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RELATIVELY CANCELLATION MODULES

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Abstract

Let R be a commutative ring with identity and M be unitary (left) R-module. We shall introduce the concepts of relatively cancellation modules (weakly relatively cancellation modules.

Clearly, the class of weakly relatively cancellation modules contains the class of relatively cancellation modules.

The principal aim of this paper is to study in some details these two concepts. We give necessary and (or) sufficient for these two types of modules to be equivalent.

1. Introduction

Gilmer [1,p.60] has been introduced the concept of cancellation ideal to be the ideal I of R which satisfies the following:
whenever AI = BI with A and B are ideals of R implies A = B. Mijbass in [2] has been generalized this concept to modules. He has been defined the cancellation module whenever AM = BM with A and B are ideals of R implies A = B.

In this paper we shall introduce the concept of relatively cancellation module by using some restrictions on the ideals A and B in the above definition, namely we shall say that. An R-module M is called relatively cancellation, whenever AM = BM with A is a prime ideal of R and B is any ideal of R implies A = B.

Clearly, the class of cancellation modules contains the class of relatively cancellation modules. However we shall give conditions under which the two classes are equivalent.

This paper consists of two parts our principal aim of the first part is to study the relationships between cancellation modules and relatively cancellation modules. Also, we study the behaviour of relatively cancellation modules under localization. It turns out that the module is relatively cancellation whenever its localization is relatively cancellation, while the converse holds in the case that the module is finitely generated. Next, we discuss the property of relatively cancellation in each of the module and its trace.

In part two, we shall introduce the concept of weakly relatively cancellation module which is a generalization of relatively cancellation modules, we shall discuss the validity of the results that we obtain in part one, we shall show that the class of cyclic module is contained in the class of weakly relatively cancellation modules. Also we shall study the relation of weakly relatively cancellation module with the trace of the module T(M). And we shall end the part by introducing the behaviour of weakly relatively cancellation module under localization we shall show that under certain conditions a module is globally weakly relatively cancellation if it is locally weakly relatively cancellation.

Finally, we remark that R in this paper stands for a commutative ring with identity and all modules are unitary.

2. Relatively Cancellation Modules

In this section we introduce the definition of relatively cancellation modules with some examples about this concept. Moreover we give some basic properties of relatively cancellation modules.

2.1 Definition :

An R-module M is called relatively cancellation whenever AM = BM, with A is a prime ideal of R and B is any ideal of R, implies A = B.

2.2 Examples :

1) Z_3 as a Z_9-module is relatively cancellation module.
2) Z_3 as a Z_{12}-module is not relatively cancellation module since: $\langle 3\rangle Z_3 = \langle 6\rangle Z_3$, but $\langle 3\rangle \neq \langle 6\rangle$
3) Z_5 as a Z_{15}-module is relatively cancellation module.

4) Q as a Z-module is not relatively cancellation module since: <p>Q = Q for any prime number p in Z.
It is clear that <p>Q $\subseteq$ Q
Now, let $x \in$ Q. Then $x = \frac{a}{b} = \frac{pa}{pb} = p \cdot \frac{a}{pb}$ $\in$ <p>Q where a, b $\in$ Z. Implies Q $\subseteq$ <p>Q. Therefore Q is not relatively cancellation module. However <p> $\neq$ Z.

5) Consider Z_{p^∞} = Q_p/Z = {$x \in$ Q; $x = \frac{m}{p^i}$ +Z, $m \in$ Z, i = 1,2,…} as Z-module is not relatively cancellation module since: Q_p = {$\frac{m}{n}$;gcd(m,n)=1, n=p^i, i =1,2,…} is a submodule of Q containing Z.
we claim that <p> $Z_{p^\infty} = Z_{p^\infty}$
Let $x = \frac{m}{p^i}$+Z=$\frac{pm}{pp^i}$+Z = $p\frac{m}{p^{i+1}}$+Z $\in$ $(p)Z_{p^\infty}$. Therefore Z_{p^∞} is not relatively cancellation module.

Recall that the element m in an R-module M (where R is an integral domain) is called torsion element if there exists $0 \neq r \in$ R such that $r\ m$ = 0. And m is called a non-torsion element if $r\,m \neq 0$, $\forall\ 0 \neq$ r $\in$R, [3].

For cyclic modules we have the following result.

2.3 Proposition :

Every cyclic module generated by a non-torsion element is relatively cancellation module.

Proof:

Let M = <m>, where m is a non-torsion element and let A <m> = B <m>, where A is prime ideal of R and B is any ideal of R. $am \in$B <m> for all $a \in$ A, then $am = bm$, where $b \in$ B, implies $am - bm = 0$.

Therefore (a – b)m = 0,but m is a non-torsion element, then a – b=0, implies a=b. Therefore A $\subseteq$ B.

Similary, B $\subseteq$ A, and hence A=B.

We shall show by an example that the condition M is generated by a non-torsion element in proposition (2.3) can not be dropped.

2.4 Example :

Let M=Z_2 be a Z-module, it is clear that Z_2=<$\bar{1}$> and $\bar{1}$ is a torsion element in Z_2.

Now, since <2>Z_2 = 0 and <0> Z_2= 0. Then <2>Z_2 = <0> Z_2, but <2> $\neq$ <0>.

Therefore Z_2 is not relatively cancellation module.

In the following theorem we give some characterizations of relatively cancellation modules.

2.5 Theorem:

Let M be an R-module. Then the following statements are equivalent:
(1) M is relatively cancellation module.
(2) If AM$\subseteq$BM, such that A is any ideal of R and B is a prime ideal of R, then A$\subseteq$B.
(3) If <a>M $\subseteq$BM, such that $a \in$R and B is a prime ideal of R, then $a \in$ B.
(4) (AM:M)=A, for all prime ideals A of R.
(5) (AM:BM) = (A:B), for all ideals B of R and for all prime ideals A of R.

Proof:

(1) $\Rightarrow$ (2): Suppose that M is relatively cancellation module and AM $\subseteq$ BM, where B is a prime ideal of R and A is any ideal of R.
Now, BM = AM + BM = (A + B)M, then B = A + B, implies A $\subseteq$ B.
(2) $\Rightarrow$ (3): Let <a>M $\subseteq$BM. Then <a> $\subseteq$B by (2). Hence $a \in$ B.
(3) $\Rightarrow$ (4); Let $x \in$ (AM:M). Then xM $\subseteq$ AM by (3) $x \in$ A. Hence (AM:M) $\subseteq$ A.
On the other side, if $x \in$ A, then xM$\subseteq$ AM. Therefore $x \in$ (AM:M) and hence (AM:M)=A.
(4) $\Rightarrow$ (5): Let $x \in$ ((AM:M):B) (since (AM:M)=A by (4)), implies $x\in$ (AM:BM), [4,prop.(2.3),p.38]. Now, if $x \in$ (AM:BM) = ((AM:M):B) and since (AM:M) = A by (4). Then $x \in$ (A:B). Therefore (AM:BM) = (A:B).
(5) $\Rightarrow$ (1): Let AM = BM and A is prime ideal of R, B is any ideal of R. Then (AM:BM) = R, implies (A:B) = R. Therefore A $\subseteq$ B.
Similarly, A $\subseteq$ B. Then A = B. Hence M is relatively cancellation module.

3. Relatively Cancellation Modules and Cancellation Modules

In this section the relationship between relatively cancellation property and cancellation property of modules will be examine more closely and we try to lie some light on this relation.

Recall that an R-module M is called faithful if $ann_R(M) = 0$, where $ann_R(M) = \{r \in R; rx = 0, \forall x \in M\}$, [5].

3.1 Proposition:

Let M be an R-module. Then M is a cancellation module iff M is faithful relatively cancellation module.

Proof:

Every cancellation module is relatively cancellation module, and every cancellation module is faithful module, [2, remark (1.4), p.8].

Conversely, Suppose that M is faithful relatively cancellation module. Let AM = BM, where A and B are two ideals in R.

If A is prime ideal of R and B is any ideal of R, implies A = B (since M is relatively cancellation module).

If A is not prime ideal of R and B is any ideal of R, AM–BM=0, implies (A–B)M=0. Then $(A - B) \subseteq ann_R(M)$. But M is faithful module, implies A – B = 0, hence A = B. Therefore M is a cancellation module.

Recall that the Jacobson radical of R is the intersection of all maximal ideals of R, $J(R)=\cap\{I:I$ is maximal ideal of R$\}$, [6]. And the Jacobson radical of M is the intersection of all maximal submodules of M, $J(M) = \cap\{N:N$ is maximal submodules of M$\}$, and J(M) = M, if M has no maximal submodules, [6].

The following proposition and it's corollary give a necessary condition for a module to be relatively cancellation module.

3.2 Proposition:

Let M be a non-zero module on R. If M is relatively cancellation module, then (J(M):M) = J(R).

Proof:

If AM = M for some prime ideal A of R. Then A = R, which is a contradiction! (since M is relatively cancellation module). Hence AM ≠ M for all maximal ideals of R. Now, $(J(M):M) = (\bigcap_{\lambda\in\wedge} A_\lambda M : M) = \bigcap_{\lambda\in\wedge}(A_\lambda M : M)$, [4,Ex.1.4,P.240]. But $A_\lambda \subseteq (A_\lambda M:M)$, then $(A_\lambda M:M) = A_\lambda \ \forall \lambda \in \wedge$. Therefore $(J(M):M) = \bigcap_{\lambda\in\wedge} A_\lambda = J(R)$.

3.3 Corollary:

If M is a relatively cancellation R-module, then $ann_R(M) \subseteq J(R)$, and therefore $ann_R(M)$ is a small ideal of R.

Proof:

By proposition (3.2), we get, (J(M):M) = J(R), $ann_R(M) \subseteq (J(M):M)$, then $ann_R(M) \subseteq J(R)$ and hence $ann_R(M)$ is small ideal of R by [5].

Recall that if R has only one maximal ideal, then R is called a local ring, [5].

3.4 Corollary:

If M is a relatively cancellation R-module, and $ann_R(M)$ is a maximal ideal of R, then R is local ring.

Proof:

It is clear, so it is omitted.

4. Relatively Cancellation Modules and localization

In this section, we give the concept of contraction. For all submodules N of M we shall denote the extension N in N_p by N^e and for all submodules L in M_p we shall denote the contraction of L in M_p by L^c where L^c means $f^{-1}(L)$; where $f : M \longrightarrow M_p$ is the natural homomorphism, [6,p.9].

In this section we shall study the behavior under localization.

Before we introduce the next proposition, we need to prove the following remark:

4.1 Lemma:

If A is a prime ideal of R, then A_p is a prime ideal of R_p.

Proof:

Let $f(a)\cdot f(b) \in A_p$, where $a, b \in R$ and $f : A \longrightarrow A_p$ be natural homomorphism, $f(a\cdot b) \in A_p$ (since f is homomorphism), $a\cdot b \in A$. Then either $a \in A$ or $b \in A$ (since A is prime ideal). Therefore $f(a) \in A_p$ or $f(b) \in A_p$. Hence A_p is a prime ideal of R_p.

4.2 Proposition :

Let M be an R-module and M is locally relatively cancellation module. Then M is relatively cancellation module.

Proof:

Let AM = BM, where A is a prime ideal of R and B is any ideal of R. Then $(AM)_p=(BM)_p$, implies $A_pM_p = B_pM_p$. Therefore $A_p = B_p$, by lemma (4.1), (since M_p is relatively cancellation module). Hence A = B, [4, prop.(3.13),p.70], which completes the proof

Recall that an R-module M is called finitely generated if there exists subset $\{x_1, x_2, \ldots, x_n\}$ of M such that $M = Rx_1 + Rx_2 + \ldots + Rx_n$, [5, p.22].

4.3 Proposition :

Let M be a finitely generated R-module and $(B^c)_p = (B_p)^c$ for all maximal ideals P of R and B is any ideal of R. Then M is relatively cancellation module if and only if M is locally relatively cancellation module.

Proof:

Suppose that M is relatively cancellation module of R and $a/s\ M_p \subseteq B_pM_p$, where $a \in R$, $s \notin P$ and B_p is a prime ideal of R_p. Since M is finitely generated, therefore there exists a subset $\{m_1, m_2, \ldots, m_n\}$ of M which generates M.

Now, $\forall\ i = 1, 2, \ldots, n$; $\frac{m_i}{1} \in M_p$. Then $\frac{a}{s}\cdot\frac{m_i}{1} \in B_pM_p$, implies $\frac{a\,m_i}{s} = \sum_{j=1}^{n}\frac{b_{ij}}{s_{ij}}\cdot\frac{m_i}{1}$.

Let $t_i = \prod_{j=1}^{n} s_{ij}$ and $b'_{ij} = s_{i1} \ldots s_{ij+1} \ldots s_{in}\, b_{ij}$

$\frac{a\,m_i}{s} = \frac{\sum_{i=1}^{n} b'_{ij} m_j}{t_i}$. Hence there exists $x_i \notin P$ such that $x_i\, t_i\, a\, m_i = x_i\, s \sum_{j=1}^{n} b'_{ij} m_j$. Put $s_i = x_i\, t_i$ and $z_i = x_i\, s$. Then $s_i\, a\, m_i = z_i \sum_{j=1}^{n} b'_{ij} m_j$.

Therefore $s_i\, a\, m_i \in z_i\, (B^c)_pM \subseteq (B_p)^cM$. Implies $s'aM \subseteq (B_p)^cM$, where $s' = s_1, s_2, \ldots, s_n$. Hence $s'a \subseteq (B_p)^c$ (since M is relatively cancellation module). $f(s'a) = f(s')\cdot f(a) \in ((B_p)^c)_p = ((B^c)_p)_p$. Since $f(s')$ is a unit element in R_p. Then $f(a) \in ((B^c)_p)_p$. But $(B^c)_p = B$, [4, prop.(3.6),p.67]. Therefore $f(a) \in B_p$, implies $\frac{as}{s} \in B_p$. Hence M_p is relatively cancellation module over R_p.

5. Relatively Cancellation Modules and the Trace of Modules

Let M and N be two R-modules. The trace of N over M denoted by $T_M(N) = \sum_{\lambda \in \wedge} \theta_\lambda(M)$, where the sum is taken on all θ_λ in Hom (M, N). In the special case if N = R, then the trace of M over R written by T(M) instead of $T_M(R)$, [4].

In this section we give some relationships between the modules having the relatively cancellation property and its trace see proposition (5.1), corollary (5.2), corollary (5.3) and corollary (5.4).

Let us start with the following concept. An ideal of a ring R is called relatively cancellation ideal if AI = BI, where A is a prime ideal of R and B is any ideal of R, then A = B. It is known that if I is an ideal of R, then I is relatively cancellation ideal if and only if I is relatively cancellation R-module.

Clearly, relatively cancellation module is a natural generalization of relatively cancellation ideal.

In the following result and it's corollaries we study the relation between relatively cancellation module and it's trace.

5.1 Proposition :

Let M and N be two R-modules and $L = \sum\theta_\lambda(M)$ be a submodule of N, where the sum is taken for any subset of Hom(M,N), such that L is relatively cancellation module. Then M is relatively cancellation module.

Proof:

Let AM = BM, where A is a prime ideal of R and B is any ideal of R. Then $\theta_\lambda(AM) = \theta_\lambda(BM)$, for each $\theta_\lambda \in Hom(M,N)$, implies $\sum_{\theta_\lambda \in Hom(M,N)} \theta_\lambda(AM) = \sum_{\theta_\lambda \in Hom(M,N)} \theta_\lambda(BM)$. But $\theta_\lambda(AM) = \theta_\lambda(BM) = B\theta_\lambda(M)$. Then $A \sum_{\theta_\lambda \in Hom(M,N)} \theta_\lambda(M) = B \sum_{\theta_\lambda \in Hom(M,N)} \theta_\lambda(M)$. Therefore AL = BL, which implies that A = B (since L is relatively cancellation submodule).

5.2 Corollary:

If M is an R-module and T(M) is a relatively cancellation ideal of R, then M is relatively cancellation module.

Proof:

The result is clear by using the definition of T(M) and proposition (5.1).

5.3 Corollary:

If M is an R-module and T(M) is a multiplication ideal of R, which contain a non-zero divisor, then M is relatively cancellation module.

Proof:

Let $a \in$ T(M) and a is a non-zero divisor. T(M) is a multiplication ideal of R, so there exists an ideal J of R, such that : $<a>$=JT(M). Implies T(M) is an invertible ideal of R, [5, prop.(6.3),p.125]. Therefore T(M) is a cancellation ideal, [7,p.879], implies T(M) is relatively cancellation module. Then M is relatively cancellation module, by corollary (5.2).

5.4 Corollary:

Let M be an R-module such that T(M) is relatively cancellation submodule. Then M*=Hom(M,R) is relatively cancellation R-module.

Proof:

Let aM* $\subseteq$ BM*, such that B is a prime ideal of R.

Now, $a\ f \in a$M* $\subseteq$ BM*, $\forall\ f \in$ M*. Thus $a\ f \in$ BM*, implies $a\ f = \sum_{j=1}^{n} b_i f_i$, where $b_i \in$ B and f, $f_i \in$ M*. Therefore $af\ (m) = \sum_{i=1}^{n} b_i f_i(m)$, $\forall m \in$ M. Then aT(M) $\subseteq$ BT(M).But T(M) is a relatively cancellation submodule. Then $a \in$ B by proposition (2.5(3)), and hence M* is relatively cancellation module.

6.The Weak Relatively Cancellation Modules

In this section we start with a concept of a weak relatively cancellation modules. We shall weakening the concept of relatively cancellation property of modules by using an extra condition on the result of the cancellation. It turns out that the class of cyclic modules is contained in the class of weak relatively cancellation modules, see proposition (6.5). Next, some characterizations of weak relatively cancellation modules will be introduced in proposition (6.6).

6.1 Definition:

Let M be an R-module. Then M is called weak relatively cancellation module if AM = BM, where A is a prime ideal of R and B is any ideal of R, then A+ann_R(M) = B + ann_R(M).

6.2 Remark:

Every relatively cancellation module is a weak relatively cancellation module.

The converse of remark (6.2) is not true, as it is seen by the following example;

6.3 Example:

Consider Z_2 as a Z-module and let m_1 be an odd prime in Z and m_2 is any odd in Z, such that $m_1 \neq m_2$. Let $<m_1>Z_2=<m_2>Z_2$, $ann_R(Z_2)=<2>$. We claim that $<m_1>+<2>= <m_2>+<2>=Z$ since m_1, m_2 are an odd numbers, then $m_1 = 2n_1 + 1$, $m_2 = 2n_2+ 1$, where n_1, $n_2 \in$ Z. $m_1 - 2n_1 \in <m_1> + <2>$ implies $2n_1 + 1 - 2n_1 <m_1> + <2>$, therefore $1 \in <m_1> + <2>$, and hence $<m_1> +<2>$=Z.

Similarly, we can prove that $<m_2>+<2>$=Z. Then $<m_1> + <2> = <m_2> + <2>$=Z. Therefore Z_2 is a weak relatively cancellation module over Z. But Z_2 is not relatively cancellation module, since $<3>Z_2 = <5>Z_2$, but $<3> \neq <5>$.

The converse of remark (6.2) holds under the condition M is faithful.

6.4 Proposition:

If M is a faithful weak relatively cancellation module, then M is relatively cancellation module.

Proof:

Is trivial, so it is omitted.

In the following proposition we shall show that the class of cyclic modules is contained in the class of weak relatively cancellation modules.

6.5 Proposition:

Every cyclic module is a weak relatively cancellation module.

Proof:

Let M = $<m>$ be a cyclic module over R with $m \in$ M, and let A$<m>$ = B $<m>$, where A is a prime ideal in R and B is any ideal in R. Then $am \in$ B$<m>$, $a \in$ A, implies $am=bm$, where

$b \in$ B. Therefore $am - bm = 0$, implies $(a-b)m = 0$. Then $a - b \in ann_R(M)$. But $a = b + a - b$. Thus $a \in B + ann_R(M)$, implies $A \subseteq B + ann_R(M)$. Then $A + ann_R(M) \subseteq B + ann_R(M)$.

Similarly, we prove that $B + ann_R(M) \subseteq A + ann_R(M)$ and hence $A + ann_R(M) = B + ann_R(M)$, which is what we wanted.

We shall give some characterization of a weak relatively cancellation modules in the following theorem.

<u>6.6 Theorem:</u>

Let M be an R-module. Then the following statements are equivalent:

(1) M is a weak relatively cancellation module.
(2) If $AM \subseteq BM$, such that A is any ideal of R and B is a prime ideal of R, then $A \subseteq B + ann_R(M)$.
(3) If $\langle a \rangle M \subseteq BM$, such that $a \in R$ and B is a prime ideal of R, then $a \in B + ann_R(M)$.
(4) $(AM : M) = A + ann_R(M)$, for all ideals A of R.
(5) $(AM:BM) = (A + ann_R(M):B)$, where A is a prime ideal of R and B is any ideal of R.

<u>Proof:</u>

(1) $\Rightarrow$ (2): Let M be a weak relatively cancellation module over R, and $AM \subseteq BM$. Then $BM = AM + BM = (A + B)M$, implies $B + ann_R(M) = A + (B + ann_R(M))$. Therefore $A \subseteq B + ann_R(M)$.

(2) $\Rightarrow$ (3): Let $\langle a \rangle M \subseteq BM$. Then $\langle a \rangle \subseteq B + ann_R(M)$ by (2). Therefore $a \in B + ann_R(M)$.

(3) $\Rightarrow$ (4): Let $x \in (AM:M)$. Then $xM \subseteq AM$ implies $x \in A + ann_R(M)$ by (3). Therefore $(AM:M) \subseteq A + ann_R(M)$.

Now, let $x \in A + ann_R(M)$. Then $xM \subseteq (A + ann_R(M))M$, implies $xM \subseteq AM + ann_R(M)M = AM$.

(4) $\Rightarrow$ (5): Let $x \in (A + ann_R(M):B)$. But $(AM:M) = A + ann_R(M)$ by (4). Then $x \in ((AM:M):B) = (AM:BM)$ [4,prop.(2.3), p.38]. Now, let $x \in (AM:BM)$. Then $x \in ((A:B):B)$, since $(AM:M) = A + ann_R(M)$, therefore $x \in (A + ann_R(M):B)$. Hence $(AM:BM) = (A + ann_R(M):B)$.

(5) $\Rightarrow$ (1): Let $AM = BM$, where A is a prime ideal of R and B is any ideal of R. Then $(AM:BM) = R$.

Therefore $(A + ann_R(M):B) = R$, implies $1 \in (A + ann_R(M):B)$. Therefore $B \subseteq A + ann_R(M)$. Then $B + ann_R(M) \subseteq A + ann_R(M)$.

Similarly, we can prove that $A + ann_R(M) \subseteq B + ann_R(M)$. Therefore $A + ann_R(M) = B + ann_R(M)$. Hence M is a weak relatively cancellation module.

7.The Trace of a Module and the Property of Weak Relatively Cancellation Modules

The main aim of this section is to generalize the results in section four of chapter one. We shall prove that if the trace of a module is a weak relatively cancellation ideal and $ann_R(T(M)) = ann_R(M)$, then M is weak relatively cancellation module, see corollary (7.2), also we shall show that the dual of a module is weak relatively cancellation module when the trace is weak relatively cancellation ideal and $ann_R(T(M) = ann_R(M^*)$, see proposition (7.3).

<u>7.1 Proposition :</u>

Let M and N be two R-modules, and $L = \sum_{\lambda \in \wedge} \theta_\lambda(M)$ be a submodule of N, where the sum is taken for any subset of Hom (M,N), L is weak relatively cancellation, and $ann_R(L) = ann_R(M)$. Then M is a weak relatively cancellation module.

<u>Proof:</u>

Let $AM = BM$, where A is prime ideal of R and R and B is any ideal of R. Then $\theta_\lambda(AM) = \theta_\lambda(BM)$, that implies $\sum_{\lambda \in \wedge} \theta_\lambda(AM) = \sum_{\lambda \in \wedge} \theta_\lambda(BM)$. But $\theta_\lambda(AM) = A\theta_\lambda(M) = \theta_\lambda(BM) = B\theta_\lambda(M)$. Then $A\sum_{\lambda \in \wedge} \theta_\lambda(M) = B\sum_{\lambda \in \wedge} \theta_\lambda(M)$. Therefore $AL = BL$ (since L is weak relatively cancellation module), implies $A + ann_R(L) = B + ann_R(L)$. Therefore $A + ann_R(M) = B + ann_R(M)$. Then M is weak relatively cancellation module.

<u>7.2 Corollary:</u>

If M an R-modules, T(M) is a weak relatively cancellation ideal in R, and $ann_R(T(M)) = ann_R(M)$. Then M is a weak relatively cancellation module.

<u>Proof:</u>

The result is clear by using proposition (7.1) and the definition of T(M).

The dual of a module will be weak relatively cancellation whenever the trace of the module satisfies this property, as it is shown in the following result.

7.3 Proposition:

If M is an R-modules, T(M) is a weak relatively cancellation ideal over R such that ann_R(T(M) = ann_R(M*), then M* is weak relatively cancellation module.

Proof:

Let aM* ⊆ BM*, where $a \in$ R, and B is prime ideal of R. Then $a\ f = \sum_{j=1}^{n} b_i f_i$, where $b_i \in$ B and $f, f_i \in$ M*.

Now, ∀ $m \in$ M, $a\ f(m) = \sum_{j=1}^{n} b_i f_i(m)$. Then aT(M) ⊆ BT(M), therefore $a \in$ B + T(M) (since T(M) is weak relatively cancellation module). But ann_R(T(M)) = ann_R(M*), implies $a \in$ B + ann_R(M*). Hence M* is weak relatively cancellation module.

8.Weak Relatively Cancellation Modules and Localization

In this section we shall try to generalize the results in section three of chapter one.

8.1 Proposition:

Let M be a finitely generated R-module and $(B_p)^c = (B^c)_p$ for all maximal ideals P of R, B is any ideal of R. Then M is a weak relatively cancellation module if and only if M is locally weak relatively cancellation module.

Proof:

Suppose that M is a weak relatively cancellation module over R and P is a maximal ideal of R.

Now, let $a/s\ M_p \subseteq B_pM_p$, where B_p is prime ideal of R_p, $a \in$ R, $s \notin$ P and $a/s \in R_p$. M is finitely generated, then there existssubset $\{m_1, m_2, \ldots, m_n\}$ of M generates M. Then $m_i/1 \in M_p$, ∀ i = 1,2,..,n; implies $a/s \cdot m_i/1 \in B_pM_p$. Therefore $a m_i / s = \sum_{j=1}^{n} b_{ij} / s_{ij} \cdot m_i / 1$, where $s_{ij} \notin$ P and $b_{ij} \in (B_p)^c$. Let $t_i = \prod_{j=1}^{n} s_{ij}$ and $b'_{ij} = s_{ij}, \ldots, s_{ij-1}\ s_{ij+1}, \ldots, s_m b_{ij}$. Then $a\ m_i/1s = \sum_{j=1}^{n} b'_{ij} m_j / t_i$, hence there exists $x_i \notin$ P such that $x_i\ t_i\ a\ m_i = x_i s \sum_{j=1}^{n} b_{ij} m_j$ put $s_i = x_i\ t_i$ and $z_i = x_i\ s$, implies $s_i\ a\ m_i = z_i \sum_{j=1}^{n} b'_{ij} m_j$, then $s_i a\ m_i \in z_i(B_p)^c$M ⊆ $(B_p)^c$M. Therefore $s'a\ m \subseteq (B_p)^c$M, where $s' = s_1, s_2, \ldots, s_n$. Then $s'a \in (B_p)^c + ann_R(M)$ (since M is weak reltively cancellation module), implies $f(s'a) = f(s')\ f(a) \in ((B_p)^c + ann_R(M))_p$, where $f : R \longrightarrow R_p$, be the natural homomorphism. Therefore $f(s')\ f(a) \in ((B_p)^c)_p + ann_R(M))_p$, [4, ex.(4),p.75]. Since M is finitely generated, then $ann_R(M)_p = ann_R(M_p)$, [2,prop.(3.14), p.43]. Then $f(s')\ f(a) \subseteq ((B_p)^c)_p + ann_R(M))_p = ((B_p)^c)_p + ann_R(M_p)$, $f(s')$ is unit element of R_p and $(B^c)_p$ = B, [4, prop.(3.6),p.67]. Therefore $f(a)=as/a \in B_p + ann_R(M_p)$, implies $a/\ s = a^s / s{\cdot}1 / s \in B_p + ann_R(M_p)$. Therefore M_p is weak relatively cancellation module over R_p, by proposition (6.6).

Conversely, suppose that M is locally weak relatively cancellation module and let AM = BM, where A is a prime ideal of R and B is any ideal of R, suppose that P is maximal ideal of R. Then $(AM)_p = (BM)_p$, implies $A_pM_p = B_pM_p$. Therefore $A_p + ann_R(M_p) = B_p + ann_R(M_p)$ (since M_p is weak relatively cancellation module). M is finitely generated, then $A_p + (ann_R(M))_p = B_p + (ann_R(M))_p$. Therefore $(A + ann_R(M))_p = (B + ann_R(M))_p$,implies A + ann_R(M) =B + ann_R(M) [4, prop.(3.13),p.70]. Hence M is weak relatively cancellation module.

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الخلاصة

نفرض ان R حلقة ابدالية ذات عنصر محايد. سوف نقدم مفهوم موديولات الحذف نسبياً. يقال عن الموديول M المعرف على الحلقة R متى ما حقق الخاصية الاتية:

اذا كان AM = BM حيث A مثالي اولي في R و B اي مثالي في R، فان A = B.

وبصورة اكثر تعميماً سنطلق على الموديول M على R الذي يحقق الخاصية الاتية:

اذا كان AM = BM حيث A مثالي اولي في R و B اي مثالي في R، فان A+ann(M)=B+ann(M)، اسم موديول حذف ضعيف نسبياً. من الواضح ان كل موديول حذف نسبياً هو موديول حذف ضعيف نسبياً. لكن العكس غير صحيح.

ان هدفنا الاساس في عملنا هذا هو دراسة الموديولات التي تحقق واحدة من هاتين الخاصيتين حيث قدمنا عدداً من العبارات المكافئة لخاصية الحذف نسبياً (الحذف الضعيفة نسبياً).