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USING DIRECT RITZ METHOD FOR SOLVING ELECTROCHEMICAL MACHINING MOVING BOUNDARY PROBLEM

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Abstract

In this paper, the objective is to derive the variational formulation of the electrochemical machining problem (ECM for short) and to evaluate the numerical solution using the direction Ritz method. This problem is of degenerate problem which has so many difficulties to be solved using other approach.

1. Introducation

As an alternative to definition to mechanical machining problem, a piece of metal can sometimes be shaped by using it as an anode in electrolytic cell with an appropriately shaped cathode. This represent a moving boundary value problem**,** because the anode surface is moved toward the cathode at constant velocity, and products of the erosion of the anode are swept away by the electrolytes, which is pumped through the space between the electrodes, [12].

2. The mathematical model of the problem

The two dimensional ECM problem will be considered for two electrodes shown in Fig. (1). the space between the electrodes is filled by an appropriate electrolyte. A voltage is placed across the electrodes and these causes are removal of material from the anode, [5]. We will solve equation ECM of an anode surrounded by circular cathode, the conductivity in the gap between the electrodes is considered constant [5]. This problem is formally identical to one-phase Stefan problem with zero heat capacity [13].

Referring to Fig.(2), suppose that the anode is the shrinking region A(t), with moving boundary Γt and Γo denote the initial anode surface at t=0. The region inside the cathode surface C is denoted by D and the region occupied by the electrolyte by Dt, so that D includes A(t) and Dt. Also, it is convenient to define the moving boundary Γt, ∀t ≥ 0 by

$$\Gamma t = s(\theta, t) \quad \ldots\ldots (1)$$

Where

$\Gamma = \{(r,\theta): r = 1, \quad 0 < \theta < 2\pi\}$.

An approximate model for the process is given by, [5]:

$$\tilde{\nabla}^2 \tilde{\phi} = 0 \text{ in Dt} \quad \ldots\ldots (2)$$

$$\left.\begin{aligned} \tilde{\phi} &= 0 \quad on\ C \\ \tilde{\phi} &= g \quad on\ \Gamma_t \end{aligned}\right\} \quad \ldots\ldots (3)$$

$$\frac{d\tilde{R}}{dt} = M\,\tilde{\nabla}\,\tilde{\phi}_a \text{ on } \Gamma t \quad \ldots\ldots (4)$$

Where Dt is defined as

Dt = {(r, θ,t) ,sit, a ≤ r ≤ s(θ,t): 0≤θ≤2π : 0≤t≤T}.

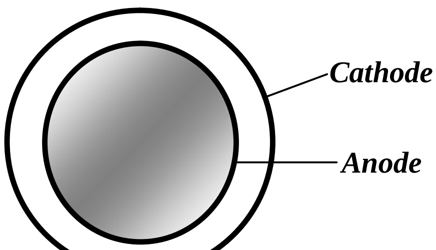

Fig. (1) : Basic Configuration of ECM.

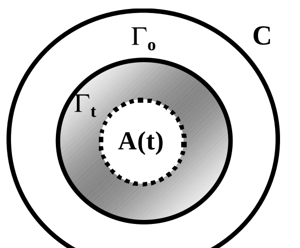

Fig. (2) : Annular ECM.

It should be noted that the annular electrochemical machining boundary conditions (3) are based on the assumption that the effects of over potentials could be ignored.

The particular problem we treated in detail consists of circular cathode of radius c with an anode inside it. We measure all lengths in unite of αc, where α is a positive non-dimensional constant.

Then∇, (the gradient operator of the non-dimensional length), and $\tilde{\nabla}$ are connected by $\nabla=\frac{\tilde{\nabla}}{\alpha c}$ We define further a non-dimensional potential by $\tilde{\phi}=g\phi$ such that eqs .(2), (3) becomes :

$\nabla^2\phi = 0$, in the electrolyte............................ (5)

With the boundary conditions

$$\left.\begin{array}{ll}\phi = 0 & \text{if } r = a \text{ on the cathode} \\ \phi = 1 & \text{if } r = s(\theta, t) \text{ on the anode}\end{array}\right\} \quad \text{.......... (6)}$$

A non-dimensional time variable T = (Mg αc) t is defined and eq. (2.24) becomes [6]:

$$\frac{dR}{dT}=\nabla\phi\Big|_{anode} \quad ,\text{...} \quad (7)$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$

The discretization of potential equation is straightforward. However, the free boundary condition on the anode must be transformed into an expression in terms of the anode speed along each ray. Rewriting ∇ϕ and $\frac{dR}{dt}$ in polar coordinated system see [9],

$$\left(\frac{\partial\phi}{\partial r},\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)=\left(\frac{dr}{dt},r\frac{d\theta}{dt}\right).$$

where r = s (θ, t) on the anode and this implies $\frac{\partial\phi}{\partial r}=\frac{dr}{dt}$ and $\frac{1}{r}\frac{\partial\phi}{\partial\theta}=r\frac{d\theta}{dt}$.

Now, $\frac{dr}{dt}=\frac{\partial s}{\partial t}+\frac{\partial s}{\partial\theta}\frac{d\theta}{dt}$

$$=\frac{\partial s}{\partial t}+\frac{\partial s}{\partial\theta}\frac{1}{r^2}\frac{\partial\phi}{\partial\theta} \quad \text{............................} \quad (8)$$

It is convenient to replace $\frac{\partial\phi}{\partial\theta}$ by $\frac{\partial\phi}{\partial r}$.

Since the tangential derivative vanishes on the anode, then:

$$\left(\frac{\partial\phi}{\partial r}\frac{\partial r}{\partial\theta}+\frac{\partial\phi}{\partial\theta}\frac{\partial\theta}{\partial\theta}\right)=0 \Big|_{r=s(\theta,t)}$$

Then:

$$\frac{\partial\phi}{\partial\theta}=-\frac{\partial s}{\partial\theta}\frac{\partial\phi}{\partial r} \quad \text{..} \quad (9)$$

Now substituting eq. (9) in eq. (8), so the gradient condition on the anode surface leads to:

$$\frac{\partial\phi}{\partial r}=\frac{\partial s}{\partial t}+\frac{\partial s}{\partial\theta}\frac{1}{s^2}\left[-\frac{\partial s}{\partial\theta}\frac{\partial\phi}{\partial r}\right],$$

Hence

$$\frac{\partial s}{\partial t}=\frac{\partial\phi}{\partial r}+\frac{1}{s^2}\frac{\partial s}{\partial\theta}\frac{\partial s}{\partial\theta}\frac{\partial\phi}{\partial r}$$

$$=\frac{\partial\phi}{\partial r}+\left(\frac{1}{s}\frac{\partial s}{\partial\theta}\right)^2\frac{\partial\phi}{\partial r}$$

Therefore:

$$\frac{\partial s}{\partial t}=\left[1+\left(\frac{1}{s}\frac{\partial s}{\partial\theta}\right)^2\right]\frac{\partial\phi}{\partial r}$$

Or equivalently:

$$\frac{\partial\phi}{\partial r}\Big|_{r=s(\theta,t)}=\frac{\partial s/\partial t}{1+\left(\frac{1}{s}\frac{\partial s}{\partial\theta}\right)^2}\equiv f(\theta,t), \text{............} (10) \quad \text{..... (7)}$$

So the final mathematical model of the problem is:

$\nabla^2\phi=0$ in the electrolyte .... (11)

$\phi=0$ on the cathode, where r = a.................. (12)

$\phi=1$ on the anode, where r = s (θ, t) .... (13)

$\phi_r=f(\theta, t)$ on the anode................................ (14)

Where

$$f(\theta, t)=\frac{\partial s/\partial t}{1+\left(\frac{1}{s}\frac{\partial s}{\partial\theta}\right)^2} \quad \text{,see[10]}$$

3. The Variational Formulation of the ECM Problem

As its known, variational methods are one of the most important approaches that could be used to solve many complicated problems of mathematical physical and chemical in general, and moving and free boundary value problems, in particular. To solve the problem under consideration through variational approach, a variational formulation corresponding to the problem must be derived:

To make such a formulation, first let <u, v> be the symmetric, non-degenerate, bilinear form defined by:

$$\langle u, v\rangle = \iiint_D u\,v\,r\,dr\,d\theta dt \;........................ (15)$$

Where u: D → and v: D → .

Because of the importance of the linear operator used in the derivation of the variational formulation related to Laplace's equation used in the ECM problem, we shall prove next the symmetry of the Laplace's operator, $L=\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$ relative to the chosen bilinear form.

To show the symmetry of Laplace's operator in polar coordinate system, we must prove that [10]:

<Lu, v> = <Lv, u>

Consider

<Lu,v>

$$= \iiint_D \left\{\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}\right\} v\,r\,dr\,d\theta dt$$

$$= \iiint_D \left\{r\frac{\partial^2 u}{\partial r^2}v+\frac{\partial u}{\partial r}v+\frac{1}{r}\frac{\partial^2 u}{\partial\theta^2}v\right\} dr\,d\theta dt$$

$$= \iiint_D \left[\frac{\partial}{\partial r}\left\{r\frac{\partial u}{\partial r}v\right\}+\frac{\partial}{\partial\theta}\left\{\frac{1}{r}\frac{\partial u}{\partial\theta}v\right\}\right] dr d\theta dt$$

$$-\iiint_D \left[r\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\frac{\partial v}{\partial\theta}v\right] dr\,d\theta dt$$

By using divergence theorem on the first integration [10], we get:

$$\langle Lu,v\rangle=\iint_{\partial D}\left[r\frac{\partial u}{\partial r}v d\theta-\frac{1}{r}\frac{\partial u}{\partial\theta}v dr\right]-$$

$$\iiint_D \left[r\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\frac{\partial v}{\partial\theta}\right] dr\,d\theta dt$$

However, since v=constant on the boundary of D and consequently the line integral equal zero. Hence:

$$\langle Lu, v\rangle=-\iiint_D \left[r\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\frac{\partial v}{\partial\theta}\right] dr\,d\theta dt$$

Similarly:

$$\langle Lv, u\rangle=-\iiint_D \left[r\frac{\partial v}{\partial r}\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}\frac{\partial u}{\partial\theta}\right] dr\,d\theta dt$$

Therefore <Lu, v> = <Lv, u>, which men that L is symmetric relative to the non-degenerate bilinear form <. >.

Now, by using Margi's theorem, which states that (There is a variational formulation corresponding to the linear equation Lu=f, if and only if the operator L is symmetric relative to the bilinear form which is non-degenerate), where the functional is given by :

$$F[u]=\frac{1}{2}\langle Lu, u\rangle-\langle f, u\rangle .... (16)$$

The functional (16) may be simplified to be in the form

$$F[\phi]=\frac{1}{2}\iiint_D\left\{\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right\}\phi r\,dr\,d\theta dt \;..(17)$$

and (17) can be reduced to more familiar form as follows :

$$F[\phi]=\frac{1}{2}\iiint_D\left\{\frac{\partial^2\phi}{\partial r^2}\phi r+\frac{\partial\phi}{\partial r}\phi+\frac{1}{r}\frac{\partial^2\phi}{\partial\theta^2}\phi\right\}dr\,d\theta dt$$

$$=\frac{1}{2}\iiint_D\left\{\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\phi\right)+\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\phi\right)\right\}dr\,d\theta dt,$$

$$-\frac{1}{2}\iiint_D\left\{r\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right\}dr\,d\theta dt$$

And by using divergence theorem, we have:

$$F[\phi]=\frac{1}{2}\int\int_{\partial D}\left\{r\frac{\partial\phi}{\partial r}\phi d\theta-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\phi dr\right\}-\frac{1}{2}\iiint_D\left\{r\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right)^2\right\}dr\,d\theta\,dt$$

Since ϕ is constant on the boundary of D, then the above line integral equals zero, and the final version of the functional F[ϕ] is given by :

$$F[\phi]=-\frac{1}{2}\iiint_D\left\{r\phi_r^2+\frac{1}{r}\phi_\theta^2\right\}dr\,d\theta dt \;............. (18)$$

As a notation, it is important to notice that the critical points of the functional (18) are the solution of the ECM problem. Therefore instead of solving (1)-(4), we may find the critical points of the functional (18) and this has its' basis on Margi's theorem [11].

4. Numerical Solution of the Problem

As a numerical application, consider the electrochemical machining moving B.V.P. governed by [2]:

$$\phi_{rr}+\frac{1}{r}\phi_r+\frac{1}{r^2}\phi_{\theta\theta}=0 \qquad \text{In Dt}$$

$\phi = 0$ if $r = a$ on the cathode

$\phi = 1$ if $r = s(\theta, t)$ on the anode

$\phi_r = f(\phi,t)$ $r = s(\theta, t)$ on the anode

Where:

$$f(\phi, t) = \frac{\partial s/\partial t}{1+\left(\frac{1}{s}\frac{\partial s}{\partial\theta}\right)^2}$$

From the chemical and physical interpretation of the problem, and for numerical solution propose the moving boundary $s(\theta,t)$ of this problem requires the following conditions to be satisfied :-

1-When θ increases, $s(\theta, t)$ increases.

2-When t increases, $s(\theta, t)$ decreases.

3-When $t = 0$, $s(\theta, t) = s_0(\theta)$, where $s_0(\theta)$ is the initial moving boundary.

The following definition of $s(\theta,t)$ may be consider, which satisfies the above three conditions,

$$s(\theta, t) = s_0 - (a_1 + a_2(\pi-\theta)^2)t\, e^{a_3 t},$$

$$0 \le \theta \le \pi,\quad 0 \le t \le T,$$

Where a_1, a_2 are constant to be determined and so is given

Now, instead of solving the problem analytically which is so difficulty or impossible, we can find the critical points of the functional numerically:

$$F[\phi]=\int_0^T\int_0^\pi\int_{s(\theta,t)}^a\left(r\phi_r^2+\frac{1}{r}\phi_\theta^2\right)dr\,d\theta\,dt \quad ....\ (19)$$

In order to use the direct variational methods, we approximate the solution $\phi(r, \theta, t)$ as the follows:

$$\phi(r,\theta,t)=\psi(r,\theta,t)+W(r,\theta,t).$$

Where $\psi(r,\theta,t)$ is any function which satisfies the non-homogeneous boundary conditions, and $W(r,\theta,t)$ any function which satisfies the homogeneous boundary conditions.

One of the choices for $W(r,\theta,t)$ which fits our needs is the following function:

$$W(r,\theta,t)=(r-a)(r-s(\theta,t))^2\sum_{i=1}^{n}\sum_{j=0}^{m}a_{ij}\cos(j\theta)\,r^{i-1}$$

where $\phi_{ij}(r,\theta) = \cos(j\theta)\, r^{i-1}$., $i=1, 2, \dots, n$; $j=0,1, 2, \dots, m$, are chosen from a complete set of functions. With $n = 2$, $m = 1$, we have:

$$W(r,\theta,t)=(r-a)(r-s(\theta,t))^2\sum_{i=1}^{2}\sum_{j=0}^{1}a_{ij}\cos(j\theta)\,r^{i-1}$$

Where a_{ij} are constant to be determined

For simplicity rewriting, this equation as follows, with the assumption that

$a_4 = a_{10}$ $a_5 = a_{11}$ $a_6 = a_{20}$ $a_7 = a_{21}$

$W(r, \theta, t)=(r-a)(r-s(\theta, t))^2(a_4 + a_5\cos\theta+a_6 r+a_7 r\cos\theta)$.

Additionally, for the non-homogeneous boundary condition which satisfied $\phi=1$ on the anode, and by using the mathematical inspection, we can take $\psi(r, \theta, t)$ to be as follows:

$$\psi(r,\theta,t)=\frac{r-a}{s(\theta,t)-a}+\frac{(r-a)(r-s(\theta,t))}{s(\theta,t)-a}\left(\frac{\dot{s}(\theta,t)}{1+\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)^2}-\frac{1}{s(\theta,t)-a}\right)$$

where $\dot{s}(\theta,t)=\frac{\partial s(\theta,t)}{\partial t}$ and $\bar{s}(\theta,t)=\frac{\partial s(\theta,t)}{\partial\theta}$, which is easily checked that $r=s(\theta,t)$, then $\psi(r, \theta, t)=1$.

Now:

$$\phi(r, \theta, t)=\psi(r, \theta, t)+W(r, \theta, t)$$

Hence:

$$\phi(r,\theta,t)=\frac{r-a}{s(\theta,t)-a}+\frac{(r-a)(r-s(\theta,t))}{s(\theta,t)-a}\left(\frac{\dot{s}(\theta,t)}{1+\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)^2}-\frac{1}{s(\theta,t)-a}\right)$$
$$+(r-a)(r-s(\theta, t))^2(a_4 + a_5\cos\theta + a_6 r + a_7 r\cos\theta).$$

Where the first two parts satisfy the non-homogeneous boundary conditions and the latest term satisfy the homogeneous conditions.

In order to minimize the functional (19), the partial derivatives of $\phi(r, \theta, t)$ with respect to r and θ are found, which are:

$$\phi_r=\frac{1}{s(\theta,t)-a}+\frac{(r-a)+(r-s(\theta,t))}{s(\theta,t)-a}\left(\frac{\dot{s}(\theta,t)}{1+\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)^2}-\frac{1}{s(\theta,t)-a}\right)$$

$+2\,(r-a)\,(r-s\,(\theta,t))\,(a_4+a_5\cos\theta+a_6\,r+a_7\,r\cos\theta)+(r-s\,(\theta,t))^2(a_4+a_5\cos\theta+a_6\,r+a_7\,r\cos\theta)+(r-a)\,(r-s\,(\theta,t))^2\,(a_6+a_7\cos\theta)$.

and

$$\phi_\theta=-\frac{(r-a)(\bar{s}(\theta,t))}{(s(\theta,t)-a)^2}+\frac{(r-a)(r-s(\theta,t))}{(s(\theta,t)-a)}\left(\frac{\left[1+\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)^2\right]\overline{(\dot{s}(\theta,t))}-2\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)\dot{s}(\theta,t)\,\frac{s(\theta,t)\bar{\bar{s}}(\theta,t)-\bar{s}^2(\theta,t)}{s^2(\theta,t)}}{\left[1+\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)^2\right]^2}+\frac{\bar{s}(\theta,t)}{(s(\theta,t)-a)^2}\right)+\left(\frac{\dot{s}(\theta,t)}{1+\left(\frac{\bar{s}(\theta,t)}{s(\theta,t)}\right)^2}-\frac{1}{s(\theta,t)-a}\right)\frac{(r-a)(s(\theta,t)-a)(-\bar{s}(\theta,t)-(r-s(\theta,t)(\bar{s}(\theta,t))}{(s(\theta,t)-a)^2}$$

$-2\,(r-a)\,(r-s\,(\theta,t))\,(\bar{s}\,(\theta,t))\,(a_4+a_5\cos\theta+a_6\,r+a_7\,r\cos\theta)-(r-a)\,(r-s\,(\theta,t))^2\,(a_5\sin\theta+a_7\,r\sin\theta)$.

where

$$\overline{(\dot{s}(\theta,t))}=\frac{\partial}{\partial\theta}\left(\frac{\partial s}{\partial t}\right)\quad and\quad \bar{\bar{s}}((\theta,t))=\frac{\partial}{\partial\theta}\left(\frac{\partial s}{\partial t}\right)$$

Therefore, minimizing the variational formulation (19), the following results are obtained:-

$a_1 = 4.947315$ $a_2 = 0.2211012$
$a_3 = -1.826011$ $a_4 = -20.54833$
$a_5 = 20.46942$ $a_6 = -3.23838$
$a_7 = 3.317843$

where the functional minimum equal to 22.30315.

Also, successive approximations to the moving boundary for different time steps t and different value of θ are presented in Fig.(3) and (4).

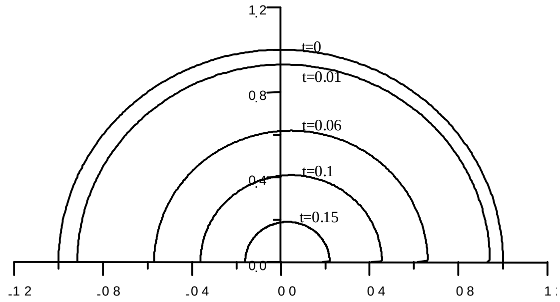

Fig.(3): Successive approximation to the moving boundary s(θ,t) with increasing t.

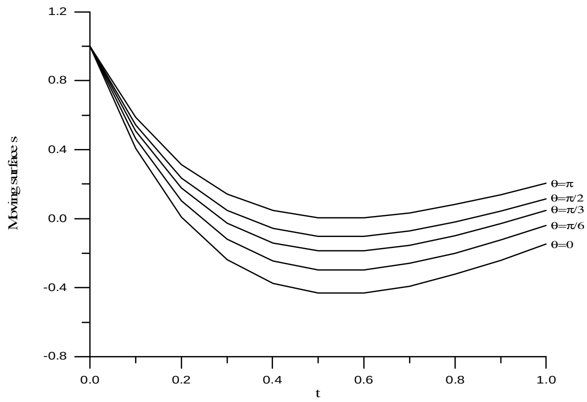

Fig.(4) : Successive approximation to the moving boundary s(θ,t) with increasing θ.

From the obtained results , one can see the accuracy of the results in which decreasing in the moving boundary with increasing t and increasing moving boundary with respect to increasing θ which satisfies conditions of the physical problem or the mathematical and numerical solution of the problem as it is given in condition (1) and (2) .

5. References

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الخلاصة

في هذا البحث ،قمنا باشتقاق الصياغة التغايرية لمسألة المكننة الكتروكيميائيا وكذلك إيجاد الحل العددي لها باستخدام طريقة ريتز المباشرة. يمكن اعتبار هذه المسألة من المسائل التي يطلق عليها بالمسائل المنحلة والتي من الصعوبة التعامل معها باستخدام الطرق العددية الأخرى.