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L-FLAT MODULES

Ahmed Bakheet*
*Department of Mathematics, College of Science, University of Al-Mustansiriyah, Baghdad-Iraq.

Abstract

In this work certain generalization of flat modules is consider relative to prime radical. We extend some results on flat modules and study the relations between these modules with other known modules.

Introduction

In this paper R stands for a commutative ring with identity 1, and a module means a unitary left R-module. The prime radical of an R-module M denoted by $L(M)$ is defined to be the intersection of all prime submodules of M, and in case M has no prime submodule then $L(M)=M$ [1]. Recall that an R-module M is said to be flat if for each finite subsets $\{x_1,x_2,...,x_n\}$ and $\{r_1,r_2,...,r_n\}$ of M and R respectively such that $\sum_{i=1}^{n} r_i x_i = 0$ there exists elements $y_1,y_2,...,y_k$ in M and s_{ij} in R, $1 \le i \le n, 1 \le j \le k$ that satisfy the following [2]:

1) $\sum_{i=1}^{n} s_{ij} r_i = 0 \qquad 1 \le j \le k$

2) $\sum_{j=1}^{k} s_{ij} y_i = x_i \qquad 1 \le i \le n$

The concepts of L-projective modules and L-pure submodules were introduced as generalizations of projective modules and pure submodules respectively, an R-module M is said to be L-projective if for each R-epimorphism $f: A \to B$ (where A and B are two R-modules) and each R-homomorphism $g: M \to B$ there exists an R-homomorphism $h: M \to A$ such that $(f \circ h - g)(M) \subseteq L(B)$. A submodule P of an R-module M is L-pure if and only if for every finite sets $\{m_i\}_{i=1}^{n}$, $m_i \in M$, $\{b_j\}_{j=1}^{n}$, $b_j \in P$ and $\{r_{ij}\}, r_{ij} \in R$ such that $b_j = \sum_{i=1}^{n} r_{ij} m_j$, there exists $x_i \in P$, $1 \le i \le n$ such that

$b_j - \sum_{i=1}^{n} r_{ij} x_i \in P \cap L(M)$, $1 \le j \le k$.[3]

A submodule N of an R-module M is said to be lie over a direct L-summand of M if there exists a direct decomposition $M = P \oplus Q$ with $P \subseteq N$ and $N \cap Q \subseteq L(M)$. [4]

L-Flat Modules

In this paper we introduce the concept of L-flat modules with some of properties and then we study the relation between this class of modules with L-projective modules and L-pure submodules.

First we give some properties for L-pure submodules

Proposition:

Let M be an R-module, then

1- If K is L-pure submodule in H and H is L-pure submodule in M, then K is L-pure in M.

2- Every direct L-summand of M is L-pure.

3- If P_i is an L-pure submodule of M_i, $i = 1,2,\ldots,n_i$ then $\oplus P_i$ is L-pure in $\oplus M_i$

Proof:

(1) For any finite sets $\{m_i\}_{i=1}^{n}$, $m_i \in M$, $\{b_j\}_{j=1}^{n}$, $b_j \in K$ and $\{r_{ij}\}, r_{ij} \in R$ with $b_j = \sum_{i=1}^{n} r_{ij} m_j$, there exists $h_i \in H$ such that $b_j - \sum_{i=1}^{n} r_{ij} h_i \in H \cap L(M)$.

$b_j = \sum_{i=1}^{n} r_{ij} h_i + t$ where $t \in H \cap L(M)$,then $b_j = \sum_{i=1}^{n} r_{ij} h_i'$, $h_i' \in H$.

Thus there exists $k_i \in K$ such that $b_j = \sum_{i=1}^{n} r_{ij} k_i + w \in L(M) \cap K$ which implies that K is L-pure in M.

(2) Let M be an R-module and P is a direct L-summand of M, there exists a submodule N of M with $M = P + N$ and $P \cap N \subseteq L(M)$.

For a finite sets $\{m_i\} \subseteq M, \{p_j\} \subseteq P$ and $\{r_{ij}\} \subseteq R$ with $p_j = \sum r_{ij} m_i$.

For each i, $m_i = a_i + b_i$ where $a_i \in P$ and $b_i \in N$ then $p_j = \sum r_{ij} a_i + \sum r_{ij} b_i$ or $\sum r_{ij} b_i = p_j - \sum r_{ij} a_i \in P \cap N \subseteq L(M)$ thus $p_j - \sum r_{ij} a_i \in P \cap L(M)$ hence P is L-pure.

(3) Let $M = M_1 \oplus M_2$, for a finite sets $\{x_j\} \subseteq P_1 \oplus P_2$, $\{m_i\} \subseteq M$ and $\{r_{ij}\} \subseteq R$ with $x_j = \sum r_{ij} m_i$ where $x_j = x_{1j} + x_{2j}$ and $m_i = m_{1i} + m_{2i}$,then $x_{1j} + x_{2j} = \sum r_{ij} m_{1i} + \sum r_{ij} m_{2i}$. Thus there exist $y_{1j} \in P_1$ and $y_{2j} \in P_2$ such that $x_{1j} - \sum r_{ij} y_{1i} \in L(M) \cap P_1$ and $x_{2j} - \sum r_{ij} y_{2i} \in L(M) \cap P_2$.

Thus $x_j - \sum r_{ij} y_i \in L(M) \cap P_1 \cap L(M) \cap P_2$

$$= L(M) \cap (P_1 \oplus P_2) .$$

Definition:

An R-module M is said to be L-flat if for each finite subsets $\{x_1, x_2, ..., x_n\}$ and $\{r_1, r_2, ..., r_n\}$ of M and R respectively such that $\sum_{i=1}^{n} r_i x_i = 0$ there exists elements $y_1, y_2, ..., y_k$ in M and s_{ij} in R, $1 \le i \le n, 1 \le j \le k$ that satisfy the following :

1) $\sum_{i=1}^{n} s_{ij} r_i = 0 \qquad 1 \le j \le k$

2) $\sum_{j=1}^{k} s_{ij} y_i - x_i \in L(M) \qquad 1 \le i \le n$

Let M be an R-module, and let F be a free R-module over M. Let π be the natural projection from F onto M. Then there exists a short exact sequence :

$$0 \longrightarrow K \xrightarrow{\ i\ } F \xrightarrow{\ \pi\ } M \longrightarrow 0$$

where K=ker π and i is the inclusion map from K into F. Such a short exact sequence is called a presentation for M.

The following theorem gives different characterizations for L-flat modules.

Theorem:

Let M be an R-module. Then the following statements are equivalent:

1) M is L-flat R-module.

2) For each presentation for M

$$0 \longrightarrow K \xrightarrow{\ i\ } F \xrightarrow{\ \pi\ } M \longrightarrow 0$$

and for each element $u \in K$, there exists an R-endomorphism θ of F (may depend on u) such that

a) $(\pi \circ \theta - \pi)(x) \in L(M) \qquad \forall x \in F$

b) $\theta(u) = 0$

Proof:

Assume (1) and let $u \in K$, there exists $r_1, r_2, ..., r_n$ in R such that

$$u = \sum_{i=1}^{n} r_i x_{\alpha_i}$$

Then $\pi(u) = \sum_{i=1}^{n} r_i \pi(x_{\alpha_i}) = \sum_{i=1}^{n} r_i a_{\alpha_i} = 0$

Since M is L-flat then there exists $\{b_i\}_{i=1}^{n} \subseteq M$ and s_{ij} in R, $1 \le i \le n$, $1 \le j \le m$ such that $\sum_{i=1}^{n} s_{ij} r_i = 0$, $1 \le j \le m$ and $\sum_{i=1}^{n} s_{ij} b_i - a_{\alpha_i} \in L(M)$, $1 \le j \le m$ Since π is an epimorphism there exists $t_{kj} \in R, k = 1, ..., \ell$ such that $bj = \pi(\sum_{i=1}^{n} t_{kj} x_{\alpha_k})$ and $\pi(\sum_{i=1}^{n} t_{kj} x_{\alpha_k}) = a_{\alpha_k}$

Define an endomorphism of F on the basis $\{x_\alpha : \alpha \in \wedge\}$ as follows

$$\theta(x_{\alpha_i}) = \sum_{i=1}^{n} \sum_{k=1}^{\ell} s_{ij} t_{kj} x_{\alpha_k} , \quad 1 \le j \le m$$

$\theta(x_\alpha) = x_\alpha$ for each $\alpha \ne \alpha_1, \alpha_2, ..., \alpha_m$

Now,

$$\pi \circ \theta(x_{\alpha_i}) - \pi(x_{\alpha_i}) = \pi(\sum_{i=1}^{n} \sum_{k=1}^{\ell} s_{ij} t_{kj} x_{\alpha_k}) - \pi(x_{\alpha_i})$$

$$= \sum_{i=1}^{n} \pi(\sum_{k=1}^{\ell} s_{ij} t_{kj} x_{\alpha_k}) - \pi(x_{\alpha_i})$$

$$= (\sum_{i=1}^{n} s_{ij} b_j - a_i) \in L(M)$$

and $\theta(u)=\theta(\sum_{i=1}^{n} r_i x_{\alpha_i})=\sum_{i=1}^{n} r_i \theta(x_{\alpha_i})$

$$=\sum_{i=1}^{n} r_i (\sum_{i=1}^{n}\sum_{k=1}^{1} s_{ij} t_{kj} x_{\alpha_k})=0 .$$

Assume (2), and let $\{x_\alpha : \alpha \in \wedge\}$ be a basis for the free module F and let $\{a_i\}_{i=1}^{n} \subseteq M$ and $\{r_i\}_{i=1}^{n} \subseteq R$ such that $\sum_{i=1}^{n} r_i a_i = 0$.

Let $u=\sum_{i=1}^{n} r_i x_i$ then

$$\pi(u)=\pi(\sum_{i=1}^{n} r_i x_i)=\sum_{i=1}^{n} r_i a_i = 0$$

Then $u \in \ker\pi$. From (2) there exists an R-endomorphism θ of F such that

a) $(\pi \circ \theta - \pi)(x) \in L(M) \quad \forall x \in F$

b) $\theta(u)=0$

assume that $\theta(x_\alpha)=\sum_{j=1}^{k} s_{ij} x_{ij}$ then we have

1) $\sum_{i=1}^{n} s_{ij} r_i = 0 \qquad 1 \le j \le k$

2) $\sum_{j=1}^{k} s_{ij} x_{ij} - a_\alpha = \pi(\theta(x_\alpha)) - \pi(x_\alpha) \in L(M)$

$1 \le i \le n$

Thus M is L-flat.

Remarks

1. It is clear that every flat module and every L-projective module (hence every projective module) is L-flat.
2. L-flat module with zero prime radical is flat.
3. L-flat module may not be flat for example:

Consider the Z-module Z_{p^∞} where p is prime number. It is known that $Z_{p^\infty}=\bigcup_{n\in N} Z_{p^n}$, let $S=\{x_n \mid n \in N\}$ and let F be the free module over S. Let a_n be a generator of the cyclic subgroups Z_{p^n} of the group Z_{p^∞} consider the short exact sequence

$$0 \longrightarrow K \xrightarrow{\ i\ } F \xrightarrow{\ \pi\ } M \longrightarrow 0$$

where $\pi(x_n)=a_n$ and $K=\ker\pi$. Let $u \in K$ then $u=(u_1,u_2,...,u_k,0,0,...)$ where $u_i \in Z$ and $u_i=0$ for $i>k$, $k \in N$. Define $\theta : F \longrightarrow F$ as follows :

if $y=\sum_{i=1}^{n} t_i x_i$, $t_i \in Z$ put $\theta(y)=\sum_{i>k} t_i x_i$ it is clear that $\theta(u)=0$. Moreover, Z_{p^∞} has no prime submodules [5], thus $L(Z_{p^\infty})=Z_{p^\infty}$.

Hence $(\pi \circ \theta - \pi)(y) \in Z_{p^\infty} = L(Z_{p^\infty})$ which show that Z_{p^∞} is L-flat. But Z_{p^∞} is not flat Z-module since Z is integral domain and Z_{p^∞} is not torsion free [6].

A ring R is said to be a nice ring if $L(R)M=L(M)$ for each R-module M. Equivalently, R is nice ring if and only if $L(N)=L(M)\cap N$, for each submodule N of M.[7]

Proposition:

Let R be a nice ring, M be an R-module and P be a submodule of M. Then

1- If M/P is L-flat then P is L-pure.

2- If M is L-flat and P is L-pure, then M/P is L-flat.

Proof :

(1) Let $p_j \in P$ and $p_j=\sum_{i=1}^{n} r_i m_i$ where $r_i \in R$, $m_i \in M$ then $\sum_{i=1}^{n} r_i \bar{m}_i = 0 \in M/P$ but M/P is L-flat, then there exists $s_{ij} \in R$ and $\bar{y}_1,\bar{y}_2,...,\bar{y}_k \in M/P$ such that

$\bar{m}_i - \sum_{j=1}^{k} s_{ij}\bar{y}_j = \bar{w}$ and $\sum_{j=1}^{k} r_i s_{ij} = 0$ where $\bar{w} \in L(M/P)$ $m_i + P = (\sum_{j=1}^{k} s_{ij} y_i + w) + P$ then

$$m_i - (\sum_{j=1}^{k} s_{ij} y_i + w) \in P$$

Put $p_i = m_i - (\sum_{j=1}^{k} s_{ij} y_i + w)$, hence

$$\sum_{i=1}^{n} r_i p_i = \sum_{i=1}^{n} r_i (m_i - \sum_{j=1}^{k} s_{ij} y_i + w)$$

$$= \sum_{i=1}^{n} r_i m_i - \sum_{i=1}^{n} r_i (\sum_{j=1}^{k} s_{ij} y_i + w)$$

$$= \sum_{i=1}^{n} r_i m_i - \sum_{i=1}^{n}\sum_{j=1}^{k} r_i s_{ij} y_i + \sum_{i=1}^{n} r_i w$$

$$= \sum_{i=1}^{n} r_i m_i + \sum_{i=1}^{n} r_i w$$

Hence $\sum_{i=1}^{n} r_i p_i - p_j \in L(M) \cap P$. Thus P is L-pure.

(2) Let $\{\bar{m}_i\}_{i=1}^{n} \subseteq M/P$ and $\{r_i\}_{i=1}^{n} \subseteq R$ such that $\sum_{i=1}^{n} r_i \bar{m}_i = 0$.

Then $\sum_{i=1}^{n} r_i m_i = p \in P$. But P is L-pure then there exists $\{p_i\}_{i=1}^{n} \subseteq P$ such that $p - \sum_{i=1}^{n} r_i p_i = w \in L(M) \cap P = L(P)$.

Hence $\sum_{i=1}^{n} r_i (m_i - p_i) = w \in L(M) \cap P = L(P)$

$\sum_{i=1}^{n} r_i (m_i - p_i) - w = 0$

$= \sum_{i=1}^{n} r_i (m_i - p_i) - (1w + 0.w + 0.w + ... + 0.w)$ (n-times) $0 = \sum_{i=1}^{n} r_i (m_i - p_i) - \sum_{i=1}^{n} w_i$ where $r_i = \{^{1 \text{ if } i=1}_{0 \text{ if } i \neq 1}$ and $w_i = w$ for each i.

and since M is L-flat then there exists $s_{ij} \in R, y_i \in M$, $1 \le i \le n$, $1 \le j \le k$ such that $(m_i - p_i) - \sum_{j=1}^{k} s_{ij} y_i \in L(M)$ then $\overline{m_i - p_i} - \sum_{j=1}^{k} s_{ij} \bar{y}_i \in L(M)/P$,but $\overline{m_i - p_i} = \bar{m}_i$.

Thus $\bar{m}_i - \sum_{j=1}^{k} s_{ij} \bar{y}_i \in L(M/P)$ which implies that *M /P* is L-flat.

Theorem :

Let R be a nice ring, then every L-pure submodule of L-flat R-module is L-flate.

Proof :

Let M be an L-flat R-module and P an L-pure submodule of M. let $\{x_1, x_2, ..., x_n\} \subseteq P$ and $\{r_1, r_2, ..., r_n\} \subseteq R$, with $\sum_{i=1}^{n} r_i x_i = 0$. Since M is L-flat then there exist $\{y_1, y_2, ..., y_k\} \subseteq M$ and s_{ij} in R such that $\sum_{i=1}^{n} s_{ij} r_i = 0$, $1 \le j \le k$ and $\sum_{j=1}^{k} s_{ij} y_i - x_i \in L(M)$, $1 \le i \le n$. But P is L-pure then there exists $y_1', y_2', ..., y_k'$ in P such that $\sum_{j=1}^{k} s_{ij} y_i' - x_i \in L(M) \cap P = L(P)$ [7]. Thus P is L-flat.

The following theorem gives different characterization for flat modules [8].

Theorem:

Let M be an R-module. Then M is flat if and only if each presentation for M

$$0 \to K \xrightarrow{i} F \xrightarrow{\pi} M \to 0$$

and for each $u \in K$ there exists an R-homomorphism $\theta \in$ End(F) that satisfies:

1. $\theta(u) = 0$
2. $(\pi \circ \theta)(F) = \pi(F)$.

Recall that a submodule N of an R-module M is called small in M if $M = K + N$ implies that $M = K$ for each submodule K of M [9].

Proposition:

Every L-flat module with small prime radical is flat.

Proof :

Let M be an L-flat R-module and $L(M)$ is small in M. Let $0 \to K \xrightarrow{i} F \xrightarrow{\pi} M \to 0$ be a presentation for M where F is a free R-module and $K = \ker(\pi)$. It follows that $M = \pi(F) = (\pi \circ \theta)(F) + L(M)$ where $\theta \in End_R(F)$. But $L(M)$ is small in M, hence $\pi(F) = (\pi \circ \theta)(F)$. Thus M is flat [8].

It is known that if M is a finitely generated R-module, then Rad(M) the radical of M is small in M [9] and hence $L(M)$ is small in M.

Corollary :

Every finitely generated L-flat module is flat.

References

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الخلاصة

في هذا البحث قدمنا تعميماً للمقاسات المسطحة نسبةً للجذر الاولي، تم توسيع بعض النتائج التي تتعلق بالمقاسات المسطحة ثم درسنا العلاقات بين هذا النوع من المقاسات مع بعض الانواع المعروفة من المقاسات.