SYMMETRY OF THE BALLISTIC EQUATIONS

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Abstract

In this paper, we characterize the possible symmetry groups of the ballistic equations, in time variable and two spatial variables.

Lie's method of continuous transformation groups is applied and it is shown that, three-parameter Lie group of transformation acting on (t, x, y)-space is admitted by the equations.

Introduction

The use of continuous transformation groups in the solution of differential equations was developed by Lie and his followers in the nineteenth century.

We will begin by reviewing a few relevant points from Lie's theory of symmetry groups of differential equations, as presented, for instance, in [1,2,3].

Consider a general system of the nth-order DEs:

$$\Delta_\upsilon(x, u^{(n)}) = 0 \qquad \upsilon = 1,\dots,m \quad ....(1)$$

In p independent variables $x = (x^1,\dots\dots,x^p)$ and q dependent variables $u = (u^1,\dots\dots,u^q)$, with $u^{(n)}$ denoting the derivatives of the u's with respect to the x's up to order n. In general, a symmetry group of a differential equation is a group which maps any solution of the differential equation to another solution of the differential equation. We consider here that symmetries defined by infinitesimal transformation, whose infinitesimals depend on independent variables, dependent variables and derivatives of dependent variables. Such symmetries are local symmetries since at any point x the infinitesimals are determined if u(x) is sufficiently smooth in some neighborhood of x.

Definition 1 [4]:

A local Lie group of transformations G is called a symmetry group of the system of DEs (1) if $\overline{f}$ = gof, where $g \in G$, is a solution whenever f is.

It is always assumed that the transformation group G is connected. Connectivity implies that it suffices to work with the associated infinitesimal generators, which form a Lie algebra of vector field,

$$v = \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \quad ....(2)$$

on the space of independent and dependent variables.

Since the transformations in G act on functions u = f(x), they also act on their derivatives, and so induce "prolonged transformations".

$$(\overline{x},\overline{u}^{(n)}) = pr^{(n)}g(x,u^{(n)})$$

The explicit formula for the prolonged group transformation is rather complicated, and so it is easier to work with the prolonged infinitesimal generators, which are vector fields,

$$pr^{(n)}v = \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\sum_{J\le n} \varphi_J^\alpha(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha} \quad ....(3)$$

On the space of independent and dependent variables and their derivatives up to order n, which are denoted by :

$u_j^\alpha = \partial^j u^\alpha / \partial x^J$, where $J=(j_1,\dots,j_n)$, $1 \le j\upsilon \le p$.

The coefficients ϕ_J^α of $pr^{(n)}v$ can be derived in terms of the coefficients ζ^i and φ^α of the original vector field (2).

Theorem 2 [4]:

A connected group of transformations G is a symmetry group of the system of DEs (1) if and only if the classical infinitesimal symmetry criterion

$$pr^{(n)}v(\Delta_\upsilon)=0, \quad \upsilon = 1,\dots,r \quad \text{whenever} \quad \Delta=0 \quad ....(4)$$

holds for every infinitesimal generator v of G.

The equations (4) are known as the determining equations of the symmetry group for the system. They form a large over determined system of partial differential equations for the coefficients ζ^i and φ^α of v, and can, in practice, be explicitly solved to determine the complete connected symmetry group of the system (1).

Basic Equations of the Problem

In this paper, we apply symmetry method to the exterior ballistics equations,

$$\frac{d^2x}{dt^2}=\alpha v\frac{dx}{dt} \quad \text{.......................................(5)}$$

$$\frac{d^2y}{dt^2}=\alpha v\frac{dy}{dt}-g \quad \text{.................................(6)}$$

Where $v=\left(\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right)^{1/2}$, α and g are non zero constants.

These equations deal with the motion of a projectile of weight w be fired with an initial v_0 at an angle of elevation ϕ; let v denote the velocity of the projectile at any point in its path and let θ denote the inclination of the velocity vector at that point and finally let ρ denote the radius of curvature of the trajectory (path) at the point in question and let kv^n denote the air resistance at that point. Then taking time as the independent variable and resolving forces in the horizontal and vertical directions at p give [5] :

$$-k\,v^n\cos\theta=\frac{w}{g}\frac{d^2x}{dt^2}$$

$$-k\,v^n\sin\theta-w=\frac{w}{g}\frac{d^2y}{dt^2}$$

or

$$\frac{d^2x}{dt^2}=-\frac{kg}{w}v^n\cos\theta=-R\cos\theta$$
$$=-\frac{R}{v}v\cos\theta=-\frac{R}{v}\frac{dx}{dt}$$

$$\frac{d^2y}{dt^2}=-\frac{kg}{w}v^n\sin\theta-g=-\frac{R}{v}v\sin\theta$$
$$=-\frac{R}{v}v\cos\theta=-\frac{R}{v}\frac{dy}{dt}-g$$

where $R=\frac{kg}{w}v^n$, k and g are constants, $n\geq 1$.

Hence, the fundamental DEs in rectangular form are :

$$\frac{d^2x}{dt^2}=-\frac{R}{v}\frac{dx}{dt} \quad \text{.......................................(7)}$$

$$\frac{d^2y}{dt^2}=-\frac{R}{v}\frac{dy}{dt}-g \quad \text{.................................(8)}$$

These equations are connected by the velocity $v=\sqrt{\dot{x}^2+\dot{y}^2}$ and they are often integrated simultaneously using numerical methods. In this research, we use infinitesimal transformation to calculate the symmetry group of the system (5), (6).

Infinitesimal Transformations

We now consider a one-parameter Lie's group of infinitesimal transformation :

$$\left.\begin{aligned}
t^* &= t+\varepsilon T(t;x,y)+O(\varepsilon^2)\\
x^* &= x+\varepsilon X(t;x,y)+O(\varepsilon^2)\\
y^* &= y+\varepsilon Y(t;x,y)+O(\varepsilon^2)\\
&\text{and the extended transformation,}\\
p^* &= p+\varepsilon P(t;x,y;p,q)+O(\varepsilon^2)\\
q^* &= q+\varepsilon Q(t;x,y;p,q)+O(\varepsilon^2)\\
r^* &= r+\varepsilon R(t;x,y;p,q;r,s)+O(\varepsilon^2)\\
s^* &= s+\varepsilon S(t;x,y;p,q;r,s)+O(\varepsilon^2)
\end{aligned}\right\} \text{.......(9)}$$

where

$$\left.\begin{aligned}
&p=\frac{dx}{dt},\ q=\frac{dy}{dt},\ r=\frac{d^2x}{dt^2},\ s=\frac{d^2y}{dt^2}\\
&p^*=\frac{dx^*}{dt^*},q^*=\frac{dy^*}{dt^*},r^*=\frac{d^2x^*}{dt^{*2}},s^*=\frac{d^2y^*}{dt^{*2}}
\end{aligned}\right\}$$
$$\text{..............................(10)}$$

Following the references [1,2,3], we derive P,Q,R and S as follows:

$$\left.\begin{aligned}
P&=X_t+(X_x-T_t)p-T_xp^2+q(X_y-pT_y)\\
Q&=Y_t+(Y_y-T_t)q-T_yq^2+p(Y_x-qT_x)
\end{aligned}\right\}$$
$$\text{..............................(11)}$$

$$\left.\begin{array}{l} R=(X_{tt}-gX_y)+p(2X_{tx}-T_{tt}+gT_y) \\ +2qX_{ty}+p^2(X_{xx}-2T_{tx})+q^2X_{yy} \\ +2pq(X_{xy}-T_{ty})-p^3T_{xx}-pq^2T_{yy}-2p^2qT_{xy} \\ +\alpha\sqrt{p^2+q^2}\left\{p(X_x-2T_t)+qX_y-3p^2T_x-3pqT_y\right\} \\ \text{and} \\ S=(Y_{tt}-gY_y)+2p(T_x+Y_{xt})+q(2Y_{ty}-T_{tt}+3gT_y) \\ +p^2Y_{xx}+q^2(Y_{yy}-T_{ty})+2pq(Y_{xy}-T_{xt})-q^3T_{yy} \\ -2pq^2T_{xy}-p^2qT_{xx} \\ +\alpha\sqrt{p^2+q^2}\left\{\begin{array}{l} pY_x+q(Y_y-Y_t-2T_t) \\ -3q^2T_y-3pqT_x \end{array}\right\} \end{array}\right\} \quad \ldots\ldots (12)$$

Assuming that equations (5) and (6) are invariant under the transformation (9) we have,

$$r^*=\alpha\sqrt{p^{*2}+q^{*2}}\,p^* \quad \ldots\ldots (13)$$

$$s^*=\alpha\sqrt{p^{*2}+q^{*2}}\,q^*-g \quad \ldots\ldots (14)$$

It is clear that, if P = Q = R = S = 0 then equations (5) and (6) are invariant, however, substituting equations (9) in (13), (14), using equations (5) and (6) and considering the coefficient of the first order of ε give,

$$R=\alpha\left\{\sqrt{p^2+q^2}\,P+\frac{(pP+qQ)p}{\sqrt{p^2+q^2}}\right\} \quad \ldots\ldots (15)$$

$$S=\alpha\left\{\sqrt{p^2+q^2}\,Q+\frac{(pP+qQ)q}{\sqrt{p^2+q^2}}\right\} \quad \ldots\ldots (16)$$

Now, simplification of (15), (16) and equating coefficients of the monomials $p^n q^m$ give, the infinitesimal elements (T, X, Y) leaving invariant equations (5) and (6).

The Determining Equations

In this section, we find the determining equations for T, X and Y as follows :
Simplifying equation (15) gives :

$$(p^2+q^2)\left\{\begin{array}{l} (X_{tt}-gX_y)+p(2X_{tx}-T_{tt}+gT_y) \\ -2qX_{ty}+p^2(X_{xx}-2T_{tx}) \\ +q^2X_{yy}+2pq(X_{xy}-T_{ty})-p^3T_{xx} \\ -pq^2T_{yy}-2p^2qT_{xy} \end{array}\right\}^2$$

$$=\alpha^2(X_t+pT_t+2p^2T_x+2pqT_y)^2\,(p^4-2p^2q^2+q^4)$$

$$+2\alpha^2\,p(p^2+q^2)\left\{\begin{array}{l} pX_t+qY_t+p^2(X_x-T_t) \\ +q^2(Y_y-T_t)-p^3T_x \\ -q^3T_y+pq(X_y+Y_x) \\ -p^2qT_y-pq^2T_x \end{array}\right\}$$

$$(X_t+pT_t+2p^2T_x+2pqT_y)$$

$$+\alpha^2p^2\left\{\begin{array}{l} pX_t+qY_t+p^2(X_x-T_t) \\ +q^2(Y_y-T_t)-p^3T_x-q^3T_y \\ +pq(X_y+Y_x)-p^2qT_x \end{array}\right\}^2 \quad \ldots\ldots (17)$$

First, we find a primary set of determining equations, and when solved, simplify the calculation of the remaining equations :

Monomails	coefficients
q^6	$X_{yy}=0$ (a)
q^4	$2(X_{tt}-gX_y)X_{yy}+4X_{ty}^2=\alpha^2X_t^2$ (b)
q^2	$(X_{tt}-gX_y)^2=0$ (c)
p^2q^6	$\alpha^2T_y^2=0$ (d)
p^4q^4	$4\alpha^2(T_x^2+T_y^2)=0$ (e)
p^2q^2	$4X_{yt}^2=\alpha^2Y_t^2$ (f)

First (d) and (e) require that T is just a function of t. Then (a) requires that X is at most linear in y, i.e.,

X = A(t,x)y + B(t,x)

where A and B are arbitrary functions. Then (b) and (c) show that $A_t = B_t = 0$. Hence, X is a function of x only. Next by (f) Y doesn't depend on t. For the sake of simplicity and to calculate the determining equations, we rewrite equation (17), using the results obtained above, as follows :

$$(p^2+q^2)(-pT_{tt}+p^2X_{xx})^2$$
$$=(\alpha pT_t)^2(p^4+2p^2q^2+q^4)$$
$$+2\alpha^2p(p^2+q^2)[p^2(X_x\text{-}T_t)$$
$$+q^2(Y_y-T_t)+pqY_x](pT_t)$$
$$+\alpha^2p^2[p^2(X_x\text{-}T_t)+q^2(Y_y-T_t)+pqY_x]^2 \quad \text{..............................(18)}$$

Hence, the second set of the determining equations for the symmetry group of the equations (5), (6) will be as in the following :

Mono mails	coefficients
p^4	$T_{tt}^2=0$ (g)
p^5	$-2T_{tt}X_{xx}=0$ (h)
p^6	$X_{xx}^2=\alpha^2T_t^2+2\alpha^2(X_x-T_t)T_t$ $+\alpha^2(X_x-T_t)^2=0$ (i)
p^2q^2	$T_{tt}^2=0$ (j)
p^3q^2	$-2T_{tt}X_{xx}=0$ (k)
p^4q^2	$X_{xx}^2=2\alpha^2T_t^2+2\alpha^2(Y_y-T_t)T_t$ $+2\alpha^2(X_x-T_t)T_t$ $+2\alpha^2(X_x-T_t)(Y_y-T_t)$ $+\alpha^2Y_x^2$ (l)
p^2q4	$0=\alpha^2T_t^2+2\alpha^2(Y_y-T_t)T_t$ $+\alpha^2(Y_y-T_t)^2$ (m)
p^5q	$0=2\alpha^2Y_xT_t+2\alpha^2(X_x-T_t)Y_x$ (n)
p^3q^3	$0=2\alpha^2Y_xT_t+2\alpha^2(Y_y-T_t)Y_x$ (o)

Simplifying (m) gives that Y doesn't depend on y.

Then (g) requires that T is at most linear in t.

Now, using the available information we simplify equation (16) to :

$$(p^2+q^2)(2gT_t+p^2Y_{xx})^2=\alpha^2(p^4+2p^2q^2+q^4)(qT_t)^2-4\alpha^2q^2(p^2+q^2)T_t[(X_x-T_t)p^2-T_tq^2+pqY_x]+\alpha^2q^2[(X_x-T_t)p^2-T_tq^2+pqY_x]^2 \quad \text{..............................(19)}$$

This gives the final set of the determining equations :

Monomails	coefficients
p^2	$(2gT_t)^2=0$ (p)
p^4	$4gT_tY_{xx}=0$ (q)
p^6	$Y_{xx}^2=0$ (r)
p^2	$(2gT_t)^2=0$ (s)
p^2q^2	$4gT_tY_{xx}=0$ (t)
p^4q^2	$Y_{xx}^2=\alpha^2T_t^2-4\alpha^2T_t(X_x-T_t)$ $+\alpha^2(X_x-T_t)^2$ (u)
p^2q^4	$0=2\alpha^2T_t^2+4\alpha^2T_t^2$ $-4\alpha^2T_t(X_x-T_t)+\alpha^2Y_x^2$ $-2\alpha^2(X_x-T_t)T_t$ (v)
q^6	$0=\alpha^2T_t^2+4\alpha^2T_t^2+\alpha^2T_t^2$ (w)
p^3q^3	$0=-4\alpha^2T_tY_x+2\alpha^2(X_x-T_t)Y_x$ (x)
pq^5	$0=-4\alpha^2T_tY_x+2\alpha^2T_tY_x$ (y)

First, (o) requires that $T = C_1$, then (u) requires that $X = C_2$. Next, (v) gives that $Y = C_3$, where C_i (i = 1,2,3) are constants.
Hence, theorem (2) guarantees that these are the only continuous classical symmetries of the equation.

Conclusion

In this paper, possible invariant solutions of the ballistic equations

$$\ddot{x} = -E\dot{x}$$

$$\ddot{y} = -E\dot{y} - g$$

Where $-E = \alpha v$, $v=\sqrt{\dot{x}^2+\dot{y}^2}$ are studied by means of infinitesimal transformation of Lie theory. The symmetry algebra of the equations is spanned by the three vector fields,

$v_1 = \partial t$ time translation

$\left.\begin{array}{l} v_2 = \partial x \\ v_3 = \partial y \end{array}\right\}$ space translation

Exponentiation shows that if $X = \begin{pmatrix} x \\ y \end{pmatrix} = F(t)$ is a solution of the equations so are

$$X^1 = F(t-\varepsilon)$$

$$X^2 = F(t) + \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

$$X^3 = F(t) + \begin{pmatrix} 0 \\ \varepsilon \end{pmatrix}$$

This list of symmetries may seem small. However it can be enlarged using Nucci (or Krause) technique [6].

Once one has determined the symmetry group of a system of differential equations, a number of applications becomes available, for example, one can directly use the defining property of such a group and construct new solutions to the system from known ones.

Finally, several cases of the ballistic equations above where E = G(v)H(y) are worthy of future investigation.

References

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الخلاصة

في هذا البحث قمنا بدراسة زمر التماثل الممكن للمعادلات البالستية ذات المتغير الواحد للزمن والمتغيرين للفضاء.

لقد طبقت طريقة لي لزمر التحويلات المستمرة فوجد ان زمرة لي بثلاث معلمات تعمل على الفضاء (t;x,y) هي الزمرة الممكنة للمعادلات.