A NOTE ON AN R- MODULE WITH APPROXIMATELY-PURE INTERSECTION PROPERTY

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Abstract

Let R be a commutative ring with identity and Let M be left R – module. A submodule N of an R- module M is said to be approximately-pure submodule of M (for short AP-pure), if $N \cap IM = IN + J(R)M \cap (N \cap IM)$, for each ideal I of R. the main purpose of this paper is to develop the properties of modules with the approximately- pure intersection property.

Keywords: pure submodule, approximately-pure submodule, module with approximately pure intersection property.

1. Introduction

In this paper we assume R is commutative ring with identity and all modules are unitary left R-module. A submodule N of an R-module M is called pure submodule, if for every finitely generated ideal I of R $IM \cap N = IN$, [1]. Following [2], an R- module M has the pure intersection property (for short PIP), if the intersection of any two pure submodules is again pure. We introduce the concept of an R-module M has approximately-pure intersection property (for short AP-PIP). We prove that if N be AP- pure submodule of an R- module M, then M, has AP-PIP if and only if $\frac{M}{N}$ has AP – PIP, see proposition (2.3).

2. Properties of module which has approximately-pure intersection property:

Recall a submodule N of an R-module M is called approximately- pure (briefly AP-pure) if $N \cap IM = IN + J(R)M \cap (N \cap IM)$, for each ideal I of R, where J(R) is the Jacobson radical of R. It is clear that each pure submodule is AP-pure submodule, but the converse is not true in general see [3, remark (1.2.14)].

Remark (2.1):

1. Let M be an R- module and let N be a summand of M, then N is a AP-pure submodule of M.
2. Let M be an R- module and let N be AP- pure submodule of M. If H is AP-pure submodule of N, then H is AP-pure submodule of M.
3. Let M be an R-module and let N be AP-pure submodule of M. if A is a submodule of M containing N, and $J(R)M \cap A = J(R)A$ then N is AP-pure submodule of A.
4. Let M be an R-module and let N be AP-pure submodule of M. If H is a submodule of N, then $\frac{N}{H}$ is AP-pure submodule of $\frac{M}{H}$.
5. Let M be an R-module. Let N and H be submodule of M, If H is AP-pure submodule of M and $\frac{N}{H}$ is AP-pure submodule of $\frac{M}{H}$, then N is AP-pure submodule of M.

Proof:

1. Clear.

2. Let I be an ideal of R, since N is AP- pure in M and H is AP-pure in N, then $N \cap IM = IN + J(R)M \cap (N \cap IM)$ and $H \cap IN = IH + J(R)N \cap (H \cap IN)$ but $H \leq N$, therefore:

$$H \cap IM \subseteq N \cap IM = IN+J(R)M \cap (N \cap IM)$$

and hence:

$$\begin{aligned} H \cap IM &\subseteq [IN+J(R)M \cap (N \cap IM)] \cap H \\ &= (H \cap IN+J(R)M \cap (N \cap IM \cap H) \\ &= (IH+J(R)N \cap (H \cap IN) + J(R)M \cap (H \cap IM) \\ &\subseteq IH+J(R)M \cap (H \cap IM) \end{aligned}$$

Since $IH + J(R)M \cap (H \cap IM) \subseteq H \cap IM$, then:

$$H \cap IM = IH + J(R)N \cap (H \cap IM)$$

3. Let I be an ideal of R, since N is AP-pure in M, then

$N \cap IM = IN + J(R)\, M \cap (N \cap IM)$.

But $A \leq M$, therefore:

$N \cap IA \subseteq N \cap IM$

$= IN + J(R)\, M \cap (N \cap IM)$

and hence

$N \cap IA \subseteq [IN+J(R)\, M \cap (N \cap IM)] \cap IA$

$= IN+J(R)\, M \cap (N \cap IA)$

$= IN+ J(R)\, M \cap (A \cap N \cap IA)$

$= IN+ (J(R)\, M \cap A) \cap (N \cap IA)$

$= IN+ J(R)\, A \cap (N \cap IA)$

Since $IN +J(R)\, A \cap (N \cap IA) \subseteq N \cap IA$, then

$N \cap IA = IN +J(R)\, A \cap (N \cap IA)$

4. Let I be an ideal of R, since N is AP-pure submodule of M, then**:**

$N \cap IM = IN +J(R)\, M \cap (N \cap IM)$

So $\frac{N}{H} \cap I\left(\frac{M}{H}\right) = \frac{N}{H} \cap \frac{IM+H}{H}$

$= \frac{(N \cap IM)+H}{H}$

$= \frac{IN+J(R)M \cap (N \cap IM)+H}{H}$

$= \frac{IN+H}{H} + \frac{J(R)M \cap (N \cap IM)+H}{H}$

$= I\left(\frac{N}{H}\right) + \frac{[J(R)M+H] \cap (N \cap IM)}{H}$

$= I\left(\frac{N}{H}\right) + \frac{J(R)M+H}{H} \cap \frac{N \cap IM}{H}$

$= I\left(\frac{N}{H}\right) + J(R)\frac{M}{H} \cap \left(\frac{N}{H} \cap \frac{IM}{H}\right)$

$= I\left(\frac{N}{H}\right) + J(R)\frac{M}{H} \cap \left(\frac{N}{H} \cap I\left(\frac{M}{H}\right)\right)$

5. Clear. ■

Definition (2.2):

An R-module M is said to have the approximately pure intersection property (for short AP-PIP) if the intersection of any two AP- pure submodules is again AP- pure.

Proposition (2.3):

1. If an R-module M has the AP- PIP, then every AP- pure submodule of M has the AP-PIP.
2. Let N be AP-pure submodule of an R-module M. M has AP- PIP if and only if $\frac{M}{N}$ has AP-PIP.

Proof:

1. Clear
2. ($\Rightarrow$)

Let $\frac{A}{N}$, $\frac{B}{N}$ be two AP-pure submodules of $\frac{M}{N}$ and let K be an ideal in R. We want to show that:

$\left(\frac{A}{N} \cap \frac{B}{N}\right) \cap K\left(\frac{M}{N}\right) = K\left(\frac{A}{N} \cap \frac{B}{N}\right) + J(R)\frac{M}{N} \cap \left[\left(\frac{A}{N} \cap \frac{B}{N}\right) \cap K\left(\frac{M}{N}\right)\right]$

We claim that each of A and B is AP-pure in M.

To show this, let I be an ideal in R and let $x \in A \cap IM$. Since $\frac{A}{N}$ is AP-pure in $\frac{M}{N}$, then:

$\frac{A}{N} \cap I\left(\frac{M}{N}\right) = I\left(\frac{A}{N}\right) + J(R)\frac{M}{N} \cap \left(\frac{A}{N} \cap I\left(\frac{M}{N}\right)\right)$,

Thus:

$\frac{A}{N} \cap \frac{IM+N}{N} = \frac{IA+N}{N} + \left(\frac{J(R)M+N}{N}\right) \cap \left(\frac{A}{N} \cap \left(\frac{IM+N}{N}\right)\right)$

and this implies that:

$\frac{A \cap (IM+N)}{N} = \frac{IA+N}{N} + \frac{(J(R)M+N) \cap (A \cap (IM+N))}{N}$

$= \frac{(IA+N)+(J(R)M+N) \cap (A \cap (IM+N))}{N}$,

Therefore:

$A \cap (IM+N) = IA+ J(R)\, M \cap (A \cap IM) + N$,

and hence:

$(A \cap IM) + N = IA + J(R)\, M \cap (A \cap IM) +N$

Since $x \in A \cap IM \subseteq A \cap (IM+N)$, then:

$x \in IA + J(R)\, M \cap (A \cap IM) + N$

Let $x = w + m + n$, where $w \in IA$ and $m \in J(R)M \cap (A \cap IM)$ and $n \in N$

Now, consider:

$n = x-w-m \in N \cap IM = IN+J(R)M \cap (N \cap IM)$
$\subseteq IA+J(R)M \cap (A \cap IM)$

and hence A is AP-pure in M. Since M has the AP-PIP, then $A \cap B$ is AP- pure in M.

Thus $(A \cap B) \cap KM = K(A \cap B) + J(R)M \cap ((A \cap B) \cap IM)$

Now, let $x \in \left(\frac{A}{N} \cap \frac{B}{N}\right) \cap K\left(\frac{M}{N}\right)$, then $x=w+N$, where $w \in KM$, and

$x = a +N = b +N$, where $a \in A$ and $b \in B$.

Thus $w - a \in N \subseteq A$, $w - b \in N \subseteq B$ and hence $w \in A \cap B$.

Thus:

$w \in (A \cap B) \cap KM = K(A \cap B) + J(R)M \cap ((A\cap B) \cap KM)$

Then $x = w + N \in K\left(\frac{A\cap B}{N}\right) = K\left(\frac{A}{N} \cap \frac{B}{N}\right)$

$\leq K\left(\frac{A}{N} \cap \frac{B}{N}\right) + J(R)\frac{M}{N} \cap \left(\left(\frac{A}{N} \cap \frac{B}{N}\right) \cap K\frac{M}{N}\right)$

($\Leftarrow$) Conversely let E and F be AP- pure submodule of M, let N be a submodule of E and N be a submodule of F then $\frac{E}{N}$ and $\frac{F}{N}$ is AP-pure submodule of $\frac{M}{N}$. Since $\frac{M}{N}$ has AP-PIP, then $\frac{E}{N} \cap \frac{F}{N} = \frac{E\cap F}{N}$ is AP- pure submodule of $\frac{M}{N}$. Therefore $E \cap F$ is AP-pure submodule of M. ■

Theorem (2.4):

Let M be an R- module, then M has the AP-PIP if and only if:

$(IA \cap IB) + J(R)M \cap ((A \cap B) \cap IM) = I(A \cap B) + J(R)M \cap ((A \cap B) \cap IM)$

for every ideal I of R and for every AP-pure submodule A and B of M.

Proof:

Suppose M has the AP-PIP then for each AP-pure submodules A and B, $A\cap B$ is AP-pure. Let I be an ideal in R, then

$(A\cap B)\cap IM = I(A\cap B)+J(R)M\cap((A\cap B)\cap IM)$

It is clear that:

$I(A\cap B) + J(R)M\ ((A\cap B)\cap IM) \subseteq (IA\cap IB) + J(R)\ M((A \cap B) \cap IM)$

But $(IA \cap IB) + J(R)\ M \cap ((A \cap B) \cap IM) \subseteq A \cap (B \cap IM) = (A \cap B) \cap IM$

$= I(A \cap B) + J(R)M \cap ((A\cap B) \cap IM)$

Thus:

$IA \cap IB + J(R)M \cap ((A \cap B) \cap IM) = I(A \cap B) + J(R)M \cap ((A \cap B) \cap IM)$

Conversely, let A and B be AP-pure submodule of M and I an ideal in R. Then:

$A \cap B \cap IM = A \cap (B \cap IM) = A \cap (IB + J(R)M \cap (B \cap IM)$

Similarly:

$A \cap B \cap IM = B \cap (IA + J(R)M \cap (B \cap IM)$

But A, B are AP- pure in M. Thus:

$A\cap B\cap IM \subseteq IA\cap IB + J(R)M\cap(A\cap B\cap IM)$

$= I(A\cap B) + J(R)\ M\ (A\cap B\cap IM)$. ■

Theorem (2.5):

Let M be an R-module, then M has the AP-PIP if and only if for every AP-pure submodules A and B of M and for every R-homomorphism $f : A \cap B \longrightarrow$, M such that:

$(A \cap Im\ f) + J(R)\ M \cap (A + Im\ f \cap IM) = \{0\}$ and $A + Im\ f$ is AP-pure in M, ker f is AP- pure in M.

Proof:

Assume that M has the AP-PIP. Let A and B be AP-pure submodules of M and $f : A \cap B \longrightarrow M$, be an R- homomorphism such that $A \cap Im\ f = \{0\}$ and $A + Im\ f$ is AP- pure in M.

Let $T = \{x + f(x), x \in A \cap B\}$

It is clear that T is a submodule of M.

To show that T is AP- pure in M. let I be an ideal in R and

$y = \sum_{i=1}^{n} r_i m_i \in T \cap IM, r_i \in R, m_i \in M$

Hence:

$y = \sum_{i=1}^{n} r_i m_i = x + f(x)$ for some $x \in A \cap B$.

Since:

$y = \sum_{i=1}^{n} r_i m_i = x+f(x) \in A\cap B+Im\ f \subseteq A+Im\ f$

and $A + Im\ f$ is AP-pure in M. Thus:

$y = \sum_{i=1}^{n} r_i m_i \in (A +Im\ f)\cap IM = I\ (A + Im\ f) + J(R)M \cap ((A+ Im\ f) \cap IM)$

Therefore $\sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} r_i(x_i + y_i) + k$

$x_i \in A, y_i \in Im\ f, \forall\ i = 1, 2, \ldots, n; k \in J(R)\ M \cap (A + Im\ f \cap IM)$

Thus $y = \sum_{i=1}^{n} r_i m_i = \sum_{i=1}^{n} r_i x_i + \sum_{i=1}^{n} r_i y_i + k$, hence

$x - \sum_{i=1}^{n} r_i x_i = \sum_{i=1}^{n} r_i y_i - f(x) + k \in (A \cap Im\ f) + J(R)M \cap ((A+ Im\ f) \cap IM) = 0$

Therefore $x = \sum_{i=1}^{n} r_i x_i \in (A \cap B) \cap IA$.

But $A \cap B$ is AP-pure in M, hence is AP-pure in A. Thus:

$(A\cap B)\cap IA = I(A\cap B)+J(R)M\cap((A\cap B)\cap IA)$.

Thus $x \in I\ (A\cap B) + J(R)\ A \cap ((A\cap B) \cap IA)$

Let $x = \sum_{i=1}^{n} r_i w_i + h$, $w_i \in A \cap B$, $h \in J(R)\, A \cap ((A \cap B) \cap IA)$.

Then $f(x) = \sum_{i=1}^{n} r_i f(w_i) + f(h)$

Now:

$y = x + f(x) = \sum_{i=1}^{n} r_i w_i + \sum_{i=1}^{n} r_i f(w_i) + f(h)$

$= \sum_{i=1}^{n} r_i(w_i + f(w_i)) + f(h) \in IT + J(R)M \cap (T \cap IM)$

Thus $T \cap IM = IT + J(R)\, M \cap (T \cap IM)$ and T is AP-pure in M.

Next, we show that ker f = $(A \cap B) \cap T$

Let $x \in$ ker f, then $x \in A \cap B$ and $f(x) = 0$

Hence $x \in T$, Now let $x \in (A \cap B) \cap T$, then $x = y + f(y)$, $y \in A \cap B$, then:

$x - y = f(y) \in A \cap \text{Im } f$

$\leq (A \cap \text{Im } f) + J(R)M \cap (A + \text{Im} f \cap IM) = 0$

Therefore $f(x) = f(y) = 0$ and $x \in$ ker f

Since M has AP-PIP, then $(A \cap B) \cap T$ = kerf is AP-pure in M.

Conversely, let A and B be AP-pure submodules of M.

Define $f = A \cap B \longrightarrow M$, by:

$f(x) = 0$, $\forall x \in A \cap B$. It is clear:

$(A \cap \text{Im } f) + J(R)M \cap (A + \text{Im } f \cap IM) = 0$

and A + Im f = A is AP- pure in M, then:

ker f = $A \cap B$ is AP-pure in M. ■

By the same argument one can prove the following:

Theorem (2.6):

Let M be an R- module, then M has the AP-PIP if and only if for every AP- pure submodules A and B of M and for every R- homomorphism $f = A \cap B \longrightarrow C$, where C is a submodule of M, such that:

$A \cap C + J(R)M \cap (A + C \cap IM) = 0$

and A + C is AP-pure in M, kerf is AP- pure in M.

Lemma (2.8):

Let $M = \bigoplus_{i \in I} M_i$ where M_i is a submodule of M $\forall$ i and let W_i be a submodule of M_i, for each i, then if W_i is AP- pure in M_i, for each i then $\bigoplus_{i \in I} W_i$ is AP-pure in M.

Proof:

Let J be an ideal in R and:

$x = \sum_{j=1}^{n} r_j x_j \,(\bigoplus_{i \in I} W_i) \cap JM$, $x_j \in (\bigoplus_{i \in I} M_i)$

then:

$x_j = \sum_{i \in I} M_{ij}$, $M_{ij} \in M_i$, for each $i \in I$,

thus:

$x = \sum_{j=1}^{n} r_j \sum_{i \in I} m_{ij} = \sum_{i \in I} \sum_{j=1}^{n} r_j m_{ij}$,

since:

$\sum_{j=1}^{n} r_j m_{ij} \in m_i$, $M = \bigoplus_{i \in I} M_i$,

then the element x can be written uniquely as $\sum_{i \in I} \sum_{j=1}^{n} r_j m_{ij}$; but $x \in \bigoplus_{i \in I} w_i$, thus $\sum_{j=1}^{n} r_j m_{ij} \in w_i$, $\forall$ i, and hence:

$\sum_{j=1}^{n} r_j m_{ij} \in w_i \cap JMi = Jw_i + J(R)M_i \cap (w_i \cap M_i)$

(Because W_i is AP- pure in M_i) So

$\sum_{j=1}^{n} r_j m_{ij} = \sum_{j=1}^{n} r_j w_{ij} + h$, $h \in J(R)\, M_i \cap (W_i \cap JM_i)$

$W_{ij} \in W_i$, for each j. Thus:

$x = \sum_{i \in I} \sum_{j=1}^{n} r_i w_{ij}$

$= \sum_{j=1}^{n} r_j \sum_{i \in I} w_{ij} + h \in J\,(\bigoplus_{i \in I} w_i) + J(R)(\bigoplus_{i \in I} M_i)$

$\cap\, (\bigoplus_{i \in I} W_i \cap J(\bigoplus_{i \in I} M_i))$. ■

The converse is true when

$J(R)M_i \cap W_i = J(R)W_i$.

Lemma (2.9):

Let $M = \bigoplus_{i \in I} M_i$ where M_i is a submodule of M, $\forall$ i and let W_i be a submodule of M_i, for each i, then if $\bigoplus_{i \in I} W_i$ is AP- pure in M, and $J(R)M_i \cap W_i = J(R)\, W_i$ then W_i is AP- pure in M_i, for each i .

Proof:

Assume that $\bigoplus_{i \in I} W_i$ is AP- pure in M. since W_j is a summand of W_i, then W_j is AP- pure in $\oplus W_j$. But $\oplus W_j$ is AP- pure in M, so W_j is

AP-pure in M since W_j is a submodule of M_j, then W_j is AP- pure in M_j.

Proposition (2.10):

Let $M = \bigoplus_{i \in I} M_i$ be an R- module where each M_i is a submodule of M. if M has the AP-PIP, then each M_i has the AP-PIP.

Proof:

Suppose that M has AP-PIP. Since M_i is a summand of M, then M_i is AP- pure in M and hence M_i has the AP-PIP.

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايـد I ولـيكن M مقاسا أيسرا احاديا على R. يسمى المقاس الجزئي N نقيـا تقريبا في M، اذا كان:

$N \cap IM = IN + J(R) M \cap (N \cap IM)$

لكل مثالي I في R. الهدف الرئيسي من هذا البحث هو تطوير خواص المقاسات التي تمتلك خاصية التقاطع النقـي تقريبا.