

HOMOTOPY PERTURBATION METHOD FOR SOLVING SPECIAL TYPES OF NONLINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract

Nonlinear phenomena play crucial rule in applied mathematics. Explicit solutions to the nonlinear equations are of fundamental importance. Various methods for obtaining explicit solutions to nonlinear equations have been proposed.

In this work, the homotopy perturbation method is employed for solving the initial value problems of special types of nonlinear first order Fredholm integro-differential equations with some illustrative examples. The final results of these examples obtained by means of the homotopy perturbation method where compared with the results obtained from the exact solutions show that this method gave effective results.

Keywords: homotopy perturbation method, nonlinear Fredholm integro-differential equations.

1-Introduction

The homotopy perturbation method was proposed by Ji-Huan He in 1999. In this method, the solution is considered as a summation of an infinite series, which usually converges rapidly to the exact solution, [6]. Many researchers used this method say, [1] used this method to compute Laplace transform, [2] solved functional integral equations by using the homotopy perturbation method, [7] devoted the homotopy perturbation method for solving special types of linear partial differential equations with variable coefficients, [4] used this method to solve special types of nonlinear Fredholm integral equations, [3] gave the solutions of singularly perturbed Volterra integral equations via this method, [5] applied this method to find the solution of the linear integro-differential equation arising in oscillating magnetic fields.

In this paper, we use the homotopy perturbation method to solve the initial value problem which consists of the nonlinear first order Fredholm integro-differential equation of the second kind:

$$u'(x) = f(x) + \lambda \int_a^b k(x, y)[u(y)]^q dy \dots (1)$$

together with the initial condition

$$u(a) = \alpha \dots \dots \dots (2)$$

where $q \in \mathbb{N}$, $q \geq 2$, $a \leq x \leq b$, λ is a real number, the kernel k is a continuous function in $[a, b] \times [a, b]$ and f is a given continuous function defined in $[a, b]$.

2-Solutions of the Nonlinear Fredholm Integro-Differential Equations

Consider the initial value problem given by equations (1)-(2). We rewrite equation (1) as

$$A(u) - f(x) = 0 \dots \dots \dots (3)$$

$$\text{where } A(u) = \frac{du}{dx} - \lambda \int_a^b k(x, y)[u(y)]^q dy$$

Then the operator A can be divided into two parts L and N where L is a linear operator while N is a nonlinear operator. Therefore equation (3) becomes:

$$L(u) + N(u) - f(x) = 0 \dots \dots \dots (4)$$

where

$$L(u) = \frac{du}{dx}$$

and

$$N(u) = -\lambda \int_a^b k(x, y)[u(y)]^q dy$$

According to [6], we can construct a homotopy $v(x, p) : [a, b] \times [0, 1] \rightarrow \mathfrak{R}$ which satisfies $H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(x)] = 0$

In other words we can construct a homotopy v which satisfies

$$H(v, p) = (1 - p) \left[\frac{dv}{dx} - \frac{du_0}{dx} \right] + p \left[\frac{dv}{dx} - \lambda \int_a^b k(x, y) [v(y)]^q - f(x) \right] = 0 \dots\dots\dots (5)$$

where $p \in [0, 1]$ and u_0 is the initial approximation to the solution of equation (1) which satisfies the initial condition given by equation (2).

By using equation (5) it follows that

$$H(v, 0) = \frac{dv}{dx} - \frac{du_0}{dx} = 0 \dots\dots\dots (6)$$

$$H(v, 1) = \frac{dv}{dx} - \lambda \int_a^b k(x, y) [v(y)]^q - f(x) = 0 \dots\dots\dots (7)$$

and the changing process of p from zero to unity is just that of $H(v, p)$ from $\frac{dv}{dx} - \frac{du_0}{dx}$ to

$$\frac{dv}{dx} - \lambda \int_a^b k(x, y) [v(y)]^q - f(x).$$

In a topology, this is called deformation, $\frac{dv}{dx} - \frac{du_0}{dx}$ and $\frac{dv}{dx} - \lambda \int_a^b k(x, y) [v(y)]^q - f(x)$ are called homotopic.

Next, we assume that the solution of equation (5) can be expressed as

$$v(x) = v_0(x) + pv_1(x) + p^2v_2(x) + \mathbf{L} \dots\dots\dots (8)$$

Therefore the approximated solution of the initial value problem given by equations (1)-(2) can be obtained as follows:

$$u(x) = \lim_{p \rightarrow 1} v(x) = \sum_{i=0}^{\infty} v_i(x) \dots\dots\dots (9)$$

The convergence of the series given by equation (9) has been proved in [6].

By substituting the approximated solution given by equation (8) into equation (5) one can get:

$$\sum_{i=0}^{\infty} p^i \frac{dv_i}{dx} - \frac{du_0}{dx} + p \frac{du_0}{dx} + p \left[-\lambda \int_a^b k(x, y) \left[\sum_{i=0}^{\infty} p^i v_i(y) \right]^q dy - f(x) \right] = 0$$

Then by equating the terms with identical powers of p one can have:

$$p^0 : \frac{dv_0(x)}{dx} - \frac{du_0(x)}{dx} = 0 \dots\dots\dots (10.a)$$

$$p^1 : \frac{dv_1(x)}{dx} + \frac{du_0(x)}{dx} - f(x) - \lambda \int_a^b k(x, y) [v_0(y)]^q dy = 0 \dots\dots\dots (10.b)$$

$$p^2 : \left\{ \begin{array}{l} \frac{dv_2(x)}{dx} - \lambda \int_a^b k(x, y) [2v_0(y)v_1(y)] dy = 0 \text{ if } q=2 \\ \frac{dv_2(x)}{dx} - \lambda \int_a^b k(x, y) [3(v_0(y))^2v_1(y)] dy = 0 \text{ if } q=3 \\ \frac{dv_2(x)}{dx} - \lambda \int_a^b k(x, y) [4(v_0(y))^3v_1(y)] dy = 0 \text{ if } q=4 \\ \mathbf{M} \end{array} \right. \dots\dots\dots (10.c)$$

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$$p^j : \left\{ \begin{array}{l} \frac{dv_j(x)}{dx} - \lambda \int_a^b k(x,y) \cdot \\ \sum_{i=0}^{j-1} [v_i(y)v_{j-k-1}(y)] dy = 0 \text{ if } q=2 \\ \frac{dv_j(x)}{dx} - \lambda \int_a^b k(x,y) \cdot \\ \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} [v_i(y)v_k(y)v_{j-k-i-1}(y)] dy = 0 \text{ if } q=3 \\ \frac{dv_j(x)}{dx} - \lambda \int_a^b k(x,y) \cdot \sum_{i=0}^{j-1} \sum_{k=0}^{j-i-1} \sum_{l=0}^{j-i-k-1} \\ [v_i(y)v_k(y)v_l(y)v_{j-1-k-i-l}(y)] dy = 0 \text{ if } q=4 \end{array} \right.$$

..... (10.e)

Since $u(a) = \alpha$ then we choose $u_0(x) = \alpha + \int_a^x f(y)dy$ and this implies that $u_0(a) = \alpha$. Also, for simplicity we set $v_0(x) = u_0(x) = \alpha + \int_a^x f(y)dy$. Therefore by substituting $x=a$ in equation (9) one can have:

$$u(a) = \sum_{i=0}^{\infty} v_i(a)$$

But $v_0(a) = \alpha$ and $u(a) = \alpha$, hence $v_i(a) = 0, i=0,1,\dots$. So, equation (10.a) is automatically satisfied. By substituting $v_0(x) = u_0(x) = \alpha + \int_a^x f(y)dy$ into equation (10.b) one can get:

$$\frac{dv_1(x)}{dx} - \lambda \int_a^b k(x,y) \left[\alpha + \int_a^y f(z)dz \right]^q dy = 0$$

By integrating both sides of the above differential equation and by using the initial condition $v_1(a) = 0$ one can obtain

$$v_1(x) = \lambda \int_a^x \int_a^b k(t,y) \left[\alpha + \int_a^y f(z)dz \right]^q dy dt$$

..... (11)

By substituting v_0 and v_1 into equation (10.c) and by solving the resulting first order linear

ordinary differential equation together with the initial condition $v_2(a) = 0$ one can get $v_2(x)$. Then by substituting $j=3, v_0, v_1$ and v_2 into equation (10.e) and by using the initial condition $v_3(a) = 0$ one can solve the resulting first order linear ordinary differential equation to get $v_3(x)$. In a similar manner one can get $v_i(x), i = 4,5,\dots$. By substituting $v_i(x), i = 0,1,\dots$ into equation (9) one can get the approximated solution of the initial value problem given by equations (1)-(2).

3-Numerical Examples

In this section we present two examples of the initial value problems of nonlinear first order Fredholm integro-differential equations that are solved by the homotopy perturbation method.

Example (1):

Consider the initial value problem that consists of the nonlinear first order Fredholm integro-differential equation:

$$u'(x) = \frac{159}{160} - \frac{1}{64}x^2 + \int_0^{\frac{1}{2}} (x^2 + y)[u(y)]^3 dy$$

.....(12)

together with the initial condition:

$$u(0) = 0$$

.....(13)

Here $a = 0, b = \frac{1}{2}, q = 3, \lambda = 1,$

$$f(x) = \frac{159}{160} - \frac{1}{64}x^2 \text{ and}$$

$$k(x,y) = x^2 + y.$$

We use the homotopy perturbation method to solve this example. To do this, let

$$v_0(x) = u_0(x) = \int_0^x \left(\frac{159}{160} - \frac{1}{64}y^2 \right) dy$$

$$= \frac{159}{160}x - \frac{1}{192}x^2.$$

In this case, let $N=0$, then

$$u(x) \cong \sum_{i=0}^N v_i(x) = v_0(x)$$

$$= \frac{159}{160}x - \frac{1}{192}x^2$$

$$\cong 0.99375000x - 5.2083333 \times 10^{-3}x^3.$$

By substituting a, b, λ, q, k and f into equation (11) one can have:

$$v_1(x) = \int_0^x \int_0^{\frac{1}{2}} (t^2 + y) \left[\int_0^y \left(\frac{159}{160} - \frac{1}{64} z^2 \right) dz \right]^3 dy dt$$

$$= \frac{4266737979617}{697596641280000} x + \frac{27711283493}{5435817984000} x^3.$$

In this case, let $N=1$, then

$$u(x) \cong \sum_{i=0}^N v_i(x) = v_0(x) + v_1(x)$$

$$= \frac{697503400251617}{697596641280000} x - \frac{600268507}{5435817984000} x^3$$

$$\cong 0.99986634x - 1.1042837 \times 10^{-4} x^3.$$

Next, we must find $v_2(x)$

$$v_2(x) = \int_0^x \int_0^{\frac{1}{2}} k(t, y) [3v_0(y)v_1(y)] dy dt$$

$$= \frac{1632136562043826561889}{12568209553851678720000000} x + \frac{2408085905449236767}{22443231346163712000000} x^3.$$

Therefore, for $N=2$,

$$u(x) \cong \sum_{i=0}^N v_i(x) = v_0(x) + v_1(x) + v_2(x)$$

$$\cong 0.99999620x - 3.1316122 \times 10^{-6} x^3.$$

In a similar manner one can get $v_i(x), i = 3, 4, \dots$. The following table gives the approximated solutions for different values of N .

N	u(x)
0	$0.99375000x - 5.2083333 \times 10^{-3} x^3$
1	$0.99986634x - 1.1042837 \times 10^{-4} x^3$
2	$0.99999620x - 3.1316122 \times 10^{-6} x^3$
3	$0.99999988x - 1.0171587 \times 10^{-7} x^3$
4	$x - 3.5762754 \times 10^{-9} x^3$
5	$x - 1.3246308 \times 10^{-10} x^3$
6	$x - 5.0906824 \times 10^{-12} x^3$
7	$x - 2.0109439 \times 10^{-13} x^3$
8	$x - 8.1147316 \times 10^{-15} x^3$
9	$x - 3.7360906 \times 10^{-15} x^3$
10	$x - 5.5280006 \times 10^{-17} x^3$

Note that from the above table one can deduce that as N increases the approximated solution of the initial value problem given by

equations (12)-(13) converges to the exact solution $u(x) = x$.

Example (2):

Consider the initial value problem that consists of the nonlinear first order Fredholm integro-differential equation:

$$u'(x) = \frac{11}{6}x - \frac{1}{5} + \int_0^1 (xy + 1)[u(y)]^2 dy \dots (14)$$

together with the initial condition:

$$u(0) = 0 \dots \dots \dots (15)$$

Here $a = 0, b = 1, q = 2, \lambda = 1,$

$$f(x) = \frac{11}{6}x - \frac{1}{5} \text{ and}$$

$$k(x, y) = xy + 1.$$

We use the homotopy perturbation method to solve this example. To do this, let

$$v_0(x) = u_0(x) = \int_0^x \left(\frac{11}{6}y - \frac{1}{5} \right) dy$$

$$= \frac{11}{12}x^2 - \frac{1}{5}x.$$

In this case, let $N=0$, then

$$u(x) \cong \sum_{i=0}^N v_i(x) = v_0(x)$$

$$= \frac{11}{12}x^2 - \frac{1}{5}x$$

$$\cong 0.91667x^2 - 0.2x.$$

By substituting a, b, λ, q, k and f into equation (11) one can have:

$$v_1(x) = \int_0^x \int_0^1 (ty + 1) \left[\int_0^y \left(\frac{11}{6}z - \frac{1}{5} \right) dz \right]^2 dy dt$$

$$= \frac{1657}{43200}x^2 + \frac{323}{3600}x.$$

In this case, let $N=1$, then

$$u(x) \cong \sum_{i=0}^N v_i(x) = v_0(x) + v_1(x)$$

$$\cong 0.95502x^2 - 0.11028x.$$

Next, we must find $v_2(x)$

$$v_2(x) = \int_0^x \int_0^1 k(t, y) [2v_0(y)v_1(y)] dy dt$$

$$= \frac{1266611}{77760000}x^2 + \frac{51047}{1296000}x.$$

Therefore, for N=2,

$$u(x) \cong \sum_{i=0}^N v_i(x) = v_0(x) + v_1(x) + v_2(x)$$

$$\cong 0.97131x^2 - 7.09997 \times 10^{-2}x.$$

In a similar manner one can get $v_i(x)$, $i = 3, 4, \dots$. The following table gives the approximated solutions for different values of N.

N	u(x)
0	$0.91667x^2 - 0.2x$
1	$0.95502x^2 - 0.11028x$
2	$0.97131x^2 - 7.09997 \times 10^{-2}x$
3	$0.98022x^2 - 4.90465 \times 10^{-2}x$
4	$0.98572x^2 - 3.54806 \times 10^{-2}x$
5	$0.98936x^2 - 2.64527 \times 10^{-2}x$
6	$0.99190x^2 - 2.01584 \times 10^{-2}x$
7	$0.99373x^2 - 1.56202 \times 10^{-2}x$
8	$0.99508x^2 - 1.22643 \times 10^{-2}x$
9	$0.996096x^2 - 9.73292 \times 10^{-3}x$
10	$0.996875x^2 - 7.79290 \times 10^{-3}x$
11	$0.997480x^2 - 6.28651 \times 10^{-3}x$
12	$0.997955x^2 - 5.10395 \times 10^{-3}x$
13	$0.998330x^2 - 4.16697 \times 10^{-3}x$
14	$0.998631x^2 - 3.41862 \times 10^{-3}x$
15	$0.998872x^2 - 2.81678 \times 10^{-3}x$
16	$0.999067x^2 - 2.32982 \times 10^{-3}x$
17	$0.999226x^2 - 1.93369 \times 10^{-3}x$
18	$0.999355x^2 - 1.60990 \times 10^{-3}x$
19	$0.999462x^2 - 1.34412 \times 10^{-3}x$
20	$0.999550x^2 - 1.12510 \times 10^{-3}x$
21	$0.999622x^2 - 9.43993 \times 10^{-4}x$
22	$0.999682x^2 - 7.93753 \times 10^{-4}x$
23	$0.999732x^2 - 6.68758 \times 10^{-4}x$
24	$0.999774x^2 - 5.64490 \times 10^{-4}x$
25	$0.999809x^2 - 4.77297 \times 10^{-4}x$
26	$0.999838x^2 - 4.04670 \times 10^{-4}x$

27	$0.999863x^2 - 3.43486 \times 10^{-4}x$
28	$0.999883x^2 - 2.91965 \times 10^{-4}x$
29	$0.999901x^2 - 2.48517 \times 10^{-4}x$
30	$0.999915x^2 - 2.11818 \times 10^{-4}x$
31	$0.999928x^2 - 1.80774 \times 10^{-4}x$
32	$0.999938x^2 - 1.54475 \times 10^{-4}x$
33	$0.999947x^2 - 1.32165 \times 10^{-4}x$
34	$0.999954x^2 - 1.13214 \times 10^{-4}x$
35	$0.999961x^2 - 9.70962 \times 10^{-5}x$
36	$0.999966x^2 - 8.33727 \times 10^{-5}x$
37	$0.999971x^2 - 7.16747 \times 10^{-5}x$
38	$0.999975x^2 - 6.16927 \times 10^{-5}x$
39	$0.999978x^2 - 5.31665 \times 10^{-5}x$
40	$0.999981x^2 - 4.58769 \times 10^{-5}x$

Note that from the above table one can deduce that as N increases, the approximated solution of the initial value problem given by equations (14)-(15) converges to the exact solution $u(x) = x^2$.

4- Conclusions

In this paper, we deal with the approximated solutions of the initial value problems of special types of nonlinear first order Fredholm integro-differential equations using the homotopy perturbation method. This method was tested on some examples and were seen to give satisfactory results as N increases.

5- References

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