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$\mathcal{R}\alpha$ –COMPACTNESS ON BITOPOLOGICAL SPACES

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Abstract

In this paper we define a new type of open sets in bitopological space which we called $\mathcal{R}\alpha$ –open sets, which leads to define a new type of compactness on bitopological spaces called " $\mathcal{R}\alpha$ –compactness" and we study the properties of this spaces, also we define the continuous functions between these spaces.

1.Introduction

The concept of "bitopological space" was introduced by Kelly [1] in 1963. A set equipped with two topologies is called a "bitopological space" and denote by (X, τ_1, τ_2), where (X, τ_1), (X, τ_2) are two topologicals paces. Since then many authors have contributed to the development of various bitopological properties. A subset A in bitopological space (X, τ_1, τ_2) is "S- open " if it is τ_1-open or τ_2- open. in 1996 Mrsevic and Reilly [5] defined a space (X, τ_1, τ_2) to be S-compact if and only if every S-open cover of X has a finite sub cover. And also they defined a space (X, τ_1, τ_2) to be pair-wise compact [5]. In this paper we introduced a new type of compactness on bitopological spaces namely " $\mathcal{R}\alpha$ –compact" and we review some remarks, propositions, theorems and examples about it.

2. Preliminaries

In this section we introduce some definitions. Which is necessary for the paper.

Definition 2.1 [1]:

Let X be a non-empty set, let τ_1, τ_2 be any two topologies on X, then (X, τ_1, τ_2) is called " ***bitopological space***".

Definition 2.2[2]:

A subset A of a topological space X is called ***"α-open set"*** if and only if $A \subseteq \overline{A^{\circ}}^{\circ}$.
The family of all α-open sets is denoted by τ_α.

Definition 2.3[2]:

The complement of α-open set is called ***"α-closed set"***. The family of all α-closed sets is denoted by αC(X).

Proposition 2.4[3]:

(i) Every open set is α-open set .
(ii) Every closed set is α-closed set.

Corollary 2.5 :

Let (X, τ) be a topological space, then τ_α is finer than τ.

Definition 2.6 [4] :

Let (X, τ) be a topological space, a family $\mathfrak{A}$ of subsets of X is said to be an ***"α-open cover of X "*** if and only if $\mathfrak{A}$ covers X and $\mathfrak{A} \subseteq \tau_\alpha$.

Definition 2.7 [4]:

Let $\mathfrak{A}$ be any α-open cover of X, a subfamily $\mathfrak{B}$ of $\mathfrak{A}$ is said to be an "***α-open sub cover of*** $\mathfrak{A}$ "if and only if it is cover X.

Definition 2.8 [4]:

Let (X, τ) be a topological space, any subspace A of X is said to be ***"α-compact"*** if and only if every α-open cover of A has a finite sub cover.

Proposition 2.9 [4]:

Every α-compact space is compact.

3. $\mathcal{R}\alpha$ –Compactness

In this section, we will define a new type of covers in bitopological spaces, in order to define a new kind of compactness on bitopological space called " $\mathcal{R}\alpha$ –compactness".

First we begin with the definition of $\mathcal{R}\alpha$ −open set in bitopological space .

Definition 3.1 :

Let (X, τ_1, τ_2) be a bitopological space, then any collection of subsets of X which is contained $\tau_{1\alpha}$ *and* $\tau_{2\alpha}$ and it is forms a topology on X called "***the supermom topology on X*** " and is denoted by $\tau_{1\alpha} \vee \tau_{2\alpha}$. Where $\tau_{1\alpha}$ is the family of all α-open sets in the space (X, τ_1) and $\tau_{2\alpha}$ is the family of all α-open sets in the space (X, τ_2).

Definition 3.2 :

A subset A of a bitopological space (X,τ_1, τ_2) is said to be an ***"$\mathcal{R}\alpha$ −open set"*** if and only if it is open in the space (X,$\tau_{1\alpha} \vee \tau_{2\alpha}$), where $\tau_{1\alpha} \vee \tau_{2\alpha}$ is the supermom topology on X contains $\tau_{1\alpha}$ *and* $\tau_{2\alpha}$.

Definition 3.3:

The complement of an $\mathcal{R}\alpha$ −open set in a bitopological space (X, τ_1, τ_2) is called ***" $\mathcal{R}\alpha$ −closed set "***.

Remark 3.4:

Let (X, τ_1, τ_2) be a bitopological space, then :

(1) Every α-open set in (X, τ_1) or (X, τ_2) is an $\mathcal{R}\alpha$ −open set in (X, τ_1, τ_2).

(2) Every α-closed set in (X, τ_1) or (X, τ_2) is an $\mathcal{R}\alpha$ −closed set in (X, τ_1, τ_2).

Note 3.5:

The opposite direction of remark (3.4) is not true as the following example shows:

Example (1):

Let X={1,2,3}, τ_1={$\emptyset$,{1},X}, and τ_2={$\emptyset$,{2,3},X} then $\tau_{1\alpha} = \tau_1 \cup$ {{1,2}, {1,3}}, and $\tau_{2\alpha} = \tau_2$. thus $\tau_{1\alpha} \vee \tau_{2\alpha}$ = {$\emptyset$,{1},{1,2},{1,3},{2,3},{2},{3},X} is the family of all $\mathcal{R}\alpha$ −open sets in (X,τ_1, τ_2). {3} is an $\mathcal{R}\alpha$ − open set in (X, τ_1, τ_2) but it is not α-open set of both (X, τ_1) and (X, τ_2). So {1,2} is an $\mathcal{R}\alpha$ −closed set in (X, τ_1, τ_2) which is not α-closed in both (X, τ_1) and (X, τ_2).

Now we introduce the definition of $\mathcal{R}\alpha$ −opencover in bitopological space (X,τ_1, τ_2).

Definition 3.6:

Let (X, τ_1, τ_2) be a bitopological space, let A be a subset of X. a sub collection of the family $\tau_{1\alpha} \vee \tau_{2\alpha}$ is called an **"$\mathcal{R}\alpha$ −opencover of A"** if the union of members of this collection contains A.

Definition 3.7:

A bitopological space (X, τ_1, τ_2) is said to be **"$\mathcal{R}\alpha$ −compact space"** if and only if every $\mathcal{R}\alpha$ −opencover of X has a finite subcover.

Theorem 3.8:

If (X, τ_1, τ_2) is $\mathcal{R}\alpha$ −compact space, then both (X, τ_1) and (X, τ_2) are α −compact.

***Proof*:**

To prove (X, τ_1) is α −compact space, we must prove for any α-open cover of X, has a finite sub cover .

Let $\{U_i\} i \in \Lambda$ be any α-open cover of X, implies $\{U_i\} i \in \Lambda$ is an $\mathcal{R}\alpha$ −opencover of X (by remark (3.4)) and since (X, τ_1, τ_2) is $\mathcal{R}\alpha$ −compact space, implies there exists a finite sub cover of X, so (X, τ_1) is α −compact. Similarly, we prove (X, τ_2) is α −compact. ■

Corollary 3.9:

If (X, τ_1, τ_2) is $\mathcal{R}\alpha$ −compact space, then both (X, τ_1) and (X, τ_2) are compact.

***Proof*:**

The proof is follows from theorem(3.8) and proposition (2.9). ■

Remark 3.10:

The converse of theorem (3.8) and it's corollary is not true, as the following example shows:

Example (2):

Let X={0,1}, τ_1=={$\emptyset$,{0},X}, and τ_2={$\emptyset$,{1},X} then $\tau_{1\alpha} = \tau_1$ *and* $\tau_{2\alpha} = \tau_2$.
Now, both (X, τ_1) and (X, τ_2) are α −compact (compact) space, but (X, τ_1, τ_2) is not

$\mathcal{R}\alpha$ −compact space since there is {{0},{1}} is an $\mathcal{R}\alpha$ −opencover of X which has no finite sub cover.

The converse of theorem (3.8) becomes valid in a special case, when $\tau_{1\alpha} \subset \tau_{2\alpha}$, as the following proposition shows:

Proposition 3.11:

If $\tau_{1\alpha}$ is a subfamily of $\tau_{2\alpha}$, then (X,τ_1, τ_2) is an $\mathcal{R}\alpha$ −compact space if and only if (X, τ_2) is α −compact.

***Proof*:**

The first direction follows from theorem (3.8).

Now, if (X, τ_2) is α −compact, we must prove (X, τ_1, τ_2) is $\mathcal{R}\alpha$ −compact. since $\tau_{1\alpha} \subset \tau_{2\alpha}$, then $\tau_{1\alpha} \vee \tau_{2\alpha} = \tau_{2\alpha}$. So (X, τ_1, τ_2) is $\mathcal{R}\alpha$ −compact space. ∎

Corollary 3.12:

Let (X, τ) be a topological space, then the bitopological space (X, τ, τ_α) is $\mathcal{R}\alpha$ −compact space if and only if (X, τ_α) is α −compact.

***Proof*:**

($\Rightarrow$) *it is* clear from theorem (3.8).

($\Leftarrow$) since τ_α is a finer than τ, then by proposition (3.11) we have (X, τ, τ_α) is $\mathcal{R}\alpha$ −compact. ∎

Proposition 3.13:

If A and B are two $\mathcal{R}\alpha$ −compact subsets of a bitopological space (X, τ_1, τ_2) then $A \cup B$ is an $\mathcal{R}\alpha$ −compact subset of X .

***Proof*:**

To prove $A \cup B$ is an $\mathcal{R}\alpha$ −compact subset of X, we must prove for any $\mathcal{R}\alpha$ −opencover of $A \cup B$, it has a finite sub cover.

Let $\{U_i\}\ i \in \Lambda$ be any $\mathcal{R}\alpha$ −opencover of $A \cup B$, then $A \cup B \subseteq \{\cup U_i, i \in \Lambda\}$ and therefore $A \subseteq \cup U_i$ and B $\subseteq \cup U_i$, implies $\{U_i\}\ i \in \Lambda$ is an $\mathcal{R}\alpha$ −opencover of A and B .

But A and B are $\mathcal{R}\alpha$ −compact subsets, therefore there exists $i_1, i_2, \dots, i_n \in \Lambda$ and $i_1, i_2, \dots, i_m \in \Lambda$ such that $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\}$ and $\{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ is a finite sub cover of A and B respectively, then $\{U_{i_1}, U_{i_2}, \dots, U_{i_n}\} \cup \{U_{i_1}, U_{i_2}, \dots, U_{i_m}\}$ is a finite sub cover of A ∪ B, therefore $A \cup B$ is an $\mathcal{R}\alpha$ −compact subset of X . ∎

Remark 3.14:

If A and B are two $\mathcal{R}\alpha$ −compact subsets of a bitopological space (X, τ_1, τ_2) then $A \cap B$ need not to be $\mathcal{R}\alpha$ −compact subset of X , for example:

Example (3):

Let X=$\mathbb{N} \cup \{0, -2\}$, (where $\mathbb{N}$ is the set of natural numbers) and let $\tau = \mathbb{P}(\mathbb{N}) \cup \{U \subseteq X; 0, -2 \in U \wedge X - U$ finite }.

Then $\tau_\alpha = \tau \cup \{U \subseteq X; (0 \in U\ or -2 \in U) \wedge X - U\ is\ finite\}$.

Now, let A= $\mathbb{N} \cup \{0\}$ and B= $\mathbb{N} \cup \{-2\}$, then both A and B are α −compact subsets of a topological space (X, τ). And since τ_α is finer than τ, therefore $\tau \vee \tau_\alpha = \tau_\alpha$. hence A and B are both $\mathcal{R}\alpha$ −compact subset of a bitopological space (X, τ, τ_α) .

Since every $\mathcal{R}\alpha$ −opencover of A (of B, respectively) must contain an $\mathcal{R}\alpha$ −openset, say V such that 0 $\in V$ ($-2 \in V$, respectively), whose complement is finite. so, V together with a finite number of $\mathcal{R}\alpha$ −opensets of the cover will cover A (cover B, respectively).

But $A \cap B = \mathbb{N}$ is not an $\mathcal{R}\alpha$ −compact, since {{n}; n $\in \mathbb{N}$} is an $\mathcal{R}\alpha$ −opencover of $A \cap B$, which has no finite sub cover.

Theorem 3.15:

The $\mathcal{R}\alpha$ −closed subset of an $\mathcal{R}\alpha$ −compact space is $\mathcal{R}\alpha$ −compact.

***Proof*:**

Let (X, τ_1, τ_2) be an $\mathcal{R}\alpha$ −compact space and let A be an $\mathcal{R}\alpha$ −closed subset of X. to show that A is an $\mathcal{R}\alpha$ −compact set.

Let $\{U_i\}\ i \in \Lambda$ be any $\mathcal{R}\alpha$ −opencover of A. Since A is $\mathcal{R}\alpha$ −closed subset of X, then X-A is an $\mathcal{R}\alpha$ −open subset of X, so {X-A}

$\cup\{U_i;\ i\in\Lambda\}$ is an $\mathcal{R}\alpha-$opencover of X, which is $\mathcal{R}\alpha-$compact space.

Therefore, there exists $i_1,i_2,\dots,i_n\in\Lambda$ such that {X-A, $U_{i_1},U_{i_2},\dots,U_{i_n}$} is a finite sub cover of X .as $A\subseteq X$ and X-A covers no part of A, then $\{U_{i_1},U_{i_2},\dots,U_{i_n}\}$ is a finite sub cover of A. so A is $\mathcal{R}\alpha-$compact set. ■

Definition 3.16:

A function $f:(X,\tau_1,\tau_2)\to(Y,\tau_1',\tau_2')$ is said to be *" $\mathcal{R}\alpha-$continuous function "* if and only if the inverse image of each $\mathcal{R}\alpha-$open subset of Y is an $\mathcal{R}\alpha-$open subset of X.

Theorem 3.17:

The $\mathcal{R}\alpha-$*continuous* image of an $\mathcal{R}\alpha-$compact space is an $\mathcal{R}\alpha-$compact space.

Proof:

Let (X, τ_1, τ_2) be an $\mathcal{R}\alpha-$compact space, and let f: (X, τ_1, τ_2) $\to(Y,\tau_1',\tau_2')$ be an $\mathcal{R}\alpha-$continuous, onto function. To show that (Y,τ_1',τ_2') is an $\mathcal{R}\alpha-$compact space. Let $\{U_i;\ i\in\Lambda\}$ be an $\mathcal{R}\alpha-$opencover of Y, then $\{f^{-1}(U_i);\ i\in\Lambda\}$ is an $\mathcal{R}\alpha-$opencover of X, which is $\mathcal{R}\alpha-$compact space.

So there exists $i_1,i_2,\dots,i_n\in\Lambda$, such that the family $\{f^{-1}\left(U_{i_j}\right); j=1,2,\dots,n\}$ covers X and since f is onto, then $\{U_{i_j}; j=1,2,\dots,n\}$ is as finite sub cover of Y.

Hence Y is an $\mathcal{R}\alpha-$compact space. ■

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الخلاصة

قمنا في هذا البحث بتعريف نوع جديد من المجموعات المفتوحة على الفضاءات التبولوجية الثنائية والتي اسـميناها المجموعات المفتوحة $\mathcal{R}\alpha-$. وبالتالي عرفنا نوع جديد من التراص على الفضاءات الثنائية والذي اسميناه التراص من نوع $\mathcal{R}\alpha-$، وقمنا بدراسة خواص هذا الفـضاء وكـذلك عرفنا الدوال المستمرة بين هذه الفضاءات.