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CONTRACTION MAPPING THEOREM OF THE SPACE OF FUZZY NUMBERS

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Abstract

In this paper we present fuzzy number and some of its properties it .Also, we define a new fuzzy metric $D(\tilde{a}, \tilde{b})$ of fuzzy numbers and discuss some of its algebraic operation of it . We prove some results to study the contraction mapping theorem of fuzzy numbers using modified approach depending on the definition of function distance between fuzzy numbers.

Introduction

Dubois D. and Prade H. in 1978 introduced the notions of fuzzy numbers and define its basic properties and algebraic operations [1].The researchers Goetschel R., Kaufmann A., Gupta M. and Zhang G. [2-4] have done much work about fuzzy numbers.

Let $\mathbb{R}$ be the set of all real numbers and $F^*(\mathbb{R})$ be the set of all fuzzy subsets defined on $\mathbb{R}$ .G. Zhang [4-6] defined the fuzzy number $\tilde{a} \in F^*(\mathbb{R})$, as follows:

(1) $\tilde{a}$ is normal, i.e., there exists a unique $x \in \mathbb{R}$, such that $\tilde{a}(x) = 1$.

(2) For every $\lambda \in (0, 1)$, $\tilde{a}_\lambda = \{x \mid \tilde{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Now, let us denote the set of all fuzzy numbers defined by G. Zhang as $\overline{F}(\mathbb{R})$.

In this paper, we will use the fuzzy distance $D : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}_+ \cup \{0\}$ between two fuzzy numbers [7], as follows:

$$D(\tilde{a}, \tilde{b}) = \sup_{\lambda \in [0,1]} \max\{|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - a_\lambda^+|\}$$

for all $\tilde{a}, \tilde{b} \in \overline{F}(\mathbb{R})$, where $\tilde{a}_\lambda = [a_\lambda^-, a_\lambda^+]$, $\tilde{b}_\lambda = [b_\lambda^-, b_\lambda^+]$, we have that D is a metric on $\overline{F}(\mathbb{R})$, and the following definition may be considered:

Definition (1.1):

A fuzzy distance D between two fuzzy numbers $\tilde{a}, \tilde{b} \in \overline{F}(\mathbb{R})$ is a function $D : \mathbb{R}_F \times \mathbb{R}_F \longrightarrow \mathbb{R}_+ \cup \{0\}$, and satisfies:

1. $D(\tilde{a}, \tilde{b}) \geq 0$ and $D(\tilde{a}, \tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$.
2. $D(\tilde{a}, \tilde{b}) = D(\tilde{b}, \tilde{a})$.
3. When for every $\tilde{c} \in \overline{F}(\mathbb{R})$, then $D(\tilde{a}, \tilde{b}) \leq D(\tilde{a}, \tilde{c}) + D(\tilde{c}, \tilde{b})$.

If D is a fuzzy distance of fuzzy numbers, we call $(\overline{F}(\mathbb{R}), D)$ a fuzzy metric space ,and we defined :

$$D(\tilde{a}, \tilde{b}) = \sup_{\lambda \in [0,1]} \max\{|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - a_\lambda^+|\}$$

for every $\tilde{a}, \tilde{b} \in \overline{F}(\mathbb{R})$.

Remark (1.1):

$D(\tilde{a}, \tilde{b})$ represents the distance between two fuzzy sets, which may be denoted also by:

$$D(\tilde{a}, \tilde{b}) = \sup_{\lambda \in [0,1]} \max\{|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - b_\lambda^+|\}$$
$$= \sup_{\lambda \in [0,1]} \max\{ \sup_{x^- \in a_\lambda^-} \inf_{y^- \in b_\lambda^-} |x^- - y^-|, \sup_{x^+ \in a_\lambda^+} \inf_{y^+ \in b_\lambda^+} |x^+ - y^+|\}$$

Definitions (1.2), [9]:

Let X be a universal set, then a fuzzy set $\tilde{A}$ of X is defined by its membership function $\mu_{\tilde{A}} : X \longrightarrow [0, 1]$. We can also write the fuzzy set $\tilde{A}$ as $\{(x, \mu_{\tilde{A}}(x)) : x \in X\}$ and denote $A_\lambda = \{x \in X : \mu_{\tilde{A}}(x) \geq \lambda\}$ as the λ-level set of $\tilde{A}$.

Definition (1.3),[9]:

Let f(x) be real valued function on a topological space (X,T). If $\{x:f(x) \geq \lambda\}$ is closed for every λ, f(x) is said to be upper semicontinous .

Definition (1.4), [9]:

$\tilde{a}$ is colled closed fuzzy number if $\tilde{a}$ is a fuzzy number and its membership function $\mu_{\tilde{a}}$ is upper semicontinous .

Definition(1,5), [9]:

Let $\tilde{a}$ be a fuzzy number.

1. $\tilde{a}$ is called nonnegative fuzzy number if $\mu_{\tilde{a}}(x) = 0$, $\forall\, x < 0$.
2. $\tilde{a}$ is called nonpositive fuzzy number if $\mu_{\tilde{a}}(x) = 0$, $\forall\, x > 0$.
3. $\tilde{a}$ is called positive fuzzy number if $\mu_{\tilde{a}}(x) = 0$, $\forall\, x \le 0$.
4. $\tilde{a}$ is called negative fuzzy number if $\mu_{\tilde{a}}(x) = 0$, $\forall\, x \ge 0$.

Remark (1.2), [8]:

Let $\tilde{a}$, $\tilde{b}$, $\tilde{c}$ $\in \overline{F}(\mathbb{R})$, then the following algebraic definitions and properties may be considered:

1. $\tilde{c} = \tilde{a} + \tilde{b}$, if $c^-_\lambda = a^-_\lambda + b^-_\lambda$ and $c^+_\lambda = a^+_\lambda + b^+_\lambda$, for every $\lambda \in (0, 1]$.
2. $\tilde{c} = \tilde{a} - \tilde{b}$, if $c^-_\lambda = a^-_\lambda - b^-_\lambda$ and $c^+_\lambda = a^+_\lambda - b^+_\lambda$, for every $\lambda \in (0, 1]$.
3. For every $k \in \mathbb{R}$ and $\tilde{a} \in \overline{F}(\mathbb{R})$, then:

$$k\tilde{a} = \begin{cases} \bigcup\limits_{\lambda\in[0,1]} \lambda[ka^-_\lambda, ka^+_\lambda], & \text{if } k \ge 0 \\ \bigcup\limits_{\lambda\in[0,1]} \lambda[ka^+_\lambda, ka^-_\lambda], & \text{if } k < 0 \end{cases}$$

4. If $\tilde{a}$ and $\tilde{b}$ are two closed fuzzy number then

$$(\tilde{a} \otimes \tilde{b})_\lambda = [\min\{a^-_\lambda b^-_\lambda, a^+_\lambda b^-_\lambda, a^-_\lambda b^+_\lambda\ a^+_\lambda b^+_\lambda\}, \max\{a^-_\lambda b^-_\lambda, a^+_\lambda b^-_\lambda,\ a^-_\lambda b^+_\lambda\ a^+_\lambda b^+_\lambda\}]$$

5. If $\tilde{a}$ and $\tilde{b}$ are two nonnegative closed fuzzy number then

$$(\tilde{a} \otimes \tilde{b})_\lambda = [a^-_\lambda b^-_\lambda, a^+_\lambda b^+_\lambda]$$

Where $\otimes$ stands for the product.

6. $\tilde{a} \le \tilde{b}$ if $a^-_\lambda \le b^-_\lambda$ and $a^+_\lambda \le b^+_\lambda$, for every $\lambda \in [0, 1]$
7. $\tilde{a} < \tilde{b}$ if $\tilde{a} \le \tilde{b}$ and there exists $\lambda \in (0, 1]$, such that $a^-_\lambda < b^-_\lambda$ or $a^+_\lambda < b^+_\lambda$
8. $\tilde{a} = \tilde{b}$ if $\tilde{a} \le \tilde{b}$ and $\tilde{b} \le \tilde{a}$.

Definitions (1.6), [8]:

Let $A \subset \overline{F}(\mathbb{R})$ and

1. If there exists $\tilde{M} \in \overline{F}(\mathbb{R})$,such that $\tilde{a} \le \tilde{M}$, for every $\tilde{a} \in A$, then A is said to be an upper bound of $\tilde{M}$.
2. If there exists $\tilde{m} \in \overline{F}(\mathbb{R})$, such that $\tilde{m} \le \tilde{a}$, for every $\tilde{a} \in A$, then A is said to be a lower bound of $\tilde{m}$.
3. A is said to be bounded if A has both an upper and lower bounds.

Definition (1.7):

A sequence $\{\tilde{a}_n\}$ is said to be bounded if there exists positive numbers M_1 and M_2 such that

$|a^+_n| \le M_1$, $|a^-_n| \le M_2$ and is termed as

$|\tilde{a}_n| \le M$, $M=\{M_1 , M_2\}$ $\quad,\forall\, n$

Fuzzy Metric Spaces

We start first with the following fundamental theorem in fuzzy metric spaces, in which its proof is similar to that given in [11].

Theorem (2.1):

$(\overline{F}(\mathbb{R}), D)$ is a metric space.

Proof:

To prove that the following three conditions are satisfied for all $\tilde{a}, \tilde{b}$ and $\tilde{c} \in \overline{F}(\mathbb{R})$:

1. $D(\tilde{a}, \tilde{b}) = D(\tilde{b}, \tilde{a})$.
2. $D(\tilde{a}, \tilde{b}) \ge 0$ and $D(\tilde{a}, \tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$.
3. When for every $\tilde{c} \in \overline{F}(\mathbb{R})$, we have:

$$D(\tilde{a}, \tilde{b}) \le D(\tilde{a}, \tilde{c}) + D(\tilde{c}, \tilde{b}).$$

For the first condition:

$$D(\tilde{a}, \tilde{b}) = \sup_{\lambda\in[0,1]} \max\{|a^-_\lambda - b^-_\lambda|, |a^+_\lambda - b^+_\lambda|\}$$

$$= \sup_{\lambda\in[0,1]} \max\{|b^-_\lambda - a^-_\lambda|, |b^+_\lambda - a^+_\lambda|\}$$

$$= D(\tilde{b}, \tilde{a})$$

For the second condition, it is clear that $D(\tilde{a},\tilde{b}) \geq 0$, since $|a_\lambda^- - b_\lambda^-| \geq 0$, and $|a_\lambda^+ - b_\lambda^+| \geq 0$, for all $\tilde{a}$, $\tilde{b} \in \overline{F}(\mathbb{R})$.

Now, to show that $D(\tilde{a},\tilde{b}) = 0$ if and only if $\tilde{a} = \tilde{b}$

$\tilde{a} = \tilde{b} \Rightarrow D(\tilde{a},\tilde{b}) = 0$

$[a_\lambda^-, a_\lambda^+] = [b_\lambda^-, b_\lambda^+] \Rightarrow a_\lambda^- = b_\lambda^-,\ a_\lambda^+ = b_\lambda^+$

$$\Rightarrow D(\tilde{a},\tilde{b}) = \sup_{\lambda\in[0,1]} \max\{|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - b_\lambda^+|\} = 0$$

Hence $D(\tilde{a},\tilde{b}) = 0 \Rightarrow \tilde{a} = \tilde{b}$

$$D(\tilde{a},\tilde{b}) = \sup_{\lambda\in[0,1]} \max\{|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - b_\lambda^+|\} = 0$$

$\Rightarrow |a_\lambda^- - b_\lambda^-| = 0,\ |a_\lambda^+ - b_\lambda^+| = 0$

$\Rightarrow a_\lambda^- = b_\lambda^-,\ a_\lambda^+ = b_\lambda^+$

$\Rightarrow [a_\lambda^-, a_\lambda^+] = [b_\lambda^-, b_\lambda^+]$

$\Rightarrow \tilde{a} = \tilde{b}$

Finally, to prove the third condition:

$D(\tilde{a},\tilde{b}) \leq D(\tilde{a},\tilde{c}) + D(\tilde{c},\tilde{b})$, $\forall\ \tilde{a}, \tilde{b}, \tilde{c} \in \overline{F}(\mathbb{R})$

$$\begin{aligned} D(\tilde{a},\tilde{b}) &= \sup_{\lambda\in[0,1]} \max\{|a_\lambda^- - b_\lambda^-|, |a_\lambda^+ - b_\lambda^+|\} \\ &= \sup_{\lambda\in[0,1]} \max\{|a_\lambda^- - c_\lambda^- + c_\lambda^- - b_\lambda^-|, |a_\lambda^+ - c_\lambda^+ + c_\lambda^+ - b_\lambda^+|\} \\ &\leq \sup_{\lambda\in[0,1]} \max\{|a_\lambda^- - c_\lambda^-| + |c_\lambda^- - b_\lambda^-|, |a_\lambda^+ - c_\lambda^+| + |c_\lambda^+ - b_\lambda^+|\} \\ &\leq \sup_{\lambda\in[0,1]} \max\{|a_\lambda^- - c_\lambda^-|, |a_\lambda^+ - c_\lambda^+|\} + \sup_{\lambda\in[0,1]} \max\{|c_\lambda^- - b_\lambda^-|, |c_\lambda^+ - b_\lambda^+|\} \\ &\leq D(\tilde{a},\tilde{c}) + D(\tilde{c},\tilde{a}) \end{aligned}$$

which completes the proof of the theorem. <

Sequences in fuzzy metric spaces has a great importance in contraction mapping theorem, and following the definition which give the seed for this aspect.

Definition (2.1), [8]:

Let $\{\tilde{a}_n\} \subset \overline{F}(\mathbb{R})$, $\tilde{a} \in \overline{F}(\mathbb{R})$, then the sequence $\{\tilde{a}_n\}$ is said to be converge to $\tilde{a}$ (and is denoted by $\lim_{n\to\infty} \tilde{a}_n = \tilde{a}$) if for any given $\varepsilon > 0$, there exists an integer $N > 0$, such that $D(\tilde{a}_n, \tilde{a}) < \varepsilon$, for all $n \geq N$.

Theorem (2.2):

Let $\{\tilde{a}_n\} \subset \overline{F}(\mathbb{R})$, $\tilde{a} \in \overline{F}(\mathbb{R})$, then the sequence $\{\tilde{a}_n\}$ is converge to a unique $\tilde{a}$, i.e., if $\lim_{n\to\infty} \tilde{a}_n = \tilde{a}$ and $\lim_{n\to\infty} \tilde{a}_n = \tilde{b}$, then $\tilde{a} = \tilde{b}$.

Proof:

Assume the contrary. Then $\tilde{a} \neq \tilde{b}$, so that $D(\tilde{a},\tilde{b}) > 0$.

Let $\frac{1}{2} D(\tilde{a},\tilde{b}) = \varepsilon$

By hypothesis $\lim_{n\to\infty} \tilde{a}_n = \tilde{a}$, then there exist an integer $N_1 > 0$, such that $D(\tilde{a}_n, \tilde{a}) < \varepsilon$, for any $n \geq N_1$.

Similarly, since $\lim_{n\to\infty} \tilde{a}_n = \tilde{b}$, then there exist an integer $N_2 > 0$ such that $D(\tilde{a}_n, \tilde{b}) < \varepsilon$, for any $n \geq N_2$

Let $N = \text{Max}\ \{N_1, N_2\}$, then from definition (1.1) and the properties of the fuzzy sets

$$\begin{aligned} D(\tilde{a},\tilde{b}) &\leq D(\tilde{a},\tilde{a}_n) + D(\tilde{a}_n, \tilde{b}) \\ &< \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

Hence $D(\tilde{a},\tilde{b}) < 2\varepsilon$, which is a contradiction

Then $\tilde{a} = \tilde{b}$. <

Theorem (2.3):

If $\lim_{n\to\infty} \tilde{a}_n = \tilde{a}$ and $\lim_{n\to\infty} \tilde{b}_n = \tilde{b}$, then $\lim_{n\to\infty} D(\tilde{a}_n, \tilde{b}_n) = D(\tilde{a},\tilde{b})$ (or $D(\tilde{a}_n, \tilde{b}_n) \longrightarrow D(\tilde{a},\tilde{b})$ as $n \longrightarrow \infty$).

Proof:

Since $\tilde{a}_n$ is convergent, then $D(\tilde{a}_n, \tilde{a}) < \varepsilon$, for all $n > k_1$

Also, since $\tilde{b}_n$ is convergent, then $D(\tilde{b}_n, \tilde{b}) < \varepsilon$, for all $n > k_2$

Let $k = \text{Max}\ \{k_1, k_2\}$, hence:

$$\begin{aligned} D(\tilde{a}_n, \tilde{b}_n) &\leq D(\tilde{a}_n, \tilde{a}) + D(\tilde{a}, \tilde{b}_n) \\ &\leq D(\tilde{a}_n, \tilde{a}) + D(\tilde{a},\tilde{b}) + D(\tilde{b}_n, \tilde{b}) \end{aligned}$$

Hence:

$$D(\tilde{a}_n, \tilde{b}_n) - D(\tilde{a},\tilde{b}) \leq D(\tilde{a}_n, \tilde{a}) + D(\tilde{b}_n, \tilde{b})$$

i.e.

$|D(\tilde{a}_n, \tilde{b}_n) - D(\tilde{a}, \tilde{b})| \le \varepsilon + \varepsilon = 2\varepsilon$
and as $\varepsilon \longrightarrow 0$, we have:
$D(\tilde{a}_n, \tilde{b}_n) \longrightarrow D(\tilde{a}, \tilde{b})$. <

Theorem (2.4) :

Every converge sequence $\{\tilde{a}_n\}$ is bounded

Proof:

Let $\{\tilde{a}_n\} \to \tilde{a}$
There exists positive number K such that
$D(\tilde{a}_n, \tilde{a}) < 1$

$D(\tilde{a}_n, \tilde{a}) = \sup_{\lambda\in[0,1]} \max\{|a^-_{n\lambda} - a^-_\lambda|, |a^+_{n\lambda} - a^+_\lambda|\} < 1$

If $\max\{|a^-_{n\lambda} - a^-_\lambda|, |a^+_{n\lambda} - a^+_\lambda|\} = |a^-_{n\lambda} - a^-_\lambda|$

Then $|a^-_{n\lambda} - a^-_\lambda| < 1 \Rightarrow |a^-_{n\lambda}| < |a^-_\lambda| + 1 \quad \forall n$

Let $M_1 = \max\{|a^-_{1\lambda}|, |a^-_{2\lambda}|, \ldots, |a^-_{n\lambda}|, |a^-_\lambda| + 1\}$

$\Rightarrow |\tilde{a}_n| \le M_1 \quad \Rightarrow \quad \{\tilde{a}_n\} < M_1$ is bounded
Similarity if
$\max\{|a^-_{n\lambda} - a^-_\lambda|, |a^+_{n\lambda} - a^+_\lambda|\} = |a^+_{n\lambda} - a^+_\lambda|$

$\{\tilde{a}_n\}$ is bounded . <

3.Some Algebraic Operations on fuzzy sequence:

In this section, some algebraic operations connecting convergent sequence are given, including multiplication of convergent sequence.

Theorem (3.1):

Let $\{\tilde{a}_n\}, \{\tilde{b}_n\} \subset \bar{F}(R)$ be a convergent sequences, i.e., $\tilde{a}_n \longrightarrow \tilde{a}$ and $\tilde{b}_n \longrightarrow \tilde{b}$, then $\{\tilde{a}_n + \tilde{b}_n\} \longrightarrow \tilde{a} + \tilde{b}$ as $n \longrightarrow \infty$.

Proof:

Since $\tilde{a}_n \longrightarrow \tilde{a}$, then there exists $N_1 > 0$, such that $D(\tilde{a}_n, \tilde{a}) < \varepsilon/2$, $\forall\, n > N_1$
Similarly, since $\tilde{b}_n \longrightarrow \tilde{b}$, then there exists $N_2 > 0$, such that $D(\tilde{b}_n, \tilde{b}) < \varepsilon/2$, $\forall\, n > N_2$
Hence:

$$D(\tilde{a}_n + \tilde{b}_n, \tilde{a} + \tilde{b}) = \sup_{\lambda\in[0,1]} \max\{|(a^-_{n\lambda} + b^-_{n\lambda}) - (\overset{\infty-}{a_\lambda} + \overset{\infty-}{b_\lambda})|, |(a^+_{n\lambda} + b^+_{n\lambda}) - (\overset{\infty+}{a_\lambda} + \overset{\infty+}{b_\lambda})|\}$$

$$= \sup_{\lambda\in[0,1]} \max\{|(a^-_{n\lambda} - \overset{\infty-}{a_\lambda}) + (b^-_{n\lambda} - \overset{\infty-}{b_\lambda})|, |(a^+_{n\lambda} - \overset{\infty+}{a_\lambda}) + (b^+_{n\lambda} - \overset{\infty+}{b_\lambda})|\}$$

$$< \sup_{\lambda\in[0,1]} \max\{\varepsilon/2 + \varepsilon/2, \varepsilon/2 + \varepsilon/2\}$$

$$< \sup_{\lambda\in[0,1]} \max\{\varepsilon, \varepsilon\} = \varepsilon$$

and as $n \longrightarrow 0$, we have $\{\tilde{a}_n + \tilde{b}_n\} \longrightarrow \tilde{a} + \tilde{b}$.

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Theorem (3.2):

If $\{\tilde{a}_n\}, \{\tilde{b}_n\}$ are two nonnegative closed fuzzy numbers and converge, then $\{\tilde{a}_n \tilde{b}_n\}$ is converge to $\tilde{a}\,\tilde{b}$ such that $\tilde{a}$ and $\tilde{b}$ are two nonnegative closed fuzzy number ,

(i,e) $\tilde{a}_n \tilde{b}_n \Rightarrow \tilde{a}\,\tilde{b}$

Proof:

Since $\{\tilde{a}_n\}$ is converge
Then there exists $M_1 > 0$ such that $|\tilde{a}_n| < M_1$ and (from theorem (2.4)) $|\tilde{a}| < M_1$.
Similarity there exists M2 > 0 such that $|\tilde{b}_n| < M_2$ and $|\tilde{b}| < M_2$.

let $M = \{M_1, M_2\}$

also there exists $K_1, K_2 \in N$ such that

$D(\tilde{a}_n, \tilde{a}) < \dfrac{\in}{2M} \quad \forall\ n > K_1$

$D(\tilde{b}_n, \tilde{b}) < \dfrac{\in}{2M} \quad \forall\ n > K_2$

Let $K = \{K_1, K_2\}$

Now $\forall\ n > K$

$$D(\tilde{a}_n \tilde{b}_n, \tilde{a}\,\tilde{b}) = \sup_{\lambda\in[0,1]} \max \{|a^-_{n\lambda} b^-_{n\lambda} - a^-_\lambda b^-_\lambda|, |a^+_{n\lambda} b^+_{n\lambda} - a^+_\lambda b^+_\lambda|\}.$$

$$= \sup_{\lambda\in[0,1]} \max\{|a^-_{n\lambda} b^-_{n\lambda} - a^-_{\lambda} b^-_{n\lambda} + a^-_{\lambda} b^-_{n\lambda} - a^-_{\lambda} b^-_{\lambda}|, |a^+_{n\lambda} b^+_{n\lambda} - a^+_{n\lambda} b^+_{\lambda} + a^+_{n\lambda} b^+_{\lambda} - a^+_{\lambda} b^+_{\lambda}|\}.$$

$$= \sup_{\lambda\in[0,1]} \max\{|b^-_{n\lambda}||a^-_{n\lambda} - a^-_{\lambda}| + |a^-_{\lambda}||b^-_{n\lambda} - b^-_{\lambda}|, |a^+_{n\lambda}||b^+_{n\lambda} - b^+_{\lambda}||b^+_{\lambda}||a^+_{n\lambda} - a^+_{\lambda}|.$$

$$\leq \sup_{\lambda\in[0,1]} \max\{M_2.\frac{\in}{2M} + M_1.\frac{\in}{2M}, M_2.\frac{\in}{2M} + M_1.\frac{\in}{2M}\}$$

$$\leq \sup_{\lambda\in[0,1]} \max\{\frac{\in}{2M}(M_1 + M_2)\} \quad \text{since}$$

$M=\{M_1, M_2\}$

$$\leq \sup_{\lambda\in[0,1]} \max\{\frac{\in}{2M}.2M\} = \in$$

$$\rightarrow \tilde{a}_n \tilde{b}_n \Rightarrow \tilde{a}\,\tilde{b}. \;<$$

The next theorem may be prove easily similar to proof of above theorem.

<u>Theorem (3.3):</u>

For every $k \geq 0$ and $\{\tilde{a}_n\} \subset \bar{F}(R)$ and coverge then $\{k\tilde{a}_n\}$ is converge,

(i.e) $\{k\tilde{a}_n\} \longrightarrow \{k\tilde{a}\}$

Before indulging and prove the fuzzy contraction mapping theorem of fuzzy number, some additional basic concepts and definitions are needed:

1. For every mapping $f : X \longrightarrow X$, we can define a fuzzy mapping $\tilde{f} : X^* \longrightarrow X^*$, as follows [10]:

   $\tilde{f}(A) = \sup\{A(w) : w \in f^{-1}(x)\}$
2. A mapping $\tilde{f} : X^* \longrightarrow X^*$ is called fuzzy contraction mapping if there exists a constant $0 \leq r < 1$, such that:

   $d^*(\tilde{f}(A), \tilde{f}(B)) \leq r d^*(A, B)$

   any such number r is called a contractivity factor for f*, [10].

<u>Remark (3.1):</u>

See [10], the space $(F^*(\mathbb{R}), D)$ is a complete fuzzy metric space.

<u>Theorem (3.4):</u>

If $f : X \longrightarrow X$ is a contraction mapping with contractivity factor r, then $\tilde{f} : \bar{F}(\mathbb{R}) \longrightarrow \bar{F}(\mathbb{R})$ is a fuzzy contraction mapping with contractivity factor r.

<u>Proof:</u>

To prove that:

$D(\tilde{f}(\tilde{a}), \tilde{f}(\tilde{b})) \leq rD(\tilde{a}, \tilde{b}), \forall\, \tilde{a}, \tilde{b} \in \bar{F}(\mathbb{R})$,

We have:

$$D(\tilde{f}(\tilde{a}), \tilde{f}(\tilde{b})) = \sup_{\lambda\in[0,1]} \max\{|f^-_{\lambda}(\tilde{a}) - f^-_{\lambda}(\tilde{b})|, |f^+_{\lambda}(\tilde{a}) - f^+_{\lambda}(\tilde{b})|\}$$

$$= \sup_{\lambda\in[0,1]} \max\{\sup_{x^-\in f^-_{\lambda}(\tilde{a})} \inf_{y^-\in f^-_{\lambda}(\tilde{b})} |x^- - y^-|, \sup_{x^+\in f^+_{\lambda}(\tilde{a})} \inf_{y^+\in f^+_{\lambda}(\tilde{b})} |x^+ - y^+|\}$$

Since there exists $x'^- \in a^-_{\lambda\lambda} \subseteq \tilde{a}$, $x'^+ \in a^+_{\lambda\lambda} \subseteq \tilde{a}$ such that $f(x'^-) = x^-$ and $f(x'^+) = x^+$

Also, there exists $y'^- \in b^-_{\lambda\lambda} \subseteq \tilde{b}$, $y'^+ \in b^+_{\lambda\lambda} \subseteq \tilde{b}$ such that $f(y'^-) = y^-$ and $f(y'^+) = y^+$

Therefore:

$$D(\tilde{f}(\tilde{a}), \tilde{f}(\tilde{b})) = \sup_{\lambda\in[0,1]} \max\{\sup_{x'^-\in \sup p(\tilde{a})} \inf_{y'^-\in \sup p(\tilde{b})} |f(x'^-) - f(y'^-)|, \sup_{x'^+\in \sup p(\tilde{a})} \inf_{y'^+\in \sup p(\tilde{b})} |f(x'^+) - f(y'^-)|\}$$

Since X is a complete metric space and f is a contraction mapping.

Hence there exists $0 \leq r < 1$, such that:

$$D(\tilde{f}(\tilde{a}), \tilde{f}(\tilde{b})) \leq \sup_{\lambda\in[0,1]} \max\{\sup_{x'^-\in \sup p(\tilde{a})} \inf_{y'^-\in \sup p(\tilde{b})} r|x'^- - y'^-|, \sup_{x'^+\in \sup p(\tilde{a})} \inf_{y'^+\in \sup p(\tilde{b})} r|x'^+ - y'^-|\}$$

$$= r \sup_{\lambda\in[0,1]} \max\{\sup_{x'^-\in \sup p(\tilde{a})} \inf_{y'^-\in \sup p(\tilde{b})} |x'^- - y'^-|, \sup_{x'^+\in \sup p(\tilde{a})} \inf_{y'^+\in \sup p(\tilde{b})} |x'^+ - y'^-|\}$$

$$= rD(\tilde{a}, \tilde{b}).$$

Hence $\tilde{f}$ is a contraction mapping. <

Conclusions and Recommendations

From the present paper, one can conclude and recommend that the study of fuzzy sets in mathematics has so many fields of applications, such as the study of compact fuzzy sets, completeness of fuzzy sets, Riemann theory of integration of fuzzy sets, etc. Also, in the fields of topological spaces, normed spaces, vector spaces, the subject of fuzzy sets may be widely applied in differintional equation and fixed point theorems.

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الخلاصة

في هذا البحث،اعطينا تعريف وبعض مــن خــواص الاعداد الضبابية .بالاضافة الى ذلك أستخدمنا تعريف جديد لدالة المسافة[7] بين الاعداد الــضبابية وناقــشنا العمليــات الجبرية على المتتابعات الضبابية .

كما قمنا ببرهان بعض النتائج حول مبرهنة التطبيقات الانكماشــية (The contraction mapping theorem) الاعداد الضبابية (Fuzzy numbers) .