SOLUTIONS FOR LINEAR FREDHOLM-VOLTTERRA INTEGRAL EQUATIONS OF THE SECOND KIND USING THE REPEATED CORRECTED TRAPEZOIDAL AND SIMPSON'S METHODS

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Abstract

In this paper we introduce numerical methods for solving linear Fredholm–Volterra integral equations of the second kind. The main idea is based on the corrected trapezoidal and Simpson's 1/3 quadrature formulas. These techniques are very effective. Numerical results are illustrated by different examples.

1-Introduction

Many problems in mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems of mechanics of continuous media reduce to integral equations of the second kind.

The books edited by Linz [1] and Kanwal [2] contain some different methods to solve the integral equations analytically. Numerical methods also take an important place in solving the integral equations (see [1],[3],[4],[5],[6]).

The literatures of Fredholm-Volterra integral equations contain few numerical methods. The commonly used methods are the projection method, time collocation method [7], [8], Nystrum method [9],[10], the Adomian decomposition method [11] and Taylor expansion method [12]. A corrected trapezoidal approach for solving Fredholm and Volterra integral equations has been presented by Nadjafi and Heidari [13] and then this has been extended by Majeed to system Fredholm and Volterra integral equation [14]. In this work, we present the repeated corrected trapezoidal formula and the repeated corrected Simpson's 1/3 formula to solve the linear Fredholm-Volterra integral equation of second kind given by:

$$u(x)=f(x)+\lambda\int_a^b k(x,y)u(y)dy+\mu\int_a^x k^*(x,y)u(y)dy \qquad \text{............................ (1.1)}$$

where $a \le x \le b$, λ and μ are real numbers, f(x), k(x, y) and k*(x, y), are given continuous functions and u is the unknown function to be determined. To do this we assume the functions $\frac{\partial k(x,y)}{\partial x}$, $\frac{\partial k(x,y)}{\partial y}$, $\frac{\partial k^*(x,y)}{\partial x}$, $\frac{\partial k^*(x,y)}{\partial y}$ and $f'(x)$ must exist.

2-The Repeated Corrected Trapezoidal Method for Solving Equation (1.1)

Recall that the repeated corrected trapezoidal formula for finding definite integral $\int_a^b f(x)dx$ is

$$\int_a^b f(x)dx=\frac{h}{2}f(a)+h\sum_{j=1}^{n-1}f(x_j)+\frac{h}{2}f(b)+\frac{h^2}{12}\left[f'(a)-f'(b)\right]-\frac{(b-a)h^4}{720}f^{(4)}(\xi), a<\xi<b. \qquad \text{............................(2.1)}$$

where f is a continuous function defined on the closed interval [a,b], $x_i=a+ih$, i=0, 1, …, n, $h=(b-a)/n$ and n is the number of subintervals of the interval [a,b].

Next, To solve the linear Fredholm-Volterra integral equation of second kind given by equation (1.1) on the finite interval [a, b], we divide the interval [a,b] into n smaller intervals of width h, where h = (b – a)/n. The i-th point of subdivision is denoted by x_i such that $x_i=a+ih$, $i=0,1,...,n$. Let u_i denote the numerical solution of equation (1.1) at each x_i. Then, we substitute x= x_i, $i=0,1,...,n$ into equation (1.1) to get

$$u(x_i)=f_i+\int_a^b k(x_i,y)u(y)dy+$$

$$\int_a^{x_i} k^*(x_i, y)u(y)dy,\ i = 0,1,...,n. \qquad ....(2.2)$$

where $f_i = f(x_i),\ i = 0,1,...,n.$

If we approximate the integrals that appeared in equation (2.2) by the repeated corrected trapezoidal formula which will yield the following system of (n+1) equation:

$$u_0 = f_0 + \left(\frac{h}{2}k_{0,0} + \frac{h^2}{12}J_{0,0}\right)u_0 + h\sum_{j=1}^{n-1} k_{0,j}u_j + \left(\frac{h}{2}k_{0,n} - \frac{h^2}{12}J_{0,n}u_n\right)u_n + \frac{h^2}{12}\left(k_{0,0}u'_0 - k_{0,n}u'_n\right),$$

$$u_i = f_i + \left(\frac{h}{2}\left(k_{i,0} + k^*_{i,0}\right) + \frac{h^2}{12}\left(J_{i,0} + J^*_{i,0}\right)\right)u_0 + h\sum_{j=1}^{i-1}\left(k_{i,j} + k^*_{i,j}\right)u_j + \left(hk_{i,i} + \frac{h}{2}k^*_{i,i} - \frac{h^2}{12}J^*_{i,i}\right)u_i + h\sum_{j=i+1}^{n-1} k_{i,j}u_j + \left(\frac{h}{2}k_{i,n} - \frac{h^2}{12}J_{i,n}\right)u_n + \frac{h^2}{12}\left(\left(k_{i,0} + k^*_{i,0}\right)u'_0 - k^*_{i,i}u'_i - k_{i,n}u'_n\right),\ i = 1,2,...,n-1,$$

and

$$u_n = f_n + \left(\frac{h}{2}\left(k_{n,0} + k^*_{n,0}\right) + \frac{h^2}{12}\left(J_{n,0} + J^*_{n,0}\right)\right)u_0 + h\sum_{j=1}^{n-1}\left(k_{n,j} + k^*_{n,j}\right)u_j + \left(\frac{h}{2}\left(k_{n,n} + k^*_{n,n}\right) - \frac{h^2}{12}\left(J_{n,n} + J^*_{n,n}\right)\right)u_n + \frac{h^2}{12}\left(\left(k_{n,0} + k^*_{n,0}\right)u'_0 - \left(k_{n,n} + k^*_{n,n}\right)u'_n\right). \qquad ....(2.3)$$

where $k^*_{i,j} = k^*(x_i, x_j),\ J^*_{i,j} = \left.\frac{\partial k^*(x_i, y)}{\partial y}\right|_{y=y_j}$,

$$k_{i,j} = k(x_i, x_j),\ J_{i,j} = \left.\frac{\partial k(x_i, y)}{\partial y}\right|_{y=y_j},\ i,j = 0,1,...,n.$$

The above system of equations consists of (n+1) equations with (2n+2) unknowns namely, u_i and u'_i , i=0,1,…,n.

Next, we must find $u'_i, i = 0,1,\mathbf{K},n$. To do this, we must differentiate equation (1.1) with respect to x to get:

$$u'(x) = f'(x) + \int_a^b H(x,y)u(y)dy + \int_a^x H^*(x,y)u(y)dy + k^*(x,x)u(x). \qquad ....(2.4)$$

where $H(x,y) = \frac{\partial k(x,y)}{\partial x}$, $H^*(x,y) = \frac{\partial k^*(x,y)}{\partial x}$.

By evaluating equation (2.4) at $x = x_i$, $i = 0,1,...,n$, one can get

$$u'(x_i) = f'(x_i) + \int_a^b H(x_i,y)u(y)dy + \int_a^{x_i} H^*(x_i,y)u(y)dy + k^*(x_i,x_i)u(x_i), \qquad ....(2.5)$$

Next, to solve equation (2.5), one must consider four cases:

Case (1):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$ and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ exist. In this case, we approximate the integrals that appeared in eq(2.5) by the repeated corrected trapezoidal formula one can get the following system of equations :

$$u'_0 = f'_0 + \left(\frac{h}{2}H_{0,0} + \frac{h^2}{12}L_{0,0}\right)u_0 + h\sum_{j=1}^{n-1} H_{0,j}u_j + \left(\frac{h}{2}H_{0,n} - \frac{h^2}{12}L_{0,n}\right)u_n + \frac{h^2}{12}\left(H_{0,0}u'_0 - H_{0,n}u'_n\right),$$

$$u'_i = f'_i + \left(\frac{h}{2}\left(H_{i,0} + H^*_{i,0}\right) + \frac{h^2}{12}\left(L_{i,0} + L^*_{i,0}\right)\right)u_0 + h\sum_{j=1}^{i-1}\left(H_{i,j} + H^*_{i,j}\right)u_j + \left(k^*_{i,i} + hH_{i,i} + \frac{h}{2}H^*_{i,i} - \frac{h}{12}L_{i,i}\right)u_i + h\sum_{j=i+1}^{n-1} H_{i,j}u_j + \frac{h}{2}H_{i,n}u_n$$

$$\frac{h^2}{12}\left(\left(H_{i,0} + H^*_{i,0}\right)u'_0 - H^*_{i,i}u'_i - H_{i,n}u'_n\right),$$

$$i = 1,2,...,n-1,$$

and

$$u'_n = f'_n + \left(\frac{h}{2}\left(H_{n,0}+H^*_{n,0}\right)+\frac{h^2}{12}\left(L_{n0}+L^*_{n,0}\right)\right)u_0 + h\sum_{j=1}^{n-1}\left(H_{n,j}+H^*_{n,j}\right)u_j + \left(\frac{h}{2}\left(H_{n,n}+H^*_{n,n}\right) - \frac{h^2}{12}\left(L_{n,n}+L^*_{n,n}\right)+k^*_{n,n}\right)u_n + \frac{h^2}{12}\left(\left(H_{n0}+H^*_{n,0}\right)u'_0 - \left(H_{n,n}+H^*_{n,n}\right)u'_n\right). \quad \text{.............................} (2.6)$$

where $H_{i,j} = H(x_i, y_j), H^*{}_{i,j} = H^*(x_i, y_j)$,

$$L^*{}_{i,j} = \left.\frac{\partial H^*(x,y)}{\partial y}\right|_{\substack{x=x_i\\ y=y_j}} = \left.\frac{\partial^2 k^*(x,y)}{\partial x \partial y}\right|_{\substack{x=x_i\\ y=y_j}},$$

$$L_{i,j} = \left.\frac{\partial H(x,y)}{\partial y}\right|_{\substack{x=x_i\\ y=y_j}} = \left.\frac{\partial^2 k(x,y)}{\partial x \partial y}\right|_{\substack{x=x_i\\ y=y_j}}, i,j = 0,1,\ldots,n.$$

The system which consists of equation (2.3) and equation (2.6) can be solved to find the unknowns $\{u_i\}_{i=0}^n$ by any suitable method.

Case (2):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$, dose not exists and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ exists. In this case, we approximate the first integral and the second integral that appeared in eq.(2,5) by the repeated trapezoidal formula and repeated corrected trapezoidal formula respectively, to get the following system:

$$u'_0 = f'_0 + \frac{h}{2}\left(H_{0,0}u_0 + 2\sum_{j=1}^{n-1} H_{0,j}u_j + H_{0,n}u_n\right),$$

$$u'_i = f'_i + \left(\frac{h}{2}H_{i,0} + \frac{h}{2}H^*_{i,0} - \frac{h^2}{12}L^*_{i,0}\right)u_0 + h\sum_{j=1}^{i-1}\left(H_{i,j}+H^*_{i,j}\right)u_j + \left(k^*_{i,i} + hH_{i,i} + \frac{h}{2}H^*_{i,i} - \frac{h^2}{12}L^*_{i,i}\right)u_i + h\sum_{j=i+1}^{n-1} H_{i,j}u_j + \frac{h}{2}H_{i,n}u_n + \frac{h^2}{12}\left(H^*_{i,0}u'_0 - H^*_{i,i}u'_i\right),\ i = 1,2,\ldots,n-1,$$

and

$$u'_n = f'_n + \left(\frac{h}{2}\left(H_{n,0}+H^*_{n,0}\right) - \frac{h^2}{12}L^*_{n,0}\right)u_0 + h\sum_{j=1}^{n-1}\left(H_{n,j}+H^*_{n,j}\right)u_j + \left(\frac{h}{2}H_{n,n} + \frac{h}{2}H^*_{n,n} - \frac{h^2}{12}L^*_{n,n} + k^*_{n,n}\right)u_n + \frac{h^2}{12}\left(H^*_{n,0}u'_0 - H^*_{n,i}u'_i\right). \quad \text{.............................} (2.7)$$

The system which consists of equation (2.3) and equation (2.7) can be solved to find the unknowns $\{u_i\}_{i=0}^n$ by any suitable method.

Case (3):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$ exists and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ dose not exists. In this case, we approximate the first integral and the second integral that appeared in eq.(2,5) by the repeated corrected trapezoidal formula and repeated trapezoidal formula respectively, to get the following system:

$$u'_0 = f'_0 + \frac{h}{2}\left(\left(H_{0,0}+L_{0,0}\right)u_0 + 2\sum_{j=1}^{n-1} H_{0,j}u_j + \left(H_{0,n} - L_{0,n}u_n\right)u_n\right) + \frac{h^2}{12}\left(H_{0,0}u'_0 - H_{0,n}u'_n\right),$$

$$u'_i = f'_i + \left(\frac{h}{2}H_{i,0} + \frac{h}{2}H^*_{i,0} - \frac{h^2}{12}L_{i,0}\right)u_0 + h\sum_{j=1}^{i-1}\left(H_{i,j}+H^*_{i,j}\right)u_j + \left(k^*_{i,i} + hH_{i,i} + \frac{h}{2}H^*_{i,i}\right)u_i + h\sum_{j=i+1}^{n-1} H_{i,j}u_j + \left(\frac{h}{2}H_{i,n} - \frac{h^2}{12}L_{i,n}\right)u_n + \frac{h^2}{12}\left(H_{i,0}u'_0 - H_{i,n}u'_n\right),\ i = 1,2,\ldots,n-1,$$

and

$$u'_n = f'_n + \left(\frac{h}{2}\left(H_{n,0} + H^*_{n,0}\right) - \frac{h^2}{12} L_{n,0} \right) u_0 + h\sum_{j=1}^{n-1}\left(H_{n,j} + H^*_{n,j}\right)u_j + \left(\frac{h}{2}\left(H_{n,n} + H^*_{n,n}\right) - \frac{h^2}{12} L_{n,n} + k^*_{n,n} \right) u_n + \frac{h^2}{12}\left(H_{n,0}u'_0 - H_{n,n}u'_n\right). \quad .... (2.8)$$

The system consists of equation (2.3) and equation (2.8) can be solved to find the unknowns $\{u_i\}_{i=0}^{n}$ by any suitable method.

Case (4):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$ and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ do not exist. In this case, we approximate the integrals that appeared in eq.(2.5) by the repeated trapezoidal formula one can get the following system of equations:

$$u'_0 = f'_0 + \frac{h}{2}\left(H_{0,0}u_0 + 2\sum_{j=1}^{n-1} H_{0,j}u_j + H_{0,n}u_n\right),$$

$$u'_i = f'_i + \frac{h}{2}\left(H_{i,0} + H^*_{i,0}\right)u_0 + h\sum_{j=1}^{i-1}\left(H_{i,j} + H^*_{i,j}\right)u_j + \left(hH_{i,i} + \frac{h}{2}H^*_{i,i} + k^*_{i,i} \right)u_i + h\sum_{j=i+1}^{n-1} H_{i,j}u_j + \frac{h}{2}H_{i,n}u_n, \; i = 1,2,...,n-1,$$

and

$$u'_n = f'_n + \frac{h}{2}\left(H_{n,0} + H^*_{n,0}\right)u_0 + h\sum_{j=1}^{n-1}\left(H_{n,j} + H^*_{n,j}\right)u_j + \left(\frac{h}{2}\left(H_{n,n} + H^*_{n,n}\right) + k^*_{n,n} \right)u_n. \quad .... (2.9)$$

The system consists of equation (2.3) and equation (2.9) can be solved to find the unknowns $\{u_i\}_{i=0}^{n}$ by any suitable method.

3-The Repeated Corrected Simpson's 1/3 Method for Solving Equation (1.1)

Recall that, the repeated corrected Simpson's 1/3 formula for finding the definite integral $\int_a^b f(x)dx$ is

$$\int_a^b f(x)dx = \frac{h}{15}\Big(7f(a) + 16\sum_{j=1}^{n} f(x_{2j-1}) + 14\sum_{j=1}^{n-1} f(x_{2j}) + 7f(b)\Big) + \frac{h^2}{15}\left(f'(a) - f'(b)\right) - \frac{(b-a)h^5}{9450} f^{(5)}(\xi) \quad .... (3.1)$$

Next, To solve the linear Fredholm-Volterra integral equation of second kind given by equation (1.1) on the finite interval [a, b], we divide the interval into 2n smaller intervals of width h, where h = (b − a)/m, where m=2n. The solution of eq.(1.1) at the even nods (x_{2i}) is given by:

$$u(x_{2i}) = f(x_{2i}) + \int_a^b k(x_{2i}, y)u(y) + \int_a^{x_{2i}} k^*(x_{2i}, y)u(y)dy, \; i = 0,1,\mathbf{K},n.. \quad .... (3.2)$$

and at the odd nods (x_{2i+1}) is given by:

$$u(x_{2i+1}) = f(x_{2i+1}) + \int_a^b k(x_{2i+1}, y)u(y) + \int_a^{x_{2i+1}} k^*(x_{2i+1}, y)u(y)dy, \; i = 0,1,\mathbf{K},n-1. \quad .... (3.3)$$

By using the repeated corrected Simpson's 1/3 formula to approximate the integrals that appeared in equations (3.2)-(3.3) one can get the following system:

$$u_0 = f_0 + \left(\frac{7h}{15}k_{0,0} + \frac{h^2}{15}J_{0,0} \right)u_0 + \frac{14h}{15}\sum_{j=1}^{n-1} k_{0,2j}u_{2j} + \frac{16h}{15}\sum_{j=1}^{n} k_{0,2j-1}u_{2j-1} + \left(\frac{7h}{15}k_{0,2n} - \frac{h^2}{15}J_{0,2n} \right)u_{2n} + \frac{h^2}{15}\left(k_{0,0}u'_0 - k_{0,2n}u'_{2n}\right),$$

$$u_1 = f_1 + \left(\frac{7h}{15}k_{1,0} + \frac{h^2}{15}J_{1,0} + \frac{h}{2}k^*_{1,0} + \frac{h^2}{12}J^*_{1,0}\right)u_0 + \left(\frac{16h}{15}k_{1,1} + \frac{h}{2}k^*_{1,1} - \frac{h^2}{12}J^*_{1,1}\right)u_1 + \frac{14h}{15}\sum_{j=1}^{n-1}k_{1,2j}u_{2j} + \frac{16h}{15}\sum_{j=2}^{n}k_{1,2j-1}u_{2j-1} + \left(\frac{7h}{15}k_{1,2n} - \frac{h^2}{15}J_{1,2n}\right)u_{2n} + \frac{h^2}{15}\left(k_{1,0}u'_0 - k_{1,2n}u'_{2n}\right) + \frac{h^2}{12}\left(k^*_{1,0}u'_0 - k^*_{1,1}u'_1\right),$$

$$u_{2i} = f_{2i} + \left(\frac{7h}{15}\left(k_{2i,0} + k^*_{2i,0}\right) + \frac{h^2}{15}\left(J_{2i,0} + J^*_{2i,0}\right)\right)u_0 + \frac{16h}{15}\sum_{j=1}^{i}\left(k_{2i,2j-1} + k^*_{2i,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{i-1}\left(k_{2i,2j} + k^*_{2i,2j}\right)u_{2j} + \left(\frac{h}{15}\left(14k_{2i,2i} + 7k^*_{2i,2i}\right) - \frac{h^2}{15}J^*_{2i,2i}\right)u_{2i} + \frac{16h}{15}\sum_{j=i+1}^{n}k_{2i,2j-1}u_{2j-1} + \frac{14h}{15}\sum_{j=i+1}^{n-1}k_{2i,2j}u_{2j} + \left(\frac{7h}{15}k_{2i,2n} - \frac{h^2}{15}J_{2i,2n}\right)u_{2n} + \frac{h^2}{15}\Big(\left(k_{2i,0} + k^*_{2i,0}\right)u'_0 - k^*_{2i,2i}u'_{2i} - k_{2i,2n}u'_{2n}\Big),\ i = 1,2,\ldots,n-1,$$

$$u_{2i+1} = f_{2i+1} + \left(\frac{7h}{15}\left(k_{2i+1,0} + k^*_{2i+1,0}\right) + \frac{h^2}{15}\left(J_{2i+1,0} + J^*_{2i+1,0}\right)\right)u_0 + \frac{16h}{15}\sum_{j=1}^{i}\left(k_{2i+1,2j-1} + k^*_{2i+1,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{i-1}\left(k_{2i+1,2j} + k^*_{2i+1,2j}\right)u_{2j} + \left(\frac{h}{15}\left(14k_{2i+1,2i} + 7k^*_{2i+1,2i}\right) + \frac{h^2}{60}J^*_{2i+1,2i}\right)u_{2i} + \left(\frac{h}{15}\left(16k_{2i+1,2i+1} + \frac{15}{2}k^*_{2i+1,2i+1}\right) - \frac{h^2}{12}J^*_{2i+1,2i+1}\right)u_{2i+1} + \frac{14h}{15}\sum_{j=i+1}^{n-1}k_{2i+1,2j-1}u_{2j-1} + \frac{16h}{15}\sum_{j=i+2}^{n}k_{2i+1,2j-1}u_{2j-1} + \left(\frac{7h}{15}k_{2i+1,2n} - \frac{h^2}{15}J_{2i+1,2n}\right)u_{2n} + \frac{h^2}{15}\left(k_{2i+1,0} - k^*_{2i+1,0}\right)u'_0 + \frac{h^2}{60}k^*_{2i+1,2i}u'_{2i} - \frac{h^2}{12}k^*_{2i+1,2i+1}u'_{2i+1} - \frac{h^2}{15}k_{2i+1,2n}u'_{2n},$$
$$i = 1,2,\ldots,n-1,$$

$$u_{2n} = \left(\frac{7h}{15}\left(k_{2n,0} + k^*_{2n,0}\right) - \frac{h^2}{15}\left(J_{2n,0} + J^*_{2n,0}\right)\right)u_0 + \frac{16h}{15}\sum_{j=1}^{n}\left(k_{2n,2j-1} + k^*_{2n,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{n-1}\left(k_{2n,2j} + k^*_{2n,2j}\right)u_{2j} + \left(\frac{7h}{15}\left(k_{2n,2n} + k^*_{2n,2n}\right) - \frac{h^2}{15}\left(J_{2n,2n} + J^*_{2n,2n}\right)\right)u_{2n} + \frac{h^2}{15}\Big(\left(k_{2n,0} + k^*_{2n,0}\right)u'_0 - \left(k_{2n,2n} + k^*_{2n,2n}\right)u'_{2n}\Big).$$

.............................(3.4)

The above system of equations consists of (m+1) equations with (2m+2) unknowns namely, u_i and $u'_i, i = 0,1,\ldots,2n$.

To find $u'_i, i = 0,1,\ldots,2n$, one must differentiate equation (1.1) with respect to x to get equation (2.4). By evaluating equation (2.4) at the even nods (x_{2i}), one can get:

$$u'(x_{2i}) = f'(x_{2i}) + \int_a^b H(x_{2i},y)u(y) + \int_a^{x_{2i}} H^*(x_{2i},y)u(y)dy + k^*(x_{2i},x_{2i}),$$
$$i = 0,1,\ldots,n. \qquad (3.5)$$

and at the odd nods (x_{2i+1}) , one can get :

$$u'(x_{2i+1}) = f'(x_{2i+1}) + \int_a^b H(x_{2i+1},y)u(y) + \int_a^{x_{2i+1}} H^*(x_{2i+1},y)u(y)dy + k^*(x_{2i+1},x_{2i+1}),$$
$$i = 0,1,\ldots,n-1. \qquad (3.6)$$

Next, to solve equations (3.5)-(3.6), one must consider four cases:

Case(1):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$ and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ exist, In this case, we approximate the integrals that appeared in equations (3.5)-(3.6) with repeated corrected Simpson's 1/3 formula which will yield the following system of equations:

$$u'_0 = f'_0 + \left(\frac{7h}{15}H_{0,0} + \frac{h^2}{15}L_{0,0}\right)u_0 + \frac{16h}{15}\sum_{j=1}^{n}H_{0,2j-1}u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{n-1}H_{0,2j}u_{2j} + \left(\frac{7h}{15}H_{0,2n} - \frac{h^2}{15}L_{0,2n}\right)u_{2n} + \frac{h^2}{15}\left(H_{0,0}u' - H_{0,2n}u'_{2n}\right),$$

$$u'_1 = f'_1 + \left(\frac{7h}{15}H_{1,0} + \frac{h^2}{15}L_{1,0} + \frac{h}{2}H^*_{1,0} + \frac{h^2}{12}L^*_{1,0}\right)u_0 + \left(k^*_{1,1} + \frac{16h}{15}H_{1,1} + \frac{h}{2}H^*_{1,1} - \frac{h^2}{12}L^*_{1,1}\right)u_1 + \frac{14h}{15}\sum_{j=1}^{n-1}H_{1,2j}u_{2j} + \frac{16h}{15}\sum_{j=2}^{n}H_{1,2j-1}u_{2j-1} + \left(\frac{7h}{15}H_{1,2n} - \frac{h^2}{15}H_{1,2n}\right)u_{2n} + \frac{h^2}{15}\left(H_{1,0}u'_0 - H_{1,2n}u'_{2n}\right) + \frac{h^2}{12}\left(H^*_{1,0}u'_0 - H^*_{1,1}u'_1\right),$$

$$u'_{2i} = f'_{2i} + \left(\frac{7h}{15}\left(H_{2i,0} + H^*_{2i,0}\right) + \frac{h^2}{15}\left(L_{2i,0} + L^*_{2i,0}\right)\right)u_0 + \frac{16h}{15}\sum_{j=1}^{i}\left(H_{2i,2j-1} + H^*_{2i,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{i-1}\left(H_{2i,2j} + H^*_{2i,2j}\right)u_{2j} + \left(k^*_{2i,2i} + \frac{h}{15}\left(14H_{2i,2i} + 7H^*_{2i,2i}\right) - \frac{h^2}{15}L^*_{2i,2i}\right)u_{2i} + \frac{16h}{15}\sum_{j=i+1}^{n}H_{2i,2j-1}u_{2j-1} + \frac{14h}{15}\sum_{j=i+1}^{n-1}H_{2i,2j}u_{2j} + \left(\frac{7h}{15}H_{2i,2n} - \frac{h^2}{15}L_{2i,2n}\right)u_{2n} + \frac{h^2}{15}\left(\left(H_{2i,0} + H^*_{2i,0}\right)u'_0 - H^*_{2i,2i}u'_{2i} - H_{2i,2n}u'_{2n}\right),\ i = 1, 2, \ldots, n-1,$$

$$u'_{2i+1} = f'_{2i+1} + \left(\frac{7h}{15}\left(H_{2i+1,0} + H^*_{2i+1,0}\right) + \frac{h^2}{15}\left(L_{2i+1,0} + L^*_{2i+1,0}\right)\right)u_0 + \frac{16h}{15}\sum_{j=1}^{i}\left(H_{2i+1,2j-1} + H^*_{2i+1,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{i-1}\left(H_{2i+1,2j} + H^*_{2i+1,2j}\right)u_{2j} + \left(\frac{14h}{15}H_{2i+1,2i} + \frac{29h}{30}H^*_{2i+1,2i} + \frac{h^2}{60}L^*_{2i+1,2i}\right)u_{2i} + \left(k^*_{2i+1,2i+1} + \frac{16h}{15}H_{2i+1,2i+1} + \frac{h}{2}H^*_{2i+1,2i+1} - \frac{h^2}{12}L^*_{2i+1,2i+1}\right)u_{2i+1} + \frac{14h}{15}\sum_{j=i+1}^{n-1}H_{2i+1,2j}u_{2j} + \frac{16h}{15}\sum_{j=i+2}^{n}H_{2i+1,2j-1}u_{2j-1} + \left(\frac{7h}{15}H_{2i+1,2n} - \frac{h^2}{15}L_{2i+1,2n}\right)u_{2n} + \frac{h^2}{60}\left(4\left(H_{2i+1,0} + H^*_{2i+1,0}\right)u'_0 + H^*_{2i+1,2i}u'_{2i} - 5H^*_{2i+1,2i+1}u'_{2i+1} - 4H_{2i+1,2n}u'_{2n}\right),\ i = 1, 2, \ldots, n-1,$$

$$u'_{2n} = f'_{2n} + \left(\frac{7h}{15}\left(H_{2n,0} + H^*_{2n,0}\right) + \frac{h^2}{15}\left(L_{2n,0} + L^*_{2n,0}\right)\right)u_0 + \frac{16h}{15}\sum_{j=1}^{n}\left(H_{2n,2j-1} + H^*_{2n,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{n-1}\left(H_{2n,2j} + H^*_{2n,2j}\right)u_{2j} + \left(k^*_{2n,2n} + \frac{7h}{15}\left(H_{2n,2n} + H^*_{2n,2n}\right) - \frac{h^2}{15}\left(L_{2n,2n} + L^*_{2n,2n}\right)\right)u_{2n} + \frac{h^2}{15}\left(\left(H_{2n,0} + H^*_{2n,0}\right)u'_0 - \left(H_{2n,2n} + H^*_{2n,2n}\right)u'_{2n}\right).$$

..............................(3.7)

The system consists of equation (3.4) and equation (3.7) can be solved to find the unknowns $\{u_i\}_{i=0}^{2n}$ by any suitable methods.

Case (2):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$, dose not exists and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ exists, In this case, we approximate the first integral that appeared in equations (3.5)-(3.6) with the repeated Simpson's 1/3 formula and approximate the second integral that appeared in equations (3.5)-(3.6) with the repeated corrected

Simpson's 1/3 formula. Therefore, we obtain the following system:

$$u_0' = f_0' + \frac{h}{3}H_{0,0}u_0 + \frac{4h}{3}\sum_{j=1}^{n}H_{0,2j-1}u_{2j-1} + \frac{2h}{3}\sum_{j=1}^{n-1}H_{0,2j}u_{2j} + \frac{h}{3}H_{0,2n},$$

$$u_1' = f_1' + \left(\frac{h}{3}H_{1,0} + \frac{h}{2}H^*_{1,0} + \frac{h^2}{12}L^*_{1,0}\right)u_0 + \left(k^*_{1,1} + \frac{4h}{3}H_{1,1} + \frac{h}{2}H^*_{1,1} - \frac{h^2}{12}L^*_{1,1}\right)u_1 + \frac{2h}{3}\sum_{j=1}^{n-1}H_{1,2j}u_{2j} + \frac{4h}{3}\sum_{j=2}^{n}H_{1,2j-1}u_{2j-1} + \frac{h}{3}H_{1,2n}u_n + \frac{h^2}{12}\left(H^*_{1,0}u_0' - H^*_{1,1}u_1'\right),$$

$$u_{2i}' = f_{2i}' + \left(\frac{h}{3}H_{2i,0} + \frac{7h}{15}H^*_{2i,0} + \frac{h^2}{15}L^*_{2i,0}\right)u_0 + \sum_{j=1}^{i}\left(\frac{4h}{3}H_{2i,2j-1} + \frac{16h}{15}H^*_{2i,2j-1}\right)u_{2j-1} + \sum_{j=1}^{i-1}\left(\frac{2h}{3}H_{2i,2j} + \frac{14h}{15}H^*_{2i,2j}\right)u_{2j} + \left(k^*_{2i,2i} + \frac{2h}{3}H_{2i,2i} + \frac{7h}{15}H^*_{2i,2i} - \frac{h^2}{15}L^*_{2i,2i}\right)u_{2i} + \frac{4h}{3}\sum_{j=i+1}^{n}H_{2i,2j-1}u_{2j-1} + \frac{2h}{3}\sum_{j=i+1}^{n-1}H_{2i,2j}u_{2j} + \frac{h}{3}H_{2i,2n}u_{2n} + \frac{h^2}{12}\left(H^*_{2i,2i}u_0' - H^*_{2i,2i}u_{2i}'\right),$$
$$i = 1, 2, \ldots, n-1,$$

$$u_{2i+1}' = f_{2i+1}' + \left(\frac{h}{3}H_{2i+1,0} + \frac{7h}{15}H^*_{2i+1,0} + \frac{h^2}{15}L^*_{2i+1,0}\right)u_0 + \sum_{j=1}^{i}\left(\frac{4h}{3}H_{2i+1,2j-1} + \frac{16h}{15}H^*_{2i+1,2j-1}\right)u_{2j-1} + \sum_{j=1}^{i-1}\left(\frac{2h}{3}H_{2i+1,2j} + \frac{14h}{15}H^*_{2i+1,2j}\right)u_{2j} + \left(\frac{2h}{3}H_{2i+1,2i} + \frac{29h}{30}H^*_{2i+1,2i} + \frac{h^2}{60}L^*_{2i+1,2i}\right)u_{2i} + \left(k^*_{2i+1,2i+1} + \frac{4h}{3}H_{2i+1,2i+1} + \frac{h}{2}H^*_{2i+1,2i+1} - \frac{h^2}{12}L^*_{2i+1,2i+1}\right)u_{2i+1} + \frac{2h}{3}\sum_{j=i+1}^{n-1}H_{2i+1,2j}u_{2j} + \frac{4h}{3}\sum_{j=i+2}^{n}H_{2i+1,2j-1}u_{2j-1} + \frac{h}{3}H_{2i+1,2n}u_{2n} + \frac{h^2}{60}\left(4H^*_{2i+1,0}u_0' + H^*_{2i+1,2i}u_{2i}' - 4H^*_{2i+1,2i+1}u_{2i+1}'\right),\ i = 1, 2, \mathbf{K}, n-1,$$

and

$$u_{2n}' = f_{2n}' + \left(\frac{h}{3}H_{2n,0} + \frac{7h}{15}H^*_{2n,0} + \frac{h^2}{15}L^*_{2n,0}\right)u_0 + \sum_{j=1}^{n}\left(\frac{4h}{3}H_{2n,2j-1} + \frac{16h}{15}H^*_{2n,2j-1}\right)u_{2j-1} + \sum_{j=1}^{n-1}\left(\frac{2h}{3}H_{2n,2j} + \frac{14h}{15}H^*_{2n,2j}\right)u_{2j} + \left(k^*_{2n,2n} + \frac{h}{3}H_{2n,2n} + \frac{7h}{15}H^*_{2n,2n} - \frac{h^2}{15}L^*_{2n,2n}\right)u_{2n} + \frac{h^2}{15}\left(H^*_{2n,0}u_0' - H^*_{2n,2n}u_{2n}'\right).$$

............................(3.8)

The system consists of equation (3.4) and equation (3.8) can be solved to find the unknowns $\{u_i\}_{i=0}^{2n}$ by any suitable method.

Case (3):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$ exists and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ dose not exists, In this case, we approximate the first integral that appeared in equations (3.5)-(3.6) with the repeated corrected Simpson's 1/3 formula and approximate the second integral that appeared in equations (3.5)-(3.6) with the repeated Simpson's 1/3 formula. Therefore, we obtain the following system:

$$u'_0 = f'_0 + \left(\frac{7h}{15}H_{0,0} + \frac{h^2}{15}L_{0,0}\right)u_0 + \frac{16h}{15}\sum_{j=1}^{n}H_{0,2j-1}u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{n-1}H_{0,2j}u_{2j} + \left(\frac{7h}{15}H_{0,2n} - \frac{h^2}{15}L_{0,2n}\right) + \frac{h^2}{15}\left(H_{0,0}u' - H_{0,2n}u'_{2n}\right),$$

$$u'_1 = f'_1 + \left(\frac{7h}{15}H_{1,0} + \frac{h^2}{15}L_{1,0} + \frac{h}{2}H^*{}_{1,0}\right)u_0 + \left(k^*{}_{1,1} + \frac{16h}{15}H_{1,1} + \frac{h}{2}H^*{}_{1,1}\right)u_1 + \frac{14h}{15}\sum_{j=1}^{n-1}H_{1,2j}u_{2j} + \frac{16h}{15}\sum_{j=2}^{n}H_{1,2j-1}u_{2j-1} + \left(\frac{7h}{15}H_{1,2n} - \frac{h^2}{15}H_{1,2n}\right)u_{2n} + \frac{h^2}{15}\left(H_{1,0}u'_0 - H_{1,2n}u'_{2n}\right),$$

$$u'_{2i} = f'_{2i} + \left(\frac{7h}{15}H_{2i,0} + \frac{h^2}{15}L_{2i,0} + \frac{h}{3}H^*{}_{2i,0}\right)u_0 + \sum_{j=1}^{i}\left(\frac{16h}{15}H_{2i,2j-1} + \frac{4h}{3}H^*{}_{2i,2j-1}\right)u_{2j-1} + \sum_{j=1}^{i-1}\left(\frac{14h}{15}H_{2i,2j} + \frac{2h}{3}H^*{}_{2i,2j}\right)u_{2j} + \left(k^*{}_{2i,2i} + \frac{14h}{15}H_{2i,2i} + \frac{h}{3}H^*{}_{2i,2i}\right)u_{2i} + \frac{16h}{15}\sum_{j=i+1}^{n}H_{2i,2j-1}u_{2j-1} + \frac{14h}{15}\sum_{j=i+1}^{n-1}H_{2i,2j}u_{2j} + \left(\frac{7h}{15}H_{2i,2n} - \frac{h^2}{15}L_{2i,2n}\right)u_{2n} + \frac{h^2}{15}\left(H_{2i,0}u'_0 - H_{2i,2n}u'_{2n}\right), i = 1,2,\ldots,n-1,$$

$$u'_{2i+1} = f'_{2i+1} + \left(\frac{7h}{15}H_{2i+1,0} + \frac{h}{3}H^*{}_{2i+1,0} + \frac{h^2}{15}L_{2i+1,0}\right)u_0 + \sum_{j=1}^{i}\left(\frac{16h}{15}H_{2i+1,2j-1} + \frac{4h}{3}H^*{}_{2i+1,2j-1}\right)u_{2j-1} + \sum_{j=1}^{i-1}\left(\frac{14h}{15}H_{2i+1,2j} + \frac{2h}{3}H^*{}_{2i+1,2j}\right)u_{2j} + \left(\frac{14h}{15}H_{2i+1,2i} + \frac{5h}{6}H^*{}_{2i+1,2i}\right)u_{2i} + \left(k^*{}_{2i+1,2i+1} + \frac{16h}{15}H_{2i+1,2i+1} + \frac{h}{2}H^*{}_{2i+1,2i+1}\right)u_{2i+1} + \frac{14h}{15}\sum_{j=i+1}^{n-1}H_{2i+1,2j}u_{2j} + \frac{16h}{15}\sum_{j=i+2}^{n}H_{2i+1,2j-1}u_{2j-1} + \left(\frac{7h}{15}H_{2i+1,2n} - \frac{h^2}{15}L_{2i+1,2n}\right)u_{2n} + \frac{h^2}{15}\left(H_{2i+1,0}u'_0 - H_{2i+1,2n}u'_{2n}\right), i = 1,2,\ldots,n-1,$$

and

$$u'_{2n} = f'_{2n} + \left(\frac{7h}{15}H_{2n,0} + \frac{h}{3}H^*{}_{2n,0} + \frac{h^2}{15}L_{2n,0}\right)u_0 + \sum_{j=1}^{n}\left(\frac{16h}{15}H_{2n,2j-1} + \frac{4h}{3}H^*{}_{2n,2j-1}\right)u_{2j-1} + \sum_{j=1}^{n-1}\left(\frac{14h}{15}H_{2n,2j} + \frac{2h}{3}H^*{}_{2n,2j}\right)u_{2j} + \left(k^*{}_{2n,2n} + \frac{7h}{15}H_{2n,2n} + \frac{h}{3}H^*{}_{2n,2n} - \frac{h^2}{15}L_{2n,2n}\right)u_{2n} + \frac{h^2}{15}\left(H_{2n,0}u'_0 - H_{2n,2n}u'_{2n}\right).$$

............................(3.9)

The system consists of equation (3.4) and equation (3.9) can be solved to find the unknowns $\{u_i\}_{i=0}^{2n}$ by any suitable method.

Case (4):

If $L(x,y) = \frac{\partial^2 k(x,y)}{\partial x \partial y}$ and $L^*(x,y) = \frac{\partial^2 k^*(x,y)}{\partial x \partial y}$ do not exist. In this case, we approximate the integral that appeared in equations (3.5)-(3.6) with the repeated Simpson's 1/3 formula which will yield the following system of equations:

$$u'_0 = f'_0 + \frac{h}{3}H_{0,0}u_0 + \frac{4h}{3}\sum_{j=1}^{n}H_{0,2j-1}u_{2j-1} + \frac{2h}{3}\sum_{j=1}^{n-1}H_{0,2j}u_{2j} + \frac{h}{3}H_{0,2n}u_{2n},$$

$$u'_1 = f'_1 + \left(\frac{h}{3}H_{1,0} + \frac{h}{2}H^*_{1,0}\right)u_0 + \left(\frac{4h}{3}H_{1,1} + k^*_{1,1} + \frac{h}{2}H^*_{1,1}\right)u_1 + \frac{2h}{3}\sum_{j=1}^{n-1} H_{1,2j}u_{2j} + \frac{4h}{3}\sum_{j=2}^{n} H_{1,2j-1}u_{2j-1} + \frac{h}{3}H_{1,2n}u_n,$$

$$u'_{2i} = f'_{2i} + \frac{h}{3}\left(H_{2i,0} + H^*_{2i,0}\right)u_0 + \frac{4h}{3}\sum_{j=1}^{i}\left(H_{2i,2j-1} + H^*_{2i,2j-1}\right)u_{2j-1} + \frac{2h}{3}\sum_{j=1}^{i-1}\left(H_{2i,2j} + H^*_{2i,2j}\right)u_{2j} + \left(\frac{h}{3}\left(4H_{2i,2i} + H^*_{2i,2i}\right) + k^*_{2i,2i}\right)u_{2i} + \frac{4h}{3}\sum_{j=i+1}^{n} H_{2i,2j-1}u_{2j-1} + \frac{2h}{3}\sum_{j=i+1}^{n-1} H_{2i,2j}u_{2j} + \frac{h}{3}H_{2i,2n}u_{2n},\ (i = 1, 2, \ldots, n-1),$$

$$u'_{2i+1} = f'_{2i+1} + \frac{h}{3}\left(H_{2i+1,0} + H^*_{2i+1,0}\right)u_0 + \frac{4h}{3}\sum_{j=1}^{i}\left(H_{2i+1,2j-1} + H^*_{2i+1,2j-1}\right)u_{2j-1} + \frac{2h}{3}\sum_{j=1}^{i-1}\left(H_{2i+1,2j} + H^*_{2i+1,2j}\right)u_{2j} + \frac{h}{3}\left(2H_{2i+1,2i} + \frac{5}{2}H^*_{2i+1,2i}\right)u_{2i} + \left(\frac{h}{3}\left(4H_{2i+1,2i+1} + \frac{3}{2}H^*_{2i+1,2i+1}\right) + k^*_{2i+1,2i+1}\right)u_{2i+1} + \frac{4h}{3}\sum_{j=i+2}^{n} H_{2i+1,2j-1}u_{2j-1} + \frac{2h}{3}\sum_{j=i+1}^{n-1} H_{2i+1,2j}u_{2j} + \frac{h}{3}H_{2i+1,2n}u_{2n},\ i = 1, 2, \ldots, n-1,$$

and

$$u'_{2n} = f'_{2n} + \frac{h}{3}\left(H_{2n,0} + H^*_{2n,0}\right)u_0 + \frac{4h}{3}\sum_{j=1}^{n}\left(H_{2n,2j-1} + H^*_{2n,2j-1}\right)u_{2j-1} + \frac{14h}{15}\sum_{j=1}^{n-1}\left(H_{2n,2j} + H^*_{2n,2j}\right)u_{2j} + \left(\frac{h}{3}\left(H_{2n,2n} + H^*_{2n,2n}\right) + k^*_{2n,2n}\right)u_{2n}. \quad \ldots\ldots(3.10)$$

The system consists of equation (3.4) and equation (3.10) can be solved to find the unknowns $\{u_i\}_{i=0}^{2n}$ by any suitable method.

4-Numerical Examples

In this section, we give the following examples to clarify the accuracy of the presented methods and compare these methods with other methods such as the repeated trapezoid, and the repeated Simpson methods. The results in Tables 1 -2 show the absolute errors $|u(x_i) - u_i|$, $i = 0, 1, \ldots, n$. where $u(x_i)$ is the exact solution evaluated at $x=x_i$ and u_i is the corresponding approximated solution. All the results are computed by using some programs written in the Matlab Package.

Example 1:

Consider the linear Fredholm-Volterra integral equation

$$u(x) = 2\cos x - x\cos 2 - 2x\sin 2 + x - 1 + \int_0^2 xyu(y)dy + \int_0^x (x-y)u(y)dy,\ 0 \le x \le 2$$

with the exact solution $u(x) = \cos(x)$.

Approximated solutions of this example are obtained by using the repeated trapezoid method(TM) and the repeated Simpson's 1/3 method (SM) in [10].

Comparison between the approximated solutions of this equation obtained by using the repeated trapezoid method (TM), the repeated Simpson's 1/3 method (SM), the repeated corrected trapezoid method (CTM) and the repeated corrected Simpson's 1/3 method (CSM) for h =0.2, 0.1 are presented in Table (1).

Table (1)
The Absolute Errors at some mesh points of Example 1.

x	h=0.2				h=0.1			
	TM	SM	CTM	CSM	TM	SM	CTM	CSM
0	0	0	0	0	0	0	0	0
0.2	3.36697×10^{-4}	8.35541×10^{-5}	1.19336×10^{-7}	3.22717×10^{-8}	8.52287×10^{-5}	2.03106×10^{-5}	7.63262×10^{-9}	5.47517×10^{-9}
0.4	$8.17363.\times10^{-4}$	3.06842×10^{-4}	5.05230×10^{-7}	3.35273×10^{-7}	2.06533×10^{-4}	4.09493×10^{-5}	3.19126×10^{-8}	1.10288×10^{-8}
0.6	1.45337×10^{-3}	1.08547×10^{-4}	1.15741×10^{-6}	3.00188×10^{-7}	3.66811×10^{-4}	6.18788×10^{-5}	7.28305×10^{-8}	1.66532×10^{-8}
0.8	2.25741×10^{-3}	6.32625×10^{-4}	2.07648×10^{-6}	6.89157×10^{-7}	5.69301×10^{-4}	8.31273×10^{-5}	1.30432×10^{-7}	2.23581×10^{-8}
1	3.24451×10^{-3}	1.91537×10^{-4}	3.26497×10^{-6}	4.86213×10^{-7}	8.17835×10^{-4}	1.04822×10^{-4}	2.04886×10^{-7}	2.81802×10^{-8}
1.2	4.43334×10^{-3}	9.79100×10^{-4}	4.72880×10^{-6}	1.06475×10^{-6}	1.11717×10^{-3}	1.27223×10^{-4}	2.96570×10^{-7}	3.41918×10^{-8}
1.4	5.84776×10^{-3}	9.06532×10^{-5}	6.47921×10^{-6}	4.28249×10^{-7}	1.47338×10^{-3}	1.50760×10^{-4}	4.06199×10^{-7}	4.05103×10^{-8}
1.6	7.51871×10^{-3}	1.36428×10^{-3}	8.53504×10^{-6}	1.48290×10^{-6}	1.89433×10^{-3}	1.76063×10^{-4}	5.34963×10^{-7}	4.73082×10^{-8}
1.8	9.48653×10^{-3}	2.64724×10^{-4}	1.09256×10^{-5}	4.01451×10^{-8}	2.39027×10^{-3}	2.04007×10^{-4}	6.84711×10^{-7}	5.48226×10^{-8}
2	1.18036×10^{-2}	1.83138×10^{-3}	1.36940×10^{-5}	1.99187×10^{-6}	2.97449×10^{-3}	2.35749×10^{-4}	8.58151×10^{-7}	6.33666×10^{-8}

<u>Example 2:</u>

As a second example, we consider the following integral equation

$$u(x) = x^4 + 4x^3 + \frac{53}{10}x^2 + \frac{23}{4}x - 7xe^x + 1 + \int_0^1 (x^2y + x)u(y)dy + \int_0^x xe^{(x-y)}u(y)dy,\ \ 0 \le x \le 1$$

with the exact solution is $u(x) = x^3+1$.

Table(2) indicates the absolute errors to this equation by using the repeated trapezoid, repeated Simpson's 1/3, repeated corrected trapezoid and the repeated corrected Simpson's 1/3 methods for $h = 0.1, 0.05$.

From Tables (1) and (2) it can be shown that the corrected quadrature methods given more accurate results than quadrature methods, Also, the corrected trapezoidal method is more accurate than Simpson's 1/3 method.

Table (2)
The Absolute Errors at some mesh points of Example 2.

x	h=0.1				h=0.05			
	TM	SM	CTM	CSM	TM	SM	CTM	CSM
0	0	0	0	0	0	0	0	0
0.1	1.72087×10^{-2}	5.58420×10^{-4}	1.64349×10^{-6}	2.18185×10^{-7}	5.63706×10^{-3}	6.99002×10^{-5}	1.02063×10^{-7}	6.41099×10^{-9}
0.2	3.72816×10^{-2}	1.23575×10^{-3}	3.55711×10^{-6}	5.39719×10^{-7}	1.22136×10^{-2}	1.51592×10^{-4}	2.20900×10^{-7}	1.38917×10^{-8}
0.3	6.09835×10^{-2}	1.96677×10^{-3}	5.81056×10^{-6}	8.18560×10^{-7}	1.99812×10^{-2}	2.48181×10^{-4}	3.60841×10^{-7}	2.27221×10^{-8}
0.4	8.93642×10^{-2}	2.96650×10^{-3}	8.50088×10^{-6}	1.29330×10^{-6}	2.92847×10^{-2}	3.63907×10^{-4}	5.27910×10^{-7}	3.32909×10^{-8}
0.5	1.23860×10^{-1}	4.00087×10^{-3}	1.17619×10^{-5}	1.72618×10^{-6}	4.05951×10^{-2}	5.04561×10^{-4}	7.30418×10^{-7}	4.61341×10^{-8}
0.6	1.66438×10^{-1}	5.52743×10^{-3}	1.57778×10^{-5}	2.40800×10^{-6}	5.45574×10^{-2}	6.78078×10^{-4}	9.79802×10^{-7}	6.19873×10^{-8}
0.7	2.19798×10^{-1}	7.13906×10^{-3}	2.08021×10^{-5}	3.13592×10^{-6}	7.20561×10^{-2}	8.95363×10^{-4}	1.29181×10^{-6}	8.18615×10^{-8}
0.8	2.87664×10^{-1}	9.54889×10^{-3}	2.71847×10^{-5}	4.16196×10^{-6}	9.43101×10^{-2}	1.17146×10^{-3}	1.68817×10^{-6}	1.07150×10^{-7}
0.9	3.75192×10^{-1}	1.22664×10^{-2}	3.54108×10^{-5}	5.42397×10^{-6}	1.23007×10^{-1}	1.52722×10^{-3}	2.19900×10^{-6}	1.39782×10^{-7}
1	4.89565×10^{-1}	1.62343×10^{-2}	4.61559×10^{-5}	7.08502×10^{-6}	1.60495×10^{-1}	1.99170×10^{-3}	2.86627×10^{-6}	1.82442×10^{-7}

5-Conclusions and Recommendations

We introduced simple two methods with high accuracy for solving second kind linear Fredhom-Volterra integral equations. So it may be easily applied by researchers and engineers. In these methods we note that, in case one we obtain system solve eq(1.1) more accurately than systems that we obtain in other cases . Because in case one we use repeated corrected quadrature formulas to approximate each integral in eq(2.4) instead of repeated quadrature formulas. These methods will be developed by authors for solving system of Fredhom - Volterra integral and integro - differential equations.

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الخلاصة

في هذه البحث نقدم طرق عددية لحل معادلات فريدهول-فولتيرا التكاملية الخطية من النوع الثاني. أن الفكرة الرئيسية مستندة على صيغ التربيعات المصححة لشبة المنحرف وسمبسون 3/1. هذه التقنيات فعالة جداً. النتائج العددية موضحة بامثلة مختلفة.