# HOMOTOPY PERTURBATION METHOD FOR SOLVING THE NONLINEAR WAVE EQUATIONS

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## Abstract
In this paper, we present the approximated solutions of special types of the nonlinear partial differential equations in one dimension namely the nonlinear wave equations by using the homotopy perturbation method. Some examples are solved to show the efficiency of this method and a comparison is made with the Adomian decomposition method.

**Keywords**: homotopy perturbation method, nonlinear wave equations, Adomian decomposition method, exact solutions.

## 1. Introduction
Consider the nonlinear wave equation in one dimension, [5]:

$$\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial x}\right)^2=\phi(x,t),\quad -\infty<x<\infty,\quad t\geq 0 \;\dots (1)$$

together with the initial condition:

$$u(x,0)=f(x),\quad -\infty<x<\infty \;\dots\dots\dots (2)$$

This above initial value problem is solved by Kaya in 1998, [4] by using Adomian decomposition method, [1]. This method is involved in the calculation of complicated Adomian polynomials which narrows down its applications. To overcome this advantage of Adomian decomposition method, we use homotopy perturbation method, [2], [3] to solve the above initial value problem.

## 2. Solutions of the Nonlinear Wave Equations:
Consider the initial value problem given by equations (1)-(2). We rewrite equation (1) as

$$A(u)-\phi(x,t)=0 \;\dots\dots\dots (3)$$

where $A(u)=\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial x}\right)^2$.

Then the operator A can be divided into two parts L and N where L is a linear operator while N is a nonlinear operator. Therefore equation (3) becomes:

$$Lu+Nu-\phi(x,t)=0 \;\dots\dots\dots (4)$$

where $Lu=\frac{\partial u}{\partial t}$ and $Nu=\left(\frac{\partial u}{\partial x}\right)^2$. By using [2], we can construct a homotopy

$$v(x,t,p):(-\infty,\infty)\times[0,\infty)\times[0,1]\longrightarrow\Re$$

which satisfies

$$H(v,p)=(1-p)\left[L(v)-L(u_o)\right]+p\left[A(v)-\phi(x,t)\right]=0$$

In other words we can construct a homotopy $v$ which satisfies

$$H(v,p)=(1-p)\left[\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}\right]+p\left[\frac{\partial v}{\partial t}+\left(\frac{\partial v}{\partial x}\right)^2-\phi(x,t)\right]=0 \;\dots\dots\dots (5)$$

where $p\in[0,1]$ and $u_0$ is the initial approximation to the solution of equation (1) which satisfies the initial condition given by equation (2).

By using equation (5) it easily follows that

$$H(v,0)=\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}=0 \;\dots\dots\dots (6)$$

$$H(v,p)=\frac{\partial v}{\partial t}+\left(\frac{\partial v}{\partial x}\right)^2-\phi(x,t)=0 \;\dots\dots\dots (7)$$

and the changing process of p from zero to unity is just that of $H(v,p)$ from $\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}$ to $\frac{\partial v}{\partial t}+\left(\frac{\partial v}{\partial x}\right)^2-\phi(x,t)$. In a topology, this is called deformation, $\frac{\partial v}{\partial t}-\frac{\partial u_0}{\partial t}$ and $\frac{\partial v}{\partial t}+\left(\frac{\partial v}{\partial x}\right)^2-\phi(x,t)$ are called homotopic.

Next, we assume that the solution of equation (5) can be expressed as

$$v(x,t) = v_o(x,t) + pv_1(x,t) + p^2 v_2(x,t) + \mathbf{L} \quad \text{.......................} (8)$$

Therefore the approximated solution of the initial value problem given by equations (1)-(2) can be obtained as follows:

$$u(x,t) = \lim_{p\to 1} v(x,t) = \sum_{i=0}^{\infty} v_i(x,t) \quad \text{......................................} (9)$$

The convergence of the series given by equation (9) has been proved in [2].

By substituting the approximated solution given by equation (8) into equation (5) one can get:

$$\sum_{i=0}^{\infty} p^i \frac{\partial v_i}{\partial t} - \frac{\partial u_0}{\partial t} + p\frac{\partial u_0}{\partial t} + p\left[\left(\sum_{i=0}^{\infty} p^i \frac{\partial v_i}{\partial x}\right)^2 - \phi(x,t)\right] = 0$$

After simple computations one can obtain:

$$\left(\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t}\right) + \left[\frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + \left(\frac{\partial v_0}{\partial x}\right)^2 - \phi(x,t)\right]p + \left[\frac{\partial v_2}{\partial t} + 2\frac{\partial v_0}{\partial x}\frac{\partial v_1}{\partial x}\right]p^2 + \left[\frac{\partial v_3}{\partial t} + \left(\frac{\partial v_1}{\partial x}\right)^2 + 2\frac{\partial v_0}{\partial x}\frac{\partial v_2}{\partial x}\right]p^3 + \mathbf{L} = 0$$

Therefore

$$p^0 : \frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0 \quad \text{....................................} (10.a)$$

$$p^1 : \frac{\partial v_1}{\partial t} + \frac{\partial u_0}{\partial t} + \left(\frac{\partial v_0}{\partial x}\right)^2 - \phi(x,t) = 0 \quad \text{........} (10.b)$$

$$p^2 : \frac{\partial v_2}{\partial t} + 2\frac{\partial v_0}{\partial x}\frac{\partial v_1}{\partial x} = 0 \quad \text{............................} (10.c)$$

$$p^3 : \frac{\partial v_3}{\partial t} + \left(\frac{\partial v_1}{\partial x}\right)^2 + 2\frac{\partial v_0}{\partial x}\frac{\partial v_2}{\partial x} = 0 \quad \text{.............} (10.e)$$

$$\vdots$$

Since $u(x,0) = f(x)$, then we choose $u_0(x,t) = f(x)$ and this implies that $u_0(x,0) = f(x)$. Also, for simplicity we set $v_0(x,t) = u_0(x,t) = f(x)$. Therefore by substituting t=0 in equation (9) one can have:

$$u(x,0) = \sum_{i=0}^{\infty} v_i(x,0)$$

But $v_0(x,0) = f(x)$ and $u(x,0) = f(x)$, hence $v_i(x,0) = 0, i = 1,2,\mathbf{K}$.

So, equation (10.a) is automatically staistiefied. By substituting $v_0(x,t) = u_0(x,t) = f(x)$ into equation (10.b) one can get:

$$\frac{\partial v_1}{\partial t} = \phi(x,t) - [f'(x)]^2$$

By integrating both sides of the above differential equation and by using the initial condition $v_1(x,0) = 0$ one can obtain

$$v_1(x,t) = \int_0^t \phi(x,y)dy - [f'(x)]^2 t \quad \text{.................} (11)$$

By substituting $v_o$ and $v_1$ into equation (10.c) and by using the initial condition $v_2(x,0) = 0$ one can get:

$$v_2(x,t) = -2f'(x)\int_0^t\left[\int_0^z \frac{\partial\phi(x,y)}{\partial x}dy\right]dz + 2f''(x)[f'(x)]^2 t^2 \quad \text{................................} (12)$$

Then by substituting $v_o$, $v_1$ and $v_2$ into equation (10.e) and by using the initial condition $v_3(x,0) = 0$ one can get:

$$v_3(x,t) = -\int_0^t\left[\int_0^z \frac{\partial\phi(x,y)}{\partial x}dy - 2f'(x)f''(x)z\right]^2 dz - 2f'(x)\left\{-2f'(x)\int_0^t\int_0^{z_1}\int_0^{z_2} \frac{\partial^2\phi(x,y)}{\partial x^2}dydz_2dz_1 - 2f''(x)\int_0^t\int_0^{z_1}\int_0^{z_2}\frac{\partial\phi(x,y)}{\partial x}dydz_2dz_1 + \frac{2}{3}f'''(x)[f'(x)]^2t^3 + \frac{4}{3}f'(x)[f''(x)]^2t^3\right\} \quad \text{...........................................} (13)$$

In a similar manner one can get $v_i(x,t),\ i = 4,5,....$ By substituting $v_i(x,t),\ i = 0,1,...$ into equation (9) one can get the approximated solution of the initial value problem given by equations (1)-(2).

## 3.Numerical Examples

In this section, we solve the same examples that appeared in [4] by using the homotopy perturbation method.

***Example (1):***

Consider the initial value problem that consists of the nonlinear differential equation:

$$\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial x}\right)^2=0,\quad -\infty<x<\infty,\quad t\geq 0 \ldots\ldots\ldots (14)$$

together with the initial condition:

$$u(x,0)=-x^2,\quad -\infty<x<\infty \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Here $\phi(x,t)=0$ and $f(x)=-x^2$. Therefore $v_0(x,t)=u_0(x,t)=-x^2$. By substituting $\phi$ and $f$ into equations (11)-(13) one can have:

$$v_1(x,t)=-[f'(x)]^2\,t=-[-2x]^2\,t=-4x^2t$$

$$v_2(x,t)=2f''(x)[f'(x)]^2t^2$$

$$=2(-2)[-2x]^2t^2=-16x^2t^2$$

$$v_3(x,t)=-\int_0^t[-2f'(x)f''(x)z]^2\,dz$$

$$-2f'(x)\left\{-\frac{2}{3}f'''(x)[f'(x)]^2\,t^3+\frac{4}{3}f'(x)[f''(x)]^2\,t^3\right\}$$

$$=-\int_0^t[-8xz]^2\,dz-2(-2x)\left\{\frac{4}{3}(-2)^2(-2x)t^3\right\}$$

$$=-64x^2t^3.$$

In a similar manner one can get $v_i(x,t),\ i=4,5,\ldots.$ Hence

$$u(x,t)=\sum_{i=0}^{\infty}v_i(x,t)=-x^2-4x^2t-16x^2t^2$$

$$-64x^2t^3-\mathbf{L}$$

$$=-x^2\left[1+4t+16t^2+64t^3+\mathbf{L}\right]$$

$$=-x^2\left[\frac{1}{1-4t}\right]=\frac{x^2}{4t-1}.$$

This is the exact solution of the initial value problem given by equation (14)-(15). Note that this example is solved in [4] by the Adomian decomposition method and it requires also infinite number of iterations to get the exact solution.

***Example (2):***

Consider the initial value problem that consists of the nonlinear differential equation:

$$\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial x}\right)^2=0,\quad -\infty<x<\infty,\quad t\geq 0 \ldots\ldots\ldots (16)$$

together with the initial condition:

$$u(x,0)=ax,\quad -\infty<x<\infty,\ a\in\Re \ldots\ldots\ldots\ldots (17)$$

Here $\phi(x,t)=0$ and $f(x)=ax$. Therefore $v_0(x,t)=u_0(x,t)=ax$. By substituting $\phi$ and $f$ into equations (11)-(13) one can have:

$$v_1(x,t)=-[f'(x)]^2\,t=-a^2t$$

$$v_2(x,t)=v_3(x,t)=0$$

In a similar manner one can get $v_i(x,t)=0,\ i=4,5,\ldots.$ Hence

$$u(x,t)=\sum_{i=0}^{\infty}v_i(x,t)=ax-a^2t$$

This is the exact solution of the initial value problem given by equation (16)-(17). Note that this example is solved in [4] by the Adomian decomposition method and it requires also only one iteration to get the exact solution.

***Example (3):***

Consider the initial value problem that consists of the nonlinear differential equation:

$$\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial x}\right)^2=1+\cosh^2 x,\quad -\infty<x<\infty,\quad t\geq 0 \ldots\ldots\ldots\ldots\ldots (18)$$

together with the initial condition:

$$u(x,0)=\sinh x,\quad -\infty<x<\infty \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

Here $\phi(x,t)=1+\cosh^2 x$ and $f(x)=\sinh x$. Therefore $v_0(x,t)=u_0(x,t)=\sinh x$. By substituting $\phi$ and $f$ into equations (11)-(13) one can have:

$$v_1(x,t)=\int_0^t(1+\cosh^2 x)dy-[\cosh x]^2\,t=t.$$

$$v_2(x,t) = -2f'(x)\int_0^t\left[\int_0^z \frac{\partial\phi(x,y)}{\partial x}dy\right]dz + 2f''(x)\left[f'(x)\right]^2 t^2$$

$$= -2\cosh x\int_0^t 2\cosh x\sinh xzdz + 2\sinh x\cosh^2 xt^2$$

$$= -2\sinh x\cosh^2 xt^2 - 2\sinh x\cosh^2 xt^2$$

$$= 0.$$

$$v_3(x,t) = -\int_0^t\begin{bmatrix}2\cosh x\sinh xz - \\ 2\cosh x\sinh xz\end{bmatrix}^2 dz - 2\cosh x\begin{cases}-2\cosh x(2\mathrm{coxh}^2 x \\ +2\sinh^2 x)\frac{t^3}{6} - \end{cases}$$

$$\left.\begin{matrix}4\cosh x\sinh^2 x\frac{t^3}{6} + \frac{2}{3}\cosh x[\cosh x]^2 t^3 \\ +\frac{4}{3}\cosh x[\sinh x]^2 t^3\end{matrix}\right\}$$

$$= 0.$$

In a similar manner one can get $v_i(x,t) = 0,\ i = 4,5,....$ Hence

$$u(x,t) = \sum_{i=0}^{\infty} v_i(x,t) = \sinh x + t.$$

This is the exact solution of the initial value problem given by equation (18)-(19). Note that this example is solved in [4] by the Adomian decomposition method and it requires infinite number of iterations to get the exact solution. But here we need only one iteration to get the exact solution.

## 4. Conclusions

The homotopy perturbation method has been applied for solving the initial value problem of the nonlinear wave equations. Comparising with Adomian decomposition method, this provides the exact solutions of the nonlinear wave equations without the tedious calculation of the Adomian polynomials. Also, it requires a less number of iterations than the Adomian decomposition method. This method can been seen as a powerful method for solving the above problems. Also, the homotopy perturbation method can solve be used to the non linear wave equation in two dimensions.

## 6. References


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