MODULES WITH THE QUASI-PURE SUM PROPERTY

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Abstract

Let R be a commutative ring with identity and M be a unitory R-module. We say that M has the quasi-pure sum property (qpsp), if the sum of any two pure submodules of M is quasi-pure. In this paper we study modules with this property and we give a characterization of some kind of rings in terms of modules with the quasi-pure sum property.

Introduction

J. Garcia in [4] studied modules with the summand sum property (ssp); i.e. the sum of any two direct summands is direct summand.

In [7] the authors studied modules that satisfy the pure sum property (psp); i.e. the sum of any two pure submodules is again pure.

In this paper, we study modules M with property that the sum of any two pure submodules of M is quasi-pure submodule of M(quasi-pure sum property, qpsp).

This work consists of two sections. In section one, we give the definition of modules with the (qpsp) with some examples and we study the direct sum of modules with the (qpsp). In section two, we classify some rings R by means of modules that have the (qpsp).

Finally, we note that all rings considered in this paper are commutative with 1 and all R–modules are unitary (left R-modules).

S1: The quasi-Pure *Sum* Property:

In this section we introduced the concept of quasi-pure sum property (qpsp), and we illustrate it by examples and we also give some basic property.

Recall that a submodule N of an R-module M is pure in M if for each finitely generated ideal (equivalently, every ideal) I of R, $IM \cap N = IN$,[3].

Also we say that a submodule N of an R-module M is quasi-pure in M if for each $x \in M$ and $x \notin N$,there exists a pure sumodule L of M such that $L \supseteq N$ and $x \notin L$,[2].

Definition (1.1):

An R-module M is said to have the quasi-pure sum property (qpsp), if the sum of any two pure submodules of M is quasi-pure in M, [2].

Examples and Remarks (1.2):

1. If an R-module M has the psp, then M has the qpsp. The converse is not true, for example $Z\oplus Z$ has the qpsp, but does not have the psp, [7]

2. An R-module M is said to have quasi-regular if every submodule of M is quasi-pure, [2]. It is clear that every quasi-regular R–module has the qpsp.

3. Consider the module $M = Z_4 \oplus \boldsymbol{Z_2}$ as a Z-module, let $A = 0 \oplus Z_2$ and $B = Z(2,1)$, the submodule generated by $(2,1)$. It is clear that each of A and B is pure in M. But $A + B$ is not quasi-pure in M. Thus M does not have the qpsp.

4. Each pure submodule of a module with qpsp, has the qpsp, [2].

5. Every commutative ring with identity considered as an R-module has the qpsp, [7]

Remark (1.3):

If an R-module M has the qpsp, and N is pure submodule of M, then $\frac{M}{N}$ has the qpsp.

Proof:

Let $\frac{A}{N}$ and $\frac{B}{N}$ be pure submodules of $\frac{M}{N}$, then A and B are pure in M, [3]. Since M has qpsp, then $A + B$ is quasi-pure in M, we claim that $\frac{A+B}{N}$ is quasi-pure in $\frac{M}{N}$, let $\bar{x} \in \frac{M}{N}$ and $\bar{x} \notin \frac{A+B}{N}$, then $x + N \notin \frac{A+B}{N}$, hence $x \notin A + B$, since $A + B$ is quasi-pure in M, hence there exists a pure submodule L of M such that $L \supseteq A + B$ and $x \notin L$. In fact that $\frac{L}{N}$ is a pure submodule of $\frac{M}{N}$, [3]. Also $\frac{L}{N} \supseteq \frac{A+B}{N}$ since $L \supseteq A + B$ and $\frac{A}{N} + \frac{B}{N}$ is quasi-pure submodule of $\frac{M}{N}$. So that $\frac{M}{N}$ has the qpsp.

Remark (1.4):

If A_P is quasi pure submodule of M_P as an R_P–module for every maximal ideal P of R, then A is quasi pure in M as R-module.

Proof:

Let A be a submodule of an R–module M, let $a \in M$ and $a \notin A$, then $a_P \in M_P$ and $a_P \notin A_P$ where P is a maximal ideal of R. Since A_P is quasi pure, then there exists a pure submodule L_P of M_P such that $a_P \notin L_P \supseteq A_P$. Thus $a \notin L \supseteq A$ and L is pure in M which implies that A is quasi pure in M.

Proposition (1.5):

Let M be an R-module. If M_p has qpsp as R_p-module for every maximal ideal P of R, then M has qpsp as R-module.

Proof:

Let A and B be pure submodule of M. Then A_P and B_P are pure submodule of M_P as R_P-modules. Since M_P has qpsp, then $A_P + B_P = (A + B)_P$ is quasi pure in M_P for every maximal ideal P in R. Thus by (1.4) $A + B$ is quasi pure in M.

Next we give the following:

Theorem (1.6) [2]:

If an R-module M has the qpsp, then for every decomposition $M = A \oplus B$ and for any R-homomorphism $f: A \to B$, $Im\, f$ is quasi-pure in B.

Remark (1.7):

If an R-module M has the qpsp, then $M \oplus M$ may not have the qpsp as is seen in the following example.

Consider the module Z_4 as Z-module. It is clear that Z_4 is pure simple and hence has the qpsp. But $Z_4 \oplus Z_4$ does not have the qpsp. To see this define a homomorphism $f: 0 \oplus Z_4 \to Z_4 \oplus 0$ by $f(0, n) = (2n, 0)$. It is clear that $Im\, f = \{(0,0), (2,0)\}$ is not quasi-pure in $Z_4 \oplus Z_4$.

The following proposition gives a necessary condition under which the direct sum of modules with the qpsp has the qpsp.

Recall that a submodule N of an R-module M is called **fully invariant** if for every endomorphism $f: M \to M$, $f(N) \subseteq N$, [10].

Proposition (1.8):

Let $M = \oplus_{i \in I} M_i$ be an R-module where each M_i is a submodule of M. If M has qpsp, then each M_i has the qpsp. The converse is true if each pure submodule of M is fully invariant.

Proof:

Assume that M has the qpsp. Since M_i is a summand of M for each $i \in I$, then M_i has the qpsp by (1.2-4).

For the converse, let A and B be pure submodules of M, then $A = \oplus_{i \in I}(A \cap M_i)$ and $B = \oplus_{i \in I}(B \cap M_i)$. Thus $A + B = \oplus_{i \in I}(A \cap M_i) + (B \cap M_i)$. Since $A \cap M_i$ and $B \cap M_i$ are pure in M_i and M_i has qpsp, then $(A \cap M_i) + (B \cap M_i)$ is quasi-pure in M_i. By [1], $A + B$ is pure in M.

Now we show that $M \oplus N$ has the qpsp if $ann\, M + ann\, N = R$ for any R-modules M and N.

Proposition (1.9):

Let M and N be R-modules with the qpsp, such that $ann\, M + ann\, N = R$, then $M \oplus N$ has the qpsp.

Proof:

Let A and B be quasi – pure submodules of $M \oplus N$. Since $ann\, M + ann\, N = R$, then $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$, where A_1 and B_1 are submodules of M, A_2 and B_2 are submodules of N, [1]. Since M and N have the qpsp, then $A_1 + B_1$ is quasi- pure in M and $A_2 + B_2$ is quasi- pure in N. Thus by (4.1.10) [1] $(A_1 + B_1) \oplus (A_2 + B_2)$ is quasi- pure in $M \oplus N$. But $(A_1 + B_1) \oplus (A_2 + B_2) = (A_1 \oplus A_2) + (B_1 \oplus B_2) = A + B$. So $A + B$ is quasi- pure in $M \oplus N$ has the qpsp.

The following proposition gives restrictions on modules whose direct sum has the qpsp.

Proposition (1.10):

Let R be an integral domain and M be a faithfull and pure simple R-modules. If $M \oplus M$ has the qpsp, then M is divisible.

Proof:

Let $0 \neq r \in R$, define $f: M \to M$ by $f(m) = rm, \forall\, m \in M$. f is an R-homomorphism. Since $M \oplus M$ has the qpsp, then by (1.6), $Im\, f = rM$ is quasi-pure. Since M is faithfull, then $rM \neq 0$. But M is pure simple and $rM = \bigcap_{\propto\in\wedge} L_{\propto}$, $L_{\propto}$ is pure containg rM, so that $rM = M$ and hence M is divisible.

Since every torsion free and divisible R-module is injective [9].

Corollary (1.11):

Let R be an integral domain and M be a torsion free and pure simple R-module. If $M\oplus M$ has the qpsp, then M is injective.

Proposition (1.12) [2].:

Let R be a ring and let M be an R-module, if R- module, if $R\oplus M$ has the qpsp, then every cyclic submodule of M is quasi- pure.

S2: Characterization of rings by means of modules having the qpsp:

In this section we classify some ring R by means of modules that have the qpsp. First recall that a ring R is called **quasi- regular** if every ideal in R is quasi –pure, [2].

Theorem (2.1):

Let R be a ring and $J(R) = 0$. The following statements are equivalent:

1. R is a regular ring.
2. All R-modules have the qpsp.
3. All flat R-modules have the qpsp.
4. All projective R-modules have the qpsp.
5. All free R-modules have the qpsp.

Proof:

(1) →(2) Since R is regular ring, then all R-modules are F- regular [8]. Hence all R-modules have the psp [7]. Thus by (1.2.1), all R-modules have the qpsp.

(2)→(3), (3)→(4), (4)→(5) are clear.

(5)→(1) Let I be an ideal in R, then there exists a free R-module F and an epimorphism $f: F \to I$. Let $i: I \to R$ be the inclusion map. Consider $i\,of: F \to R$. Since $F\oplus R$ have the qpsp, then by (1.6) $Im\, iof = Im\, i = I$ is quasi- pure in R, hence R is quasi- regular. Since $J(R) = 0$, then by [2,cor.(4.2.6)] R is a regular ring.

For a principal ideal domain we have the following theorem.

Theorem (2.2):

Let R be a principal ideal domain. The following statements are equivalent.

1. R is a field.
2. All . R-modules have the psp.
3. All R-modules have the qpsp.
4. All $f.g$ flat R-modules have the qpsp.

Proof:

(1)→(2) Follows from [7].

(2)→(3) Clearly

(3)→(4) Clearly

(4)→(1) Let $a \in R$ and let $I = Ra$. Define $f: R \longrightarrow Ra$ by $f(r) = ra$. It is clear that f is an epimorphism. Let $i: Ra \to R$ be the inclusion map. Consider $i\,of: R \to R$. Since $R\oplus R$ is a $f.g$ flat R-module [8], then $R\oplus R$ has the qpsp. Thus by (1.6), $Im\, i\, of = Ra$ is quasi- pure in R. Hence R is quasi- regular PID. Thus by [2, cor.(4.2.6)] R is a field.

By a similar argument one can prove the equivalence of the other statements.

Recall that an R-module M is called **a multiplication module** if for each submodule N of M, there exists an ideal I in R such that $N = IM$, [5]. And an R-module M is called **self generator** if for every submodule N of M, there exists a family $\{\phi_\alpha\}_{\alpha\in\wedge}$ of endomorphisms of M, such that $N = \sum_{\alpha\in\wedge}\phi_\alpha(M)$, [7].

Theorem (2.3):

Let R be a ring and M be a $f.g$ faithful multiplication R-module. The following statements are equivalent:

1. R is quasi – regular.
2. $\oplus_I M$ has the qpsp, for every index set I.

Proof:

(1)→(2) Clear from [2].

(2)→(1) Let I be an ideal of R. Since M is a multiplication R-module, then M is self generator and hence there exists an epimorphism $f: (m_{\propto})_{\propto\in\wedge} \in \oplus_\wedge M \mapsto \sum_{\alpha\in\wedge}\phi_\alpha(m_\alpha) \in IM$, for some index set $\wedge$, where $\{\phi_\alpha\}_{\alpha\in\wedge}$ is a family of an R-endomorphisms from M into N. Let $i: IM \to M$ be the inclusion map. Consider $i\,of: \oplus M \to M$. Since $(\oplus_\wedge)\oplus M$ has the qpsp, then by (1.6) $Im\, i\, of = IM$ is quasi- pure in M, then by [2,pro.(4.1.4)] $IM = \bigcap_{\propto} L_{\propto}$ where $L_{\propto}$ is pure in M containg IM, $\forall\propto$. Put $L_{\propto} = I_{\propto}M$. Thus $IM = \bigcap_{\propto} L_{\propto} = \bigcap_{\propto} I_{\propto}M$, since M is $f.g$ faithful, then by 1/2 cancellation property [6]. $I = \bigcap_{\propto} I_{\propto}$. Claim $I_{\propto}$ is pure in R and $I_\alpha \geq I$. $(I_\alpha \cap J)M = I_\alpha M \cap JM = L_\alpha \cap JM = I_\alpha JM$. Thus $I_\alpha \cap J = I_\alpha J$ [3], which implies that I_α is pure in R. Also since $I_\alpha M = L_\alpha \supseteq IM$. Thus $I_{\propto} \supseteq I$. So that I is quasi- pure. Thus R is quasi- regular.

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الخلاصة

لتكن R حلقة ابدالية ذات عنصر محايد وليكن M مقاسا على R. نقول إن المقاس المعرف على R بأنه يحقق خاصية الجمع المباشر شبه النقي (qpsp) إذا كان مجموع أي مقاسين جزئيين نقيين مقاس جزئ شبه نقي. في بحثنا هذا سوف ندرس هذا المفهوم، وسوف نعطي تشخيصا للحلقات التي مقاساتها تحقق الخاصية نفسها.