SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY HOMOTOPY ANALYSIS METHOD

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Abstract

In this article, the homotopy analysis method (HAM) has been employed to obtain the solution of fractional integro-differential equations of the form

$$D_*^{\alpha} y(t) = p(t)y(t) + f(t) + \int_0^t k(t,s)F(y(s))\,ds\,,\ 0 < \alpha < 1$$

Where the fractional derivative is described in the Caputo sense. We shall employ here two approaches based on homotopy analysis method first for the linear fractional integro-differential equations and second for the nonlinear fractional integro-differential equations. This indicates the validity and great potential of the homotopy analysis method for solving such types of equations.

Keywords: homotopy analysis method, fractional integro differential equations.

1. Introduction

Fractional integro-differential equations arise in modeling processes in applied science (such as physics, engineering, finance, biology,…) many problems in acoustics, electromagnetic, viscoelasticity, hydrology and other areas of application can be modeled by fractional differential equations., [Mittal, 2008].

In this paper we shall consider the fractional order integro-differential equation of the type:

$$D_*^{\alpha} y(t) = p(t)y(t) + f(t) + \int_0^t k(t,s)F(y(s))ds, \quad t \in [0,1]$$

$$y(0) = \beta$$

where D_*^{α} is Caputo's fractional derivative and α is a parameter describing the order of the fractional derivative, and F(y(x)) is a non-linear continuous function. Such kind of equations arises in the mathematical modeling of various physical phenomena, such as heat conduction in materials with memory. Moreover, these equations are encountered in combined conduction, convection and radiation problems (see for example [Caputo, 1967], [Olmstead and Handelsman, 1976], [Mainardi, 1997]).

The homotopy analysis method (HAM) was first proposed by Liao in his Ph.D. thesis [Liao, 1992]. A systematic and clear exposition on HAM is given in [Liao, 2003]. In recent years, this method has been successfully employed to solve many types of non-linear, homogenous or nonhomogenous equations and system of equations as well as problems in science and engineering [Jafari, 2009].

The HAM is based on homotopy, a fundamental concept in topology and differential geometry. Briefly speaking, by means of the HAM, one construct a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping and an auxiliary parameter is used to ensure the convergence of solution series. The method enjoys great freedom in choosing initial approximations and auxiliary linear operators. By means of this kind of freedom, a complicated non-linear problem can be transferred into an infinite number of simpler, linear sub- problems [Jaradat, 2008].

In this paper, we present two approaches for solving fractional integro-differential equations based on HAM the first one is the classical HAM presented by [Liao, 1992] is used to solve the non-linear fractional integro-differential equations but we shall use it for the linear fractional integro-differential equations only and the second approach is the reliable algorithm of HAM given by [Odibat, 2010] is used to solve nonlinear fractional integro-differential equation because this approach gives accurate result for nonlinear case in comparison with the first approach.

Illustrative examples are presented to demonstrate the validity and applicability of the two approaches. MATHCAD 2001i computer software used to carry out the computations.

2. Basic Definitions [Manardi, 1997], [Gorenflo, 1997] [Zurigat, 2010]

Definition (2.1):

A real function f(x), x > 0, is said to be in the space $c_\mu, \mu \in \mathbb{R}$ if there exist a real number $p > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x)$ is continuous in $[0,\infty)$, clearly $c_\mu \subset c_\beta$ if $\beta < \mu$.

Definition (2.2):

A function f(x), x > 0 is said to be in the space c_μ^m, $m \in \mathbb{N} \cup \{0\}$, if $f^m \in c_\mu$.

Definition (2.3):

The left sided-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in c_\mu$, $\mu \geq -1$ is defined as:

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(t)}{(x-t)^{1-\alpha}} dt \ , \ \alpha > 0, x > 0$$

$$J^0 f(x) = f(x)$$

In the present section, some of the most important concepts related to HAM are given for completeness.

Definition (2.4):

Let $f \subseteq c^m$, $m \in \mathbb{N} \cup \{0\}$ then the Caputo fractional derivative of f(x) is defined

$$D_\alpha^\mu f(x) = \begin{cases} J^{m-\alpha} f^{(m)}(x), & m-1 < \alpha < m, \ m \in \mathbb{N} \\ \dfrac{d^m f(x)}{dx^m}, & \alpha = m \end{cases}$$

Hence, we have the following properties:

1. $J^\alpha J^v f = J^{\alpha+v}$, $\alpha, v \geq 0$, $f \in c_\mu, \mu \geq -1$.
2. $J^\alpha x^y = \dfrac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma+1)} x^{\gamma+\alpha}, \alpha > 0, \gamma > -1,$ $x > 0$.
3. $J^\alpha D_*^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+) \dfrac{x^k}{k!}$, $x > 0$, $m-1 < \alpha \leq m$.

3. The Homotopy Analysis Method

Consider the non-linear equation in operator form:

$N[y(x)] = 0$

where:

N = Non-linear operator

Y(x) = Unknown function

x = The independent variable.

Let $y_0(x)$ denote an initial guess of the exact solution y(x), $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function and L an auxiliary linear operator with the property L[y(x)]=0 when y(x)=0. Then using $q \in [0,1]$ as an embedding parameter, we construct such a homotopy.

$$(1-q)L[\varphi(x,q) - y_0(x)] = qhH(x)N[\varphi(x,q)] \quad \text{......................(1)}$$

It should be emphasized that we have great freedom to choose the initial guess $y_0(x)$, the auxiliary linear operator L, the non-zero auxiliary function H(x). When q = 0, the zero-order deformation Equation (1) becomes:

$$\phi(x,0) = y_0(x) \quad \text{.....................................(2)}$$

and when q = 1, since $h \neq 0$ and $H(t) \neq 0$, the zero-order deformation Equation (1) is equivalent to:

$$\phi(x,1) = y(x) \quad \text{.......................................(3)}$$

Thus, according to (2) and (3), as the embedding parameter q increases from 0 to 1, $\varphi(x,q)$ varies continuously from the initial approximation $y_0(t)$ to the exact solution y(x). Such a kind of continuous variation is called deformation in homotopy.

By Taylor's theorem, $\varphi(x,q)$ can be expanded in a power series of q as follows:

$$\varphi(x,q) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) q^m \quad \text{............(4)}$$

where

$$y_m(x) = \frac{1}{m!} \frac{\partial^m \varphi(x,q)}{\partial q^m}\bigg|_{q=0} \quad \text{.................... (5)}$$

If the initial guess $y_0(x)$, the auxiliary linear parameter L, the nonzero auxiliary parameter h and the power series (4) of $\varphi(x,q)$ converges at q=1. Then, we have under these assumptions the solution series:

$$y(x) = \varphi(x,1) = y_0(x) + \sum_{m=1}^{\infty} y_m(x) \quad \text{....... (6)}$$

For brevity, define the vector:

$$\bar{y}_n(x) = \{y_0(x), y_1(x), y_2(x), ..., y_n(x)\} \quad (7)$$

According to the definition (4), the governing equation of $y_m(x)$ can be derived from the zero-order deformation equation (1) by differentiating the zero-order deformation equation (1) m times with respective to q and then dividing by m! and finally setting q = 0, we have the so called mth-order deformation equation:

$$L[y_m(x) - \chi_m y_{m-1}(x)] = hH(x)R_m(\bar{y}_{m-1}(x)) \quad \text{...................... (8)}$$

where:

$$R_m(\bar{y}_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x,q)]}{\partial q^{m-1}}\bigg|_{q=0} \quad \text{...................... (9)}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1 \\ 1, & m \succ 1 \end{cases}$$

Note that the high- order deformation equation (7) is governed by the linear operator L and the term $R_m(\bar{y}_{m-1}(x))$ can be expressed simply by (8) for any nonlinear operator N.

According to the definition (8), the right-hand side of equation (7) is only dependent upon $y_{m-1}(x)$. Thus, we gain $y_1(x)$, $y_2(x)$, by mean of solving the linear high-order deformation equation (7) one after the other in order.

4.The First Approach for Solving Fractional Integro Differential Equations

In this section we shall construct a series solution corresponding to the fractional integro differential equations of the form:

$$D_*^{\alpha} y(t) = p(t)y(t) + f(t) + \int_0^t k(t,s)F(y(s))ds, \quad t \in [0,1]$$

$$y(0) = \beta$$

where f, k, F and p are given functions.

For this purpose, let:

$$N[y(t)] = D_*^{\alpha} y(t) - p(t)y(t) - f(t) - \int_0^t K(t,s)F(y(s))ds$$

The corresponding mth-order deformation equation (8) reads:

$$L[y_m(t) - \chi_m y_{m-1}(t)] = hH(t)R_m(\bar{y}_{m-1}(t)) \quad \text{............................. (10)}$$

$$y_m(0) = 0$$

One has:

$$R_m(\bar{y}_{m-1}(x)) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(x,q)]}{\partial q^{m-1}}\bigg|_{q=0}$$

The corresponding homotopy –series solution is given by:

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)$$

It is worth to present a simple iterative scheme for $y_m(t)$ to this end, let F(y(t)) = y(t) the linear operator L is chosen to be L = $\frac{d^{\alpha}}{dt^{\alpha}}$, and $y_0(t) = \beta$ is taken to be an initial guess, and in order to make a comparison with the Adomain Decomposition method a nonzero auxiliary parameter h = −1 and an auxiliary function H(t) = 1 are taken. This is substituted into (10) hence we have the recurrence relation:

$$y_0(t) = \beta \quad \text{... (11)}$$

$$y_1(t) = J^{\alpha}[-D_*^{\alpha}(\beta) + \beta p(t) + f(t) + \beta \int_0^t, K(t,s)\, ds] \quad \text{......................... (12)}$$

and for m ≥ 2

$$y_m(t) = J^{\alpha}[\, p(t) y_{m-1} + \int_0^t K(t,s)\, y_{m-1}\, ds] \quad \text{.. (13)}$$

Thus we gain $y_1(t)$, $y_2(t)$, $y_3(t)$, … . One after one in order.

5. Second Approach for Solving Fractional Integro Differential Equations

The HAM which provides an analytical approximate solution is applied to various nonlinear problems, [Liao, 1992], [Liao, 1995], [Liao, 1997], [Liao, 2003], [Liao, 2004], [Liao, 2007], [Abbasbandy (a), 2008], [Abbasbandy (b), 2008], [Abbasbandy (c), 2008], [Liao, 2009], [Hashim, 2009]. In this section, we present a reliable approach given in [Odibat, 2010] for the HAM. This new modification can be implemented for integer order and fractional order nonlinear equation. To illustrate the basic idea for this new algorithm we consider the following nonlinear fractional integro differential equation.

$$D_*^{\alpha}y(t) = p(t)y + f(t) + N(y), \ \ t > 0 \ \ldots\ldots (14)$$

where $m-1 < \alpha \le m$, N is a nonlinear operator given by

$$N(y) = \int_0^t K(t,s)F(y(s))ds$$

p(t) and g(t) is a known analytic function and D_*^{α} is the Caputo fractional derivative of order α.

In view of the homotopy technique, we can construct the following homotopy:

$$(1-q)L[\varphi(t,q) - \phi_0(t)] = qhH[D_*^{\alpha}\phi(t,q) - p(t)\phi(t,q) - N(\phi(t,q)) - f(t)] \ \ldots (15)$$

Where q ∈ [0, 1] is the embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, φ_0 is an initial guess of y(t) and L is an auxiliary linear operator defined as $L = \frac{d^{\alpha}}{dt^{\alpha}}$, when q = 0 equation (15) becomes,

$$L[\phi(t,0) - \phi_0(t)] = 0 \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

Its obvious that when q = 1, equation (15) becomes the original nonlinear equation (14).then as q varies from 0 to 1, the solution y(x, q) varies from the initial guess $y_0(t)$ to the solution $\varphi(t,1)$.the basic assumption of this new approach is that the solution of equation (15) can be expressed as a power series in q,

$$\varphi = \phi_0 + q\phi_1 + q^2\phi_2 + \ldots \ \ \ldots\ldots\ldots\ldots\ldots (17)$$

Substituting the series (17) into the homotopy (15) and then equating the coefficients of the like powers of q, we obtain the high- order deformation equations

$$L[\phi_1] = hH(D_*^{\alpha}\phi_0 - p(t)\phi_0 - N(\phi_0) - f(t))$$

$$L[\phi_2] = L[\phi_1] + hH(D_*^{\alpha}\phi_1 - p(t)\phi_1 - N(\phi_0,\phi_1)) \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

$$L[\varphi_3] = L[\phi_2] + hH(D_*^{\alpha}\phi_2 - p(t)\phi_2 - N(\phi_0,\phi_1,\phi_2))$$

$$L[\varphi_4] = L[\phi_3] + hH(D_*^{\alpha}\phi_3 - p(t)\phi_3 - N(\phi_0,\phi_1,\phi_2,\phi_3))$$

Where:

$$N(\phi_0 + q\phi_1 + q^2\phi_2 + \ldots) = N_0(\phi_0) + qN_1(\phi_0,\phi_1) + q^2N_2(\phi_0,\phi_1,\phi_2)$$

and

$$N_0(\phi_0) = \int_0^t k(t,s)A_0(s)ds,$$

$$N_1(\phi_0,\phi_1) = \int_0^t k(t,s)A_1(s)ds$$

$$N_2(\phi_0,\phi_1,\phi_2) = \int_0^t k(t,s)A_2(s)ds$$

Generally

$$N_n(\phi_0,\phi_1,\phi_2,\ldots,\phi_n) = \int_0^t k(t,s)A_n(s)ds$$

where

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}F\left(\sum_{i=0}^{\infty}\lambda^i y_i\right)\right]_{\lambda=0}, \ \ n = 0,1,2,\ldots$$

The approximate solution of equation (14), therefore, can be readily obtained,

$$y = \lim_{q\to 1}\phi = \phi_0 + \phi_1 + \ldots \ \ \ldots\ldots\ldots\ldots\ldots (19)$$

The success of the technique is based on the proper selection of the initial guess ϕ_0 applying the operator J^{α} .to both sides of equation (14) give

$$y(t) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{t^k}{k!} + J^{\alpha}(p(t)y) + J^{\alpha}(f(t) + J^{\alpha}(N(y(t))) \ldots\ldots\ldots (20)$$

Neglecting the nonlinear term $J^{\alpha}(N(y(t))$ and $J^{\alpha}(p(t)y)$ on the right hand side, we can use the remaining part as the initial guess of the solution that is

$$\phi_0(t) = \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{t^k}{k!} + +J^{\alpha}(f(t)) \text{ ...... (21)}$$

and by solving equations (18) we get ϕ_1, ϕ_2, ϕ_3, …; respectively.

6. Illustrative Examples

In this section we will apply the two approaches based on HAM presented in section four and five to the linear/nonlinear fractional integro-differential equation, respectively:

Example (1):

Consider the following fractional integro-differential equation

$$y^{(0.75)}(t) = \left(-\frac{t^2 e^t}{5}\right) y(t) + \frac{6t^{2.25}}{\Gamma(3.25)} + \int_0^t e^t s y(s) ds$$

$$y(0) = 0$$

where the exact solution is $y(t) = t^3$.

And according to equations (11), (12) and (13), we have the following approximations

$$y_0(t) = 0$$

$$y_1(t) = J^{0.75}\left\{\frac{6t^{2.25}}{\Gamma(3.25)}\right\}$$

$$y_1(t) = t^3$$

$$y_2(t) = \frac{-1}{5} J^{0.75}(t^2 e^t y_1(t)) + J^{0.75}\left(\int_0^t e^t s y_1(s)\, ds\right)$$

$$y_2(t) = 0$$

for $m \geq 3$

$$y_m(t) = 0$$

Hence we have got the exact solution $y(t) = t^3$.

Example (2):

Consider the following nonlinear equation:

$$D_*^{\alpha} y(t) = 1 + \int_0^t e^{-x} y^2(x)\, dx \text{ .... (22)}$$

$$y(0) = 1$$

According to the second approach given by section five and by equation (21) and with $h = -1$, $H = 1$in equation (18), we have:

$$\phi_0(t) = y(0) + J^{\alpha}(1) = 1 + J^{\alpha}(1)$$

$$\phi_1(t) = J^{\alpha}(N_0(\phi_0))$$

$$\phi_2(t) = J^{\alpha}(N_1(\phi_0, \phi_1))$$

$$\phi_3(t) = J^{\alpha}(N_2(\phi_0, \phi_1, \phi_2))$$

where:

$$N_0(\phi_0(t)) = \int_0^t e^{-x}(\phi_0(x))^2\, dx$$

$$N_1(\phi_0, \phi_1) = \int_0^t e^{-x}(2\phi_0(x)\phi_1(x))\, dx$$

$$N_2(\phi_0, \phi_1, \phi_2) = \int_0^t e^{-x}(2\phi_0(x)\phi_2(x) + (\phi_1(x))^2)\, dx$$

$$\vdots$$

In order to avoid difficult fractional integral, we can simplify the integration by taking the truncated Taylor expansions for the exponential term

$$e^{-x} \cong 1 - x + \frac{x^2}{2} - \frac{x^3}{6}$$

Following Fig. (1) shows the approximate solution of (22) for α = 0.5, 1, which is obtained after four iterations. The solution for α = 1 is the only case for which we know the exact solution as mentioned in [Mittal, 2008] and our solution is in good agreement with the exact values.

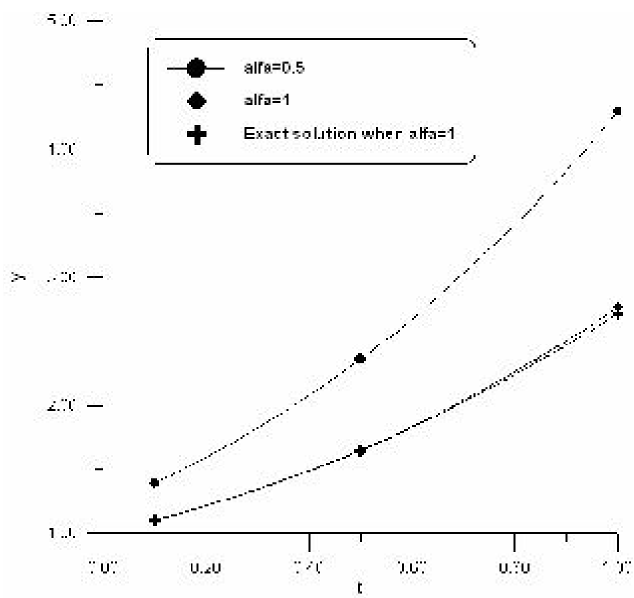

Fig. (1) : The approximate solution of example (2) with the exact solution.

Conclusion

In the paper we have taken two examples the first example is the same as solved by [Rawashedeh, 2005] using the collocation spline method.He obtained the solution having absolute error of order 10^{-4} and [Mittal, 2008] by using the Adomain decomposition method and the second example is nonlinear which is the same solved by [Mittal, 2008].

Setting h = −1 and H = 1 in (10) and (18) which gives the formulae used for solving example (1) and (2) respectively gives the same solution given by [Mittal, 2008] using the Adomain decomposition method this illustrates that the Adomain decomposition method is indeed special case of the HAM.

The HAM provides us with a simple way to adjust and control the convergence region of solution series by choosing a proper value for the auxiliary parameter h, auxiliary function H(x), auxiliary operator L.

6. References

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الخلاصة

في هذا البحث طريقة تحليل الهوموتوبي (Homotopy analysis method) تم تطبيقها للحصول على حل المعادلة التفاضلية التكاملية ذات الرتب الكسورية ومن النوع

$$D_*^{\alpha} y(t) = p(t)y(t) + f(t) + \int_0^t k(t,s)F(y(s))ds,$$

$$0 < \alpha < 1$$

حيث أن المشتقة الكسورية هي من نوع كابوتو (Caputo fractional derivative). سوف نقوم بتطبيق اسلوبين بالأعتماد على طريقة تحليل الهوموتوبي الأول سوف يتم تطبيقه على المعادلات التفاضلية التكاملية الخطية ذات الرتب الكسورية والثاني على المعادلات التفاضلية التكاملية الغير خطية ذات الرتب الكسورية وهذا يشير الى الامكانية العالية وصحة استخدام طريقة تحليل الهوموتوبي في حل هذا النوع من المعادلات.