}(x) \le 1\right\}$$

Convex fuzzy sets are of great importance in defining fuzzy numbers. This property is viewed as a generalization of the classical concept of convexity of nonfuzzy sets. The definition of convexity for fuzzy set does not necessarily mean that the membership function of a convex fuzzy set is also convex function, where a fuzzy set $\tilde{A}$ on is convex if, [6]:

$$\mu_{\tilde{A}}(\lambda x_1 + (1-\lambda)x_2) \ge \text{Min}\left\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\right\}$$

for all $x_1, x_2 \in$, and all $\lambda \in [0, 1]$. Also, a non-empty fuzzy set $\tilde{A}$ can always be normalized (i.e., the greatest membership value) by dividing $\mu_{\tilde{A}}(x)$, $\forall\ x \in X$ by $\sup_{x \in X} \mu_{\tilde{A}}(x)$ and as a matter of convenience, we will generally assume that fuzzy sets are normalized, [4].

Among the basic concepts in fuzzy set theory which will play a central role in solving fuzzy differential equations is the concept of an α-level, where if we are given a fuzzy set $\tilde{A}$ defined on the universal set X and any number $\alpha \in [0, 1]$ the α-level, A_α is the crisp set (non fuzzy set) that contains all elements of X whose membership grades in $\tilde{A}$ are greater than or equal to a pre specified value of α, i.e.

$A_\alpha = \{x : \mu_{\tilde{A}}(x) \ge \alpha, \forall\ x \in X\}$

Kandel [4] applied the concept of fuzzy differential equations to the analysis of fuzzy dynamical problems, but the boundary value problems was treated, rigorously by Lakshmikantham, V., Murty, K. N. and Turner, J. in 2001, [1]. Henderon, J. and Peterson, A. in 2004 obtained a theorem of the existence and uniqueness of solutions for the boundary value problems of fuzzy differential equations.

Pearson in 1997, introduced the analytical method for solving linear system of fuzzy differential equations with the cooperation of complex numbers while there is no such study for evaluating the analytical solution of fuzzy boundary value problems explicitly, except of the work of Al-Saedy A. J. in 2006 [2] and Al-Adhami R. H. in 2007 [1].

In this paper a modified approximate method is presented for solving fuzzy boundary value problems, using the collocation method. This approximate method is given with illustrative example.

2-Fuzzy Number

In this section, we shall give some basic concepts for fuzzy numbers and fuzzy functions before we present the modified approach for solving fuzzy boundary value problems using the collocation method in order to make our paper of self contents.

First, we will give the definition of a fuzzy number and its representation using two approaches as a α-level sets (which will be in this case as a closed subsets of the real line).

Definition (1), [7]:

A fuzzy number $\tilde{M}$ is a convex normalized fuzzy set $\tilde{M}$ of the real line R, such that:

1. There exists exactly one $x_0 \in R$, with $\mu_{\tilde{M}}(x_0) = 1$ (x_0 is called the mean value of $\tilde{M}$).
2. $\mu_{\tilde{M}}(x)$ is piecewise continuous.

Definition (2), [3]:

A fuzzy number $\tilde{M}$ is of LR-type if there exists functions, L (called the left function), R (called the right function) such that $L(x) \le \mu_{\tilde{M}}(x) \le R(x)$, $\forall x \in X$ (Universal set) and scalars $a > 0$, $b > 0$, with:

$$\mu_{\tilde{M}}(x) = \begin{cases} L\left(\dfrac{m-x}{a}\right), & \text{for } x \le m \\ R\left(\dfrac{x-m}{b}\right), & \text{for } x \ge m \end{cases}$$

m is a real number called the mean value of $\tilde{M}$, a and b are called the left and right spreads of m, respectively. Symbolically $\tilde{M}$ is denoted by $(m, a, b)_{LR}$.

Now, in applications, the representation of a fuzzy number in terms of its membership function is so difficult to use, therefore two approaches are given for representing the fuzzy number in terms of its α-level sets, as in the following remark:

Remark (1):

A fuzzy number $\tilde{M}$ may be uniquely represented in terms of its α-level sets, as the following closed intervals of the real line:

$M_\alpha = [m - \sqrt{1-\alpha}, m + \sqrt{1-\alpha}]$

or

$M_\alpha = [\alpha m, \dfrac{1}{\alpha} m]$

Where m is the mean value of $\tilde{M}$ and $\alpha \in [0, 1]$. This fuzzy number may be written as $M_\alpha = [\underline{\tilde{M}}, \overline{\tilde{M}}]$, where $\underline{\tilde{M}}$ refers to the greatest lower bound of M_α and $\overline{\tilde{M}}$ to the least upper bound of M_α.

Remark (2):

Similar to the second approach given in remark (1), one can fuzzyfy any crisp or nonfuzzy function f, by letting:

$\underline{f}(x) = \alpha f(x)$, $\overline{f}(x) = \dfrac{1}{\alpha} f(x)$, $x \in X$, $\alpha \in (0, 1]$

and hence the fuzzy function $\tilde{f}$ in terms of its α-levels is given by $f_\alpha = [\underline{f}, \overline{f}]$.

3-The Collocation Method for Solving Fuzzy Boundary Value Problems

Consider the n-th order linear ordinary differential equation with non-constant coefficients:

$$c_n(x)y^{(n)}(x) + c_{n-1}(x)y^{(n-1)}(x) + \ldots + c_1(x)y'(x) + c_0(x)y(x) = \tilde{f}(x),\ x \in [a, b] \quad \ldots\ldots(1)$$

where $\tilde{f}$ is the fuzzyfying function of the crisp function f which may be written in terms of its α-levels as $f_\alpha = [\underline{f}, \overline{f}]$, $\underline{f}(x) = \alpha f(x)$, $\overline{f}(x) = \dfrac{1}{\alpha} f(x)$, $x \in X$, $\alpha \in (0, 1]$, $c_n(x) \neq 0$, $\forall x \in [a, b]$; with certain fuzzy boundary conditions.

Let $\tilde{\phi}(x)$ be the approximate solution of eq.(1), defined by:

$$\tilde{\phi}(x) = \psi(x) + \sum_{i=1}^{N} \tilde{a}_i B_i(x),\ N \in \quad \ldots\ldots(2)$$

where $\psi(x)$ is a function which satisfies nonhomogeneous boundary conditions, B_i, $\forall i = 1, 2, \ldots, N$; is sequence of functions which satisfies the homogeneous conditions and $\tilde{a}_i$, $\forall i = 1, 2, \ldots, N$; are fuzzy numbers to be determined.

To find the approximate solution $\tilde{\phi}$, substitute $\tilde{\phi}$ in the differential equation (1) and hence the problem is reduced to the problem of evaluating of the constants $\tilde{a}_i$'s, for all i = 1, 2, …, N; which gives residue function:

$$R(\tilde{\phi}, x) = c_n\left\{\psi(x) + \sum_{i=1}^{N} \tilde{a}_i B_i(x)\right\}^{(n)} + c_{n-1}\left\{\psi(x) + \sum_{i=1}^{N} \tilde{a}_i B_i(x)\right\}^{(n-1)} + \ldots +$$

$$c_1\left\{\psi(x)+\sum_{i=1}^{N}\tilde{a}_i B_i(x)\right\}' +$$
$$c_0\left\{\psi(x)+\sum_{i=1}^{N}\tilde{a}_i B_i(x)\right\} - \tilde{f}(x)$$
$$= c_n\left\{\psi^{(n)}(x)+\sum_{i=1}^{N}\tilde{a}_i B_i^{(n)}(x)\right\} +$$
$$c_{n-1}\left\{\psi^{(n-1)}(x)+\sum_{i=1}^{N}\tilde{a}_i B_i^{(n-1)}(x)\right\} +\ldots+$$
$$c_1\left\{\psi'(x)+\sum_{i=1}^{N}\tilde{a}_i B_i'(x)\right\} +$$
$$c_0\left\{\psi(x)+\sum_{i=1}^{N}\tilde{a}_i B_i(x)\right\} - \tilde{f}(x) \quad \text{..........} (3)$$

Therefore, $R(\tilde{\phi}, x)$ is now a function of the unknowns $\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N$ which may be rewritten as $R(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N; x)$ and therefore:
$R(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N; x) \cong 0$, for all $x \in [a, b]$
Hence eq.(3) may be rewritten for the approximate solution as:

$$R(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N; x) = \sum_{i=1}^{N}\Big\{c_n \tilde{a}_i B_i^{(n)}(x) +$$
$$c_{n-1}\tilde{a}_i B_i^{(n-1)}(x) + ... +$$
$$c_1\tilde{a}_i B_i'(x) + c_0 \tilde{a}_i B_i(x)\Big\} + c_n\psi^{(n)}(x) +$$
$$c_{n-1}\psi^{(n-1)}(x) + \ldots + c_1\psi'(x) +$$
$$c_0\psi(x) - \tilde{f}(x) \quad \text{....................................} (4)$$

To evaluate the coefficients $\tilde{a}_i$'s, $i = 1, 2, \ldots, N$; we evaluate eq.(4) at n-distinct points $x_1, x_2, \ldots, x_N \in [a, b]$, which will produce the following linear system:

$$R(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N; x_1) = 0$$
$$R(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N; x_2) = 0$$
$$\vdots$$
$$R(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_N; x_n) = 0$$

Therefore, we have:

$$\sum_{i=1}^{N}\Big\{c_n \tilde{a}_i B_i^{(n)}(x_i) + c_{n-1}\tilde{a}_i B_i^{(n-1)}(x_i) + ... +$$
$$c_1\tilde{a}_i B_i'(x_i) + c_0\tilde{a}_i B_i(x_i)\Big\} = \tilde{f}(x_i) - c_n\psi^{(n)}(x_i) -$$
$$c_{n-1}\psi^{(n-1)}(x_i) - \ldots - c_1\psi'(x_i) - c_0\psi(x_i),\ i = 1, 2, \ldots, N$$

or in matrix form:

$$A\tilde{a} = \tilde{D} \quad \text{..} (5)$$

where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix}$$

where:

$a_{11} = c_n B_1^{(n)}(x_1) + c_{n-1} B_1^{(n-1)}(x_1) + \ldots + c_1 B_1'(x_1) + c_0 B_1(x_1)$

$a_{12} = c_n B_2^{(n)}(x_1) + c_{n-1} B_2^{(n-1)}(x_1) + \ldots + c_1 B_2'(x_1) + c_0 B_2(x_1)$

$\vdots$

$a_{1N} = c_n B_N^{(n)}(x_1) + c_{n-1} B_N^{(n-1)}(x_1) + \ldots + c_1 B_N'(x_1) + c_0 B_N(x_1)$

$a_{21} = c_n B_1^{(n)}(x_2) + c_{n-1} B_1^{(n-1)}(x_2) + \ldots + c_1 B_1'(x_2) + c_0 B_1(x_2)$

$a_{22} = c_n B_2^{(n)}(x_2) + c_{n-1} B_2^{(n-1)}(x_2) + \ldots + c_1 B_2'(x_2) + c_0 B_2(x_2)$

$\vdots$

$a_{2n} = c_n B_N^{(n)}(x_2) + c_{n-1} B_N^{(n-1)}(x_2) + \ldots + c_1 B_N'(x_2) + c_0 B_N(x_2)$

$\vdots$

$a_{N1} = c_n B_1^{(n)}(x_N) + c_{n-1} B_1^{(n-1)}(x_N) + \ldots + c_1 B_1'(x_N) + c_0 B_1(x_N)$

$a_{N2} = c_n B_2^{(n)}(x_N) + c_{n-1} B_2^{(n-1)}(x_N) + \ldots + c_1 B_2'(x_N) + c_0 B_2(x_N)$

$\vdots$

$a_{NN} = c_n B_N^{(n)}(x_N) + c_{n-1} B_N^{(n-1)}(x_N) + \ldots + c_1 B_N'(x_N) + c_0 B_N(x_N)$

and:

$$D = \begin{bmatrix} \tilde{f}(x_1) - c_n\psi^{(n)}(x_1) - c_{n-1}\psi^{(n-1)}(x_1) - \ldots - c_1\psi'(x_1) - c_0\psi(x_1) \\ \tilde{f}(x_2) - c_n\psi^{(n)}(x_2) - c_{n-1}\psi^{(n-1)}(x_2) - \ldots - c_1\psi'(x_1) - c_0\psi(x_2) \\ \vdots \\ \tilde{f}(x_N) - c_n\psi^{(n)}(x_N) - c_{n-1}\psi^{(n-1)}(x_N) - \ldots - c_1\psi'(x_N) - c_0\psi(x_N) \end{bmatrix},$$

$$\tilde{a} = \begin{bmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{a}_N \end{bmatrix}$$

In order to solve the resulting system (5), one must first use remark (1) to rewrite the fuzzy numbers $\tilde{a}_i$, $\forall$ i = 1, 2, …, N; in terms of its α-level sets as $a_{i_\alpha} = [\underline{a_i}, \overline{a_i}]$, $f_\alpha = [\underline{f}, \overline{f}]$, $\forall$ α ∈ (0,1]; and similarly for the fuzzy boundary conditions. Then solving the related nonfuzzy linear systems for the lower and upper values of the α-level sets $\underline{a_i}$ and $\overline{a_i}$, $\forall$ i = 1, 2, …, N, respectively.

4-Illustrative Example

Consider the second order fuzzy boundary value problem:

$$y''(x) + 2y'(x) + y(x) = \tilde{f}(x),\ x \in [0, 1] \quad \text{.............................. (6a)}$$

with fuzzy boundary conditions:

$$y(0) = \tilde{1},\ y(1) = \tilde{2} \quad \text{............................... (6b)}$$

where $\tilde{f}$ is a fuzzyfying function of the crisp function f(x) = 2x.

In order to solve the fuzzy boundary value problem (6a) and (6b), use remark (1) to rewrite first the fuzzy function $\tilde{f}$ in terms of its α-levels as $f_\alpha = [\underline{f}, \overline{f}]$, where and $\underline{f}(x) = 2\alpha x$, $\overline{f}(x) = \dfrac{2x}{\alpha}$, α ∈ (0, 1] and the fuzzy boundary conditions in terms of its α-levels, as:

$y_\alpha(0) = [1-\sqrt{1-\alpha},\ 1+\sqrt{1-\alpha}\,]$

and

$y_\alpha(1) = [2-\sqrt{1-\alpha},\ 2+\sqrt{1-\alpha}\,]$, α ∈ (0, 1]

Therefore, to solve this problem using the collocation method, consider the fuzzy approximate solution $\tilde{\varphi}$ with α-levels:

$\varphi_\alpha(x) = [\underline{\varphi}(x),\ \overline{\varphi}(x)]$, α ∈ (0, 1]

Hence, to find the solution in the lower case of solution $\underline{y}$, consider the problem:

$$\underline{y}'' + 2\underline{y}' + \underline{y} = 2\alpha x \quad \text{............................ (7)}$$

with lower bound of boundary conditions:

$\underline{y}(0) = 1-\sqrt{1-\alpha}$, $\underline{y}(1) = 2-\sqrt{1-\alpha}$, α∈(0, 1]

Now, let:

$$\underline{\varphi}(x) = \psi(x) + \sum_{i=1}^{3} \tilde{a}_i B_i(x)$$

where:

$\psi(x) = x + 1 - \sqrt{1-\alpha}$

which satisfies $\psi(0) = 1 - \sqrt{1-\alpha}$ and $\psi(1) = 2 - \sqrt{1-\alpha}$, i.e., satisfies the non-homogeneous boundary condition. The functions B_i, i = 1, 2, 3; which satisfy the homogeneous boundary conditions $\underline{y}(0) = 0$ and $\underline{y}(1) = 0$ may be chosen as:

$B_1(x) = x(x-1)$

$B_2(x) = x^2(x-1)$

$B_3(x) = x^3(x-1)$

and $\underline{\varphi}(\underline{a_1}, \underline{a_2}, \underline{a_3}; x)$ will takes the form:

$$\begin{aligned}\underline{\varphi}(\underline{a_1}, \underline{a_2}, \underline{a_3}; x) &= x + 1 - \sqrt{1-\alpha} + x(x-1)(\underline{a_1} + \underline{a_2}x + \underline{a_3}x^2)\\ &= x + 1 - \sqrt{1-\alpha} + \underline{a_1}(x^2 - x) + \underline{a_2}(x^3 - x^2) + \underline{a_3}(x^4 - x^3)\end{aligned}$$

and upon substituting in eq.(7), yields:

$$2\underline{a_1} + \underline{a_2}(6x-2) + \underline{a_3}(12x^2-6x) + 2\{1 + \underline{a_1}(2x-1) + \underline{a_2}(3x^2-2x) + \underline{a_3}(4x^3-3x^2)\} + x + 1 - \sqrt{1-\alpha} + \underline{a_1}(x^2-x) + \underline{a_2}(x^3-x^2) + \underline{a_3}(x^4-x^3) = 2\alpha x$$

or equivalently:

$$\underline{a_1}(x^2+3x) + \underline{a_2}(x^3+5x^2+2x-2) + \underline{a_3}(x^4+7x^3+6x^2-6x) = 2\alpha x - x - 3 + \sqrt{1-\alpha} \quad ..(8)$$

Now, evaluate eq.(8) at $x_1 = 0$, $x_2 = 1/2$, $x_3 = 1$; which will yield to the following linear system of algebraic equations:

$$\begin{bmatrix} 0 & -2 & 0 \\ 1.75 & 0.375 & -0.563 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} \underline{a_1} \\ \underline{a_2} \\ \underline{a_3} \end{bmatrix} = \begin{bmatrix} -3+\sqrt{1-\alpha} \\ \alpha-3.5+\sqrt{1-\alpha} \\ 2\alpha-4+\sqrt{1-\alpha} \end{bmatrix}$$

Solving this system, yields:

$\underline{a_1} = -2.45 + 0.723077\sqrt{1-\alpha} + 0.5615385\alpha$

$\underline{a_2} = 1.5 - 0.5\sqrt{1-\alpha}$

$\underline{a_3} = -0.4 + 0.138462\sqrt{1-\alpha} - 0.030769\alpha$

Therefore:

$$\underline{\varphi}(x) = x + 1-\sqrt{1-\alpha} + -2.45 + 0.723077\sqrt{1-\alpha} + 0.5615385\alpha)(x^2 - x) + (1.5 - 0.5\sqrt{1-\alpha})(x^3 - x^2) + (-0.4 + 0.138462\sqrt{1-\alpha} - 0.030769\alpha)(x^4 - x^3)$$

Similarly, for the upper solution $\overline{y}$, consider the problem:

$$\overline{y}'' + 2\overline{y}' + \overline{y} = \frac{2x}{\alpha} \quad \text{..............................} (9)$$

with upper bound of boundary conditions:

$\overline{y}(0) = 1 + \sqrt{1-\alpha}$, $\overline{y}(1) = 2 + \sqrt{1-\alpha}$, $\alpha \in (0, 1]$

We let:

$$\overline{\varphi}(x) = \psi(x) + \sum_{i=1}^{3} \overline{a}_i B_i(x)$$

where:

$\psi(x) = x + 1 + \sqrt{1-\alpha}$

which satisfies

$\psi(0) = 1 + \sqrt{1-\alpha}$ and $\psi(1) = 2 + \sqrt{1-\alpha}$

and letting also:

$B_1(x) = x(x-1)$, $B_2(x) = x^2(x-1)$, and $B_3(x) = x^3(x-1)$

Therefore, $\overline{\varphi}(x)$ will take the form:

$$\overline{\varphi}(\overline{a}_1, \overline{a}_2, \overline{a}_3; x) = x + 1 + \sqrt{1-\alpha} + \overline{a}_1 (x^2 - x) + \overline{a}_2 (x^3 - x^2) + \overline{a}_3 (x^4 - x^3)$$

and upon substituting in eq.(6), yields:

$$\overline{a}_1 (x^2+3x) + \overline{a}_2 (x^3+5x^2+2x-2) + \overline{a}_3 (x^4+7x^3+6x^2-6x) = \frac{2x}{\alpha} - 3 - \sqrt{1-\alpha} \quad \text{...............................} (10)$$

Hence, evaluating eq.(10) at $x_1 = 0$, $x_2 = 1/2$, $x_3 = 1$, which will yield to the following linear system of algebraic equations:

$$\begin{bmatrix} 0 & -2 & 0 \\ 1.75 & 0.375 & -0.563 \\ 4 & 6 & 8 \end{bmatrix} \begin{bmatrix} \overline{a}_1 \\ \overline{a}_2 \\ \overline{a}_3 \end{bmatrix} = \begin{bmatrix} -3-\sqrt{1-\alpha} \\ \frac{1}{\alpha} - 3.5 - \sqrt{1-\alpha} \\ \frac{2}{\alpha} - 4 - \sqrt{1-\alpha} \end{bmatrix}$$

and solving this system yields:

$$\overline{a}_1 = -2.45 - 0.723077\sqrt{1-\alpha} + \frac{0.561538}{\alpha}$$

$$\overline{a}_2 = 1.5 + 0.5\sqrt{1-\alpha}$$

$$\overline{a}_3 = -0.4 - 0.138462\sqrt{1-\alpha} - \frac{0.030769}{\alpha}$$

Therefore:

$$\overline{\varphi}(x) = x + 1 + \sqrt{1-\alpha} + \left(-2.45 - 0.723077\sqrt{1-\alpha} + \frac{0.561538}{\alpha}\right)(x^2 - x) + (1.5 + 0.5\sqrt{1-\alpha})(x^3 - x^2) + \left(-0.4 - 0.138462\sqrt{1-\alpha} - \frac{0.030769}{\alpha}\right)(x^4 - x^3)$$

Combining $\underline{\varphi}$ and $\overline{\varphi}$ yields the fuzzy solution of the fuzzy boundary value problem (6) as $\varphi_\alpha(x) = [\underline{\varphi}(x), \overline{\varphi}(x)]$, $\forall\ \alpha \in (0, 1]$, $x \in [0, 1]$. In addition, it is clear that for $\alpha = 1$, we get $\underline{\varphi}(x) = \overline{\varphi}(x)$, which is the same as the crisp solution of the related nonfuzzy boundary value problem. Also, the fuzzy solution $\tilde{\varphi}$ in terms of the lower bound of solution $\underline{\varphi}$ and upper bound of solution $\overline{\varphi}$ and for different α-levels (where $\alpha \in (0, 1]$) are presented in Fig.(1):

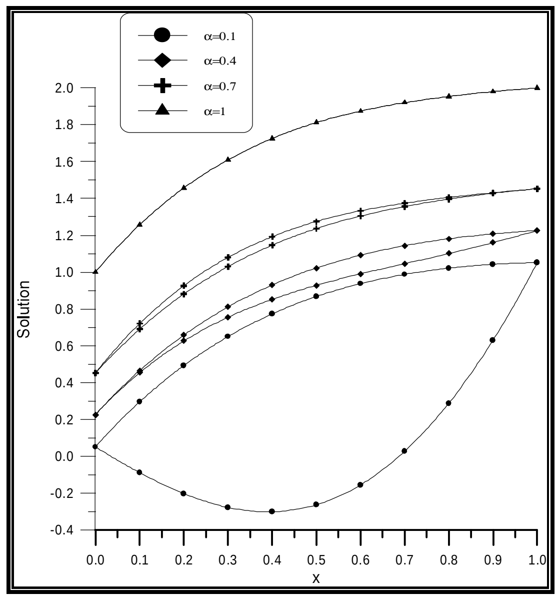

Fig.(1) : The upper and lower solutions for $\alpha = 0.1, 0.4, 0.7$ and 1 of eqs. (6a) and (6b).

5-References

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الخلاصة

في هــذا البحــث، تــم دراســة طريقــة الحــشد (the collocation method) لحــل معــادلات تفاضــلية حدودية ضبابية غيــر متجانــسة (nonhomogeneous fuzzy boundary value problems)، حيــث كانــت الضبابية في الشروط الحدودية والطرف الغيــر متجــانس المعادلة التفاضلية. أعتمدت طريقة الحــل علــى تحويــل المعادلة التفاضلية الحدودية الضبابية الى مسألة مكافئة غير ضبابية (crisp problem) باســتخدام مبــأ مجموعــات مستويات القطع (α-level sets).