St-closed Submodule

Muna A. Ahmed\textsuperscript{1} and Maysaa R. Abbas\textsuperscript{2}

Department of Mathematics, College of Science for Women, Baghdad University, Baghdad-Iraq.
\textsuperscript{1} E-mail: math.200600986@yahoo.com.
\textsuperscript{2} E-mail: maysaa.alsaher@yahoo.com.

Abstract
Throughout this paper R represents commutative ring with identity and M is a unitary left R-module, the purpose of this paper is to study a new concept, (up to our knowledge), named St-closed submodules. It is stronger than the concept of closed submodules, where a submodule N of an R-module M is called St-closed (briefly \(N \leq_{\text{Stc}} M\)) in M, if it has no proper semi-essential extensions in M, i.e if there exists a submodule K of M such that N is a semi-essential submodule of K then \(N = K\). An ideal I of R is called St-closed if I is an St-closed R-submodule. Various properties of St-closed submodules are considered.

Keywords: Prime submodules, Essential submodules, Semi-essential submodules, Closed submodules, St-closed submodules, Fully prime modules and fully essential modules.

Introduction
Let R be a commutative ring with identity and let M be a unitary left R-module, and all R-modules under study contains prime submodules. It is well known that a nonzero submodule N of M is called essential (briefly \(N \leq_{\text{E}} M\)), if \(N \cap L \neq (0)\) for each nonzero submodule L of M [8], and a nonzero submodule N of M is called semi-essential (briefly \(N \leq_{\text{sem}} M\)), if \(N \cap P \neq (0)\) for each nonzero prime R-submodule P of M [2]. Equivalently, a submodule N of an R-module M is called semi-essential if whenever \(N \cap P = (0)\), then \(P = (0)\) for every prime submodule P of M [11], where a submodule P of M is called prime, if whenever \(rm \in P\) for \(r \in R\) and \(m \in M\), then either \(m \in P\) or \(r \in (P_M)\) [14].

A submodule N of M is called closed submodule (briefly \(N \leq_{\text{c}} M\)), if N has no proper essential extensions in M, i.e if \(N \leq_{\text{E}} K \leq M\) then \(N = K\) [6]. In our work we introduce a new concept (up to our knowledge), named St-closed submodules, which is stronger than the concept of closed submodules, where a submodule N of an R-module M is called St-closed if N has no proper semi-essential extensions in M, i.e if \(N \leq_{\text{sem}} K \leq M\) then \(N = K\). This paper consist of three sections, in section one we investigate the main properties of St-closed submodules, such as the transitivity property. Also we study the relationships between St-closed submodules, closed submodules and y-closed submodules. In S\textsubscript{2} we study the behavior of the class of St-closed submodules in the class of multiplication modules. In S\textsubscript{3} we study submodules satisfying the chain conditions on St-closed submodules.

\(S_1\): St-closed submodules
In this section we investigate the main properties of St-closed submodules such as the transitive property. Moreover, we study the relationships between St-closed submodules and other submodules.

Definition (1.1):
Let M be an R-module, a submodule N of M is called St-closed in M (briefly \(N \leq_{\text{Stc}} M\)), if N has no proper semi-essential extensions in M, i.e if there exists a submodule K of M such that N is a semi-essential submodule of K then \(N = K\). An ideal I of R is called an St-closed, if it is St-closed R-submodule.

Examples and Remarks (1.2):
1) Consider the Z-module \(M = Z_8 \oplus Z_2\). In this module there are eleven submodules which are \((\bar{0}, \bar{0})\), \((\bar{1}, \bar{0})\), \((\bar{0}, \bar{1})\), \((\bar{1}, \bar{1})\), \((\bar{2}, \bar{0})\), \((\bar{2}, \bar{1})\), \((\bar{4}, \bar{0})\), \((\bar{4}, \bar{1})\), \((\bar{0}, \bar{1})\), \((\bar{4}, \bar{0})\), \((\bar{4}, \bar{1})\), and M. The submodules \((\bar{0}, \bar{1})\), \((\bar{4}, \bar{1})\), and M are St-closed in M, since they have no proper semi-essential extensions in M. On the other hand, the submodules \((\bar{0}, \bar{0})\), \((\bar{1}, \bar{1})\), \((\bar{1}, \bar{0})\), \((\bar{2}, \bar{0})\), \((\bar{2}, \bar{1})\), \((\bar{4}, \bar{0})\), \((\bar{0}, \bar{1})\), \((\bar{4}, \bar{0})\), \((\bar{4}, \bar{1})\), \((\bar{2}, \bar{0})\), \((\bar{4}, \bar{1})\), are not St-closed
submodules in M, since they have semi-essential extensions in M.

2) Every R-module M is an St-closed submodule in M.

3) (0) may not be St-closed submodule of M, for example (0) is not St-closed submodule in the Z-module, Z₂.

4) If a submodule N of an R-module M is a semi-essential and an St-closed, then N = M.

5) If N is an St-closed submodule in M then (N / R)M need not be St-closed ideal in R, for example; (8) is an St-closed submodule in the Z-module Z₂₄, while ((8)₂ Z₂₄) = 8Z is not St-closed ideal in Z.

6) A direct summand of an R-module M is not necessary St-closed submodule in M, for example: Consider the Z-module, Z₁₂, where Z₁₂ = (3) ⊕ (4). The direct summand (4) = {0, 4, 8} is an St-closed submodule in Z₁₂, since (4) has no proper semi-essential extensions in Z₁₂. But the direct summand (3) = {0, 3, 6, 9} of Z₁₂ is not St-closed submodule since (3) is a semi-essential submodule of Z₁₂. Also the Z-module, Z₃₆ = (4) ⊕ (9), it is clear that (9) is a direct summand of Z₃₆ but not St-closed submodule in Z₃₆.

7) Let M be an R-module, if M = A ⊕ B, then even though A or B or both of them are prime submodules of M, then neither A nor B are necessary St-closed submodules in M. For example: the Z-module Z₃₀ = (3) ⊕ (6) = (2) ⊕ (15), both of (2) and (5) are prime submodules of Z₃₀ and direct summand, but neither (2) nor (5) are St-closed submodules in Z₃₀. In fact both of (2) and (5) are semi-essential submodules of Z₃₀.

8) Let M be an R-module, and let A be an St-closed submodule of M. If B is a submodule of M such that A ≡ B, then it is not necessary that B is an St-closed submodule in M. For example, the Z-module Z is an St-closed submodule in Z, and Z ≡ 3Z, but 3Z is not St-closed submodule in Z, since 3Z is a semi-essential submodule of Z.

Remarks (1.3):

1) Every St-closed submodule in an R-module M is a closed submodule in M.

Proof (1):

Let N be an St-closed submodule in M, and let K ≤ M with N ≤ K ≤ M. Since N ≤ K, then N ≤ K [2, Example (2), P.49]. But N is an St-closed submodule in M, thus N = K, that is N is a closed submodule in M.

The converse is not true in general, for example: In the Z-module Z₂₄ we note that (3) is a closed submodule in Z₂₄, but it is not St-closed. Also (9) is a closed submodule in Z₃₆, but it is not St-closed in Z₃₆.

2) Let N be an St-closed submodule of M. If B is a relative M-complement of N, then N is a relative M-complement of B, where a relative complement for K in M is any submodule L of M which is maximal with respect to the property K ∩ L = (0) [6].

Proposition (1.4):

Let M be an R-module, and let (0) ≠ C ≤ M, then there exists an St-closed submodule H in M such that C ≤ H.

Proof:

Consider the set V = {K} | K is a submodule of M such that C ⊆ K. It is clear that V ≠ ∅. By Zorn’s Lemma, V has a maximal element say H. In order to prove that H is an St-closed submodule in M; assume that there exists a submodule D of M such that H ⊆ D ≤ M. Since C ⊆ H and H ⊆ D, so by [11, Proposition (1.5)], C ≤ D. But this a contradicts the maximality of H, thus H = D. That is H is an St-closed submodule in M with C ≤ H.

We cannot prove the transitive property for St-closed submodules. However under some conditions we can prove this property as we see in the following result.

Proposition (1.5):

Let A and B be submodules of an R-module C. If A is an St-closed in B and B is an St-closed in C, then A is St-closed in C provided that B contained in (or containing) any semi-essential extension of A.

Proof:

Let L ≤ C such that A ≤ L ≤ C. By assumption we have two cases: If L ≤ B, since A is an St-closed submodule in B then A = L, hence A is an St-closed submodule in C. If B ≤ L, since A ≤ L, so by [2, Proposition 4], B ≤ L. But B is an St-closed in C, thus
B = L. That is A \leq_{\text{sem}} B. On the other hand, A is an St-closed submodule in B, so A = B, hence A is an St-closed submodule in C.

Recall that an R-module M is called chained if for each submodules A and B of M either A \leq B or B \leq A [13].

**Corollary (1.6):**

Let M be a chained module, and let A and B be submodules of M such that A \leq B \leq M. if A is an St-closed submodule in B and B is an St-closed submodule in M then A is an St-closed submodule in M.

**Proof:**

Let L \leq M such that A \leq_{\text{sem}} L \leq M. since M is a chained module, then either L \leq B or B \leq L, and the result follows as the same argument which used in the proof of the Proposition (1.5).

We can put other condition to get the transitive property of St-closed submodules, but before that we need to recall some definitions and give some remarks.

Recall that a nonzero R-module M is called fully essential, if every nonzero semi-essential submodule of M is essential submodule of M [12], and an R-module M is called fully prime, if every proper submodule of M is a prime submodule [3], and every fully prime module is a fully essential module [11].

**Proposition (1.7):**

Let N be a nonzero closed submodule of an R-module M. If every semi-essential extensions of N is a fully essential submodule of M, then N is an St-closed submodule in M.

**Proof:**

Let N be a nonzero closed submodule of M, and let L \leq M such that N \leq_{\text{sem}} L \leq M. By assumption L is a fully essential module, therefore N \leq L. But N \leq_{\text{co}} M, thus N = L. That is N \leq_{\text{Stc}} M.

**Remark (1.8):**

If an R-module M is fully prime, then every nonzero closed submodule in M is an St-closed submodule in M.

**Proof:**

Let N be a nonzero closed submodule of M, and let N \leq_{\text{sem}} L \leq M. Then by [11, Proposition (2.1)], N \leq L. But N \leq_{\text{co}} M, thus N = L, and we are don.

**Proposition (1.9):**

Let C be an R-module and let (0) \neq A \leq B \leq C. Assume that every semi-essential extension of A is a fully essential submodule of M. If A \leq_{\text{Stc}} B and B \leq_{\text{Stc}} C, then A \leq_{\text{Stc}} C.

**Proof:**

Since A \leq_{\text{Stc}} B and B \leq_{\text{Stc}} C, then by Remark (1.3) (1), A \leq_{\text{c}} B and B \leq_{\text{c}} C. this implies that A \leq_{\text{c}} C, [6, Proposition (1.5), P.18]. And by Proposition (1.7), A is an St-closed submodule in C.

In a similar proof of Proposition (1.9), and by using Remark (1.8) instead of Proposition (1.7) we can prove the following.

**Proposition (1.10):**

Let M be a fully prime module, and let (0) \neq A \leq_{\text{Stc}} B and B \leq_{\text{Stc}} M, then A \leq_{\text{Stc}} M.

The following remarks verify the hereditary of St-closed property between two submodules of an R-module M.

**Remark (1.11):**

Let A and B are submodules of an R-module M such that A \leq B \leq M. If A is an St-closed submodule in M, then B need not be St-closed submodule in M. For example; the Z-module Z is an St-closed submodule of Z and 2Z \leq Z, while 2Z is not St-closed submodule in Z.

**Remark (1.12):**

If A and B are submodules of an R-module M such that A \leq B \leq M. If A is an St-closed submodule in M, then B need not be St-closed submodule in M. For example; the Z-module Z and the submodules A = (0) and B = 2Z. Note that (0) is an St-closed submodule in Z, but 2Z is not St-closed submodule in Z.

**Proposition (1.13):**

If every submodule of M is an St-closed, then every submodule of M is a direct summand of M.

**Proof:**

Since every submodule of M is an St-closed, and by Remarks (1.3) (1), every St-closed submodule is a closed, so every submodule of M is a closed. Hence the result follows from [8, Exercises (6- c), P.139].

It is well known that the intersection of two closed submodules need not be closed
submodule for example: Consider the Z-module $M = Z \oplus Z_2$. If we take $A = < (1, 0) >$ and $B = < (1, 1) >$, it is clear that both of them are direct summands of $M$, so they are closed in $M$. But if $A \cap B = < (2, 0) >$ and $(A \cap B) \leq B$, then $A \cap B$ is not closed in $M$ [6, Example (1.6), P.19]. However, we have the following.

**Proposition (1.14):**
Let $A$ and $B$ be St-closed submodules in an $R$-module $M$, then $A \cap B$ is an St-closed submodule in $M$.

**Proof:**
Let $L \leq M$ such that $A \cap B \leq_{sem} L \leq M$. By [2, Corollary (6), P.49] $A \leq_{sem} L$ and $B \leq_{sem} L$. Since $A$ and $B$ are St-closed submodules in $M$, then $A = L = B$, hence $A \cap B = L$.

**Proposition (1.15):**
Let $M$ be an $R$-module, and let $A$ and $B$ be submodules of $M$ such that $A \leq B \leq M$. If $A$ is an St-closed submodule in $M$, then $A$ is an St-closed submodule in $B$.

**Proof:**
Suppose that $A \leq_{sem} L \leq B$, so $L \leq M$. But $A$ is an St-closed submodule in $M$, therefore $A = L$.

**Corollary (1.16):**
Let $A$ and $B$ be submodules of an $R$-module $M$. If $A \cap B$ is an St-closed submodule in $M$, then $A \cap B$ is an St-closed submodule in $A$ and $B$.

**Corollary (1.17):**
If $N$ and $K$ are St-closed submodules in an $R$-module $M$, then $N$ and $K$ are St-closed submodules in $N + K$.

**Proof:**
Since $N \leq N + K \leq M$, so by Proposition (1.15) we are done.

We can proof the following proposition by using [12, Lemma (1.15)]. In fact this Lemma in [12] is true when we instead the condition "fully prime" by the condition "fully essential".

**Proposition (1.18):**
Let $M = M_1 \oplus M_2$ be a fully essential $R$-module where $M_1$ and $M_2$ be submodules, and let $A$ and $B$ be nonzero submodules of $M_1$ and $M_2$ respectively. If $A$ and $B$ are St-closed submodules in $M_1$ and $M_2$ respectively. Then $A \oplus B$ is an St-closed submodule in $M_1 \oplus M_2$, provided that $\text{ann } M_1 + \text{ann } M_2 = R$.

**Proof:**
Assume that $A \oplus B \leq_{sem} L \leq M$. Since $\text{ann } M_1 + \text{ann } M_2 = R$, so by the same proof of [1, Proposition (4.2)], $L = L_1 \oplus L_2$, where $L_1 \leq M_1$ and $L_2 \leq M_2$. Therefore $A \oplus B \leq_{sem} L_1 \oplus L_2$, and by [12, Lemma (1.15)], $A \leq_{sem} L_1$ and $B \leq_{sem} L_2$. But both of $A$ and $B$ are St-closed submodules in $M$. So that $A = L_1$ and $B = L_2$, hence $A \oplus B = L_1 \oplus L_2$.

**Proposition (1.19):**
Let $M = M_1 \oplus M_2$ be an $R$-module where $M_1$ and $M_2$ be submodules of $M$, and let $A$, $B$ be St-closed submodule in $M_1$ and $M_2$ respectively. Then $A \oplus B$ is an St-closed submodule in $M_1 \oplus M_2$, provided that $\text{ann } M_1 + \text{ann } M_2 = R$. And all semi essential extensions of $A \oplus B$ are fully essential modules.

**Proof:**
Assume that $A \oplus B \leq_{sem} L \leq M$. By the same argument of Proposition (1.18) we have $A \oplus B \leq_{sem} L_1 \oplus L_2$, where $L = L_1 \oplus L_2$. Since $L$ is a fully essential module, then $A \oplus B \leq L_1 \oplus L_2$, this implies that $A \leq L_1$ and $B \leq L_2$. It is clear that both of $A$ and $B$ are closed submodules in $M$, thus $A = L_1$ and $B = L_2$, hence $A \oplus B = L_1 \oplus L_2$.

**Theorem (1.20):**
Let $M = M_1 \oplus M_2$ be a fully prime $R$-module where $M_1$ and $M_2$ be submodules of $M$ and let $A$, $B$ be nonzero submodules of $M_1$ and $M_2$ respectively. Then $A \oplus B$ is an St-closed submodule in $M_1 \oplus M_2$ if and only if $A$ and $B$ are St-closed submodules in $M_1$ and $M_2$ respectively.

**Proof:**
Assume that $A \leq_{sem} K \leq M_1$. Since $B \leq_{sem} B$, we can easily show that $K \oplus B$ is a fully prime module. In fact if $X$ is a proper submodule of $K \oplus B$, and since $M$ is a fully prime module, then $X$ is a prime submodule of $M$. By [7, Lemma (3.7)], $X$ is a prime submodule of $K \oplus B$, and by [12, Lemma (1.15)], $A \oplus B \leq_{sem} K \oplus B \leq M$. But $A \oplus B \leq_{Stc} M$, thus $A \oplus B = K \oplus B$, that is
A = K. In similar way we can prove that 
\( B \leq_{Stc} M \).

\( \Rightarrow \) Since in a fully prime module the St-closed submodule and closed submodule are equivalent, so the result follows from [6, Exercises (15), P.20].

Recall that the prime radical of an R-module M is denoted by \( \text{rad}(M) \), and it is the intersection of all prime submodules of M [10].

**Proposition (1.21):**

Let \( f: M \to M' \) be an R-epimorphism from an R-module M to an R-module M', and let B be a submodule of M such that \( \ker f \subseteq \text{rad}(M) \cap B \). If B is an St-closed submodule in M then \( f(B) \) is an St-closed submodule in M'.

**Proof:**

Let \( K' \) be a submodule of M' such that \( f(B) \leq \text{sem} K' \leq M' \). Since \( \ker f \subseteq \text{rad}(M) \), then \( f^{-1}(B) = B \) since ker f \( \subseteq B \). This implies that \( B \leq_{sem} f^{-1}(K') \). But B is an St-closed submodule in M, then \( B = f^{-1}(K') \). Since f is epimorphism so \( f(B) = K' \), and we are done.

**Corollary (1.22):**

Let A and B be submodules of an R-module M, such that \( A \subseteq \text{rad}(M) \cap B \). If B is an St-closed submodule in M then \( B = A \) is an St-closed submodule in \( M/A \).

Recall that a submodule \( Z(M) = \{ x \in M: \text{ann}(x) \leq_{e} R \} \). If \( Z(M) = M \), then M is called the singular module. If \( Z(M) = 0 \) then M is called a nonsingular module, [6]. A submodule N of an R-module M is called y-closed submodule of M, if \( M/N \) is an nonsingular submodule [6, P.42]. We cannot find a direct relation between St-closed and y-closed submodules. However, under some conditions we can find some cases of this relationship as the following proposition shows.

**Proposition (1.23):**

If M is a fully prime R-module, then every nonzero y-closed submodule is an St-closed submodule.

**Proof:**

Let A be a nonzero y-closed submodule in M, then by [9, Remarks and Examples (2.1.1) (3)], A is a closed submodule in M and by Remark (1.8), A is an St-closed submodule in M.

**Proposition (1.24):**

Let \( M \) be a nonsingular R-module, if a submodule \( N \) of \( M \) is an St-closed, then \( N \) is a y-closed submodule.

**Proof:**

Let \( N \) be an St-closed submodule in \( M \), by Remarks (1.3) (1) \( N \) is a closed submodule in \( M \). But M is a nonsingular module, so by [9, Proposition (2.1.2)], \( N \) is a y-closed submodule of \( M \).

**Another proof:**

Assume that \( M \) is a nonsingular R-module, and let \( N \) be an St-closed submodule in \( M \). Let \( Z(M/N) \equiv B \), where B is a submodule of \( M \) with \( N \leq B \). Clearly \( B \) is a singular module. Now \( N \) \( \subseteq B \) and \( M \) is a nonsingular module, therefore \( B \) is a nonsingular submodule of \( M \). Then by [6, Proposition (1.21), P.32], \( N \leq e B \), hence \( N \leq_{sem} B \). But A is an St-closed submodule in \( M \), thus \( N = B \), and \( Z(M/N) = (0) \). So \( M/N \) is a nonsingular module, and by the definition of y-closed submodule, \( N \) is a y-closed submodule in \( M \).

**Theorem (1.25):**

Let \( M \) be a fully prime R-module, and let \( N \) be a nonzero submodule of \( M \). Consider the following statement:

1. \( N \) is a y-closed submodule.
2. \( N \) is a closed submodule.
3. \( N \) is an St-closed submodule.

Then (1) \( \Rightarrow \) (2) \( \Leftrightarrow \) (3), and if \( M \) is a nonsingular module, then (3) \( \Rightarrow \) (1).

**Proof:**

(1) \( \Rightarrow \) (2) [9, Remarks and Examples (2.1.1), 3]
(2) \( \Leftrightarrow \) (3) Since \( M \) is a fully prime module then by, Remark (1.8), \( N \) is an St-closed submodule. The converse is clear.
(3) \( \Rightarrow \) (1) Since \( M \) is a nonsingular module, then by Proposition (1.24), \( N \) is a y-closed submodule.

**S2: St-closed submodules in multiplication modules**

In this section we study the behavior of the St-closed submodules in the class of
multiplication modules. Also we study the hereditary property of the St-closed submodules between R-modules and R itself.

Recall that An R-module M is called multiplication module, if every submodule N of M is of the form IM for some ideal I of R [4]. Recall that a nonzero prime submodule N of an R-module M is called minimal prime submodule of M such that P ⊆ N, then P = N [5].

**Proposition (2.1):**
Let M be a faithful and multiplication R-module, and let N be a nonzero prime submodule of M. If N is an St-closed submodule in M, then N is a minimal prime submodule of M.

**Proof:**
Suppose that N is not minimal prime submodule of M. By [2, Prop(3), P.53], N is a semi-essential submodule of M. But N is an St-closed, thus N = M. On the other hand N is a prime submodule that is N must be a proper submodule of M, so we get a contradiction.

**Proposition (2.2):**
Let M be a nonzero multiplication R-module with only one nonzero maximal submodule N, then N cannot be St-closed submodule in M.

**Proof:**
Assume that N is an St-closed submodule in M, so by [11, Proposition (2.13)] N ≤_sem M. By Examples and Remarks (1.2) (4) N = M, but this contradicts with a maximality of N, therefore N is not St-closed submodule in M.

**Remark (2.3):**
In Proposition (2.2), we get the same result when we replace the condition "nonzero multiplication" by the condition "finitely generated", and by using [11, Proposition (2.14)] instead of [11, Proposition (2.13)].

**Proposition (2.4):**
Let M be a faithful and multiplication module such that M satisfies the condition (*), if I is an St-closed ideal in J then IM is an St-closed submodule in JM.

**Proof:**
Assume that IM ≤_sem L ≤ JM. We have to show that IM = L. Since M is a multiplication module, then L = TM for some ideal T of R. Now IM ≤_sem TM ≤ JM, since M is a faithful and multiplication module and satisfying the condition (*), so by [11, Proposition (2.10)] I ≤_sem T ≤ J. But I is an St-closed ideal in J, then I = T. This implies that IM = TM = L, hence IM is an St-closed submodule in JM.

**Proposition (2.5):**
Let M be a finitely generated, faithful and multiplication module. If IM is an St-closed submodule in JM, then I is an St-closed ideal in J.

**Proof:**
Assume that I ≤_sem E ≤ J, then by [11, Proposition (2.11)] IM ≤_sem EM ≤ JM. Since IM is St-closed in JM, then IM = EM. This implies that I = E, [5, Theorem (3.1)]. Thus I is an St-closed submodule in J.

From Proposition (2.4) and Proposition (2.5) we get the following theorem.

**Theorem (2.6):** Let M be a finitely generated, faithful and multiplication module such that M satisfies the condition (*), then I is an St-closed ideal in J if and only if IM is an St-closed submodule in JM.

**Corollary (2.7):**
Let M be a finitely generated, faithful and multiplication R-module, and let N be a submodule of M. If M satisfies the condition (*), then the following statements are equivalent:
1. N is an St-closed submodule in M.
2. (N_R M) is an St-closed ideal in R.
3. N = IM for some St-closed ideal I in R.

**Proof:**
(1) ⇒ (2) Assume that N is an St-closed submodule in M. Since M is a multiplication module, then N = (N_R M) M [5]. Put (N_R M) ≡ I, so we get IM is an St-closed submodule in M. By Theorem (2.6), I is an St-closed ideal in R.

(2) ⇒ (3) Since M is a multiplication module, then N = (N_R M) M [5], and we are done.

(3) ⇒ (1) Since I is an St-closed ideal in R, so by Theorem (2.6), IM = N is an St-closed submodule in RM = M.
S3: Chain condition on St-closed submodules

In this section we study the chain condition on St-closed submodules, we give some results and examples about this concept. We start by the following definitions.

**Definition (3.1):**

An R-module M is said to have the ascending chain condition of St-closed submodules (briefly ACC on St-closed submodules), if every ascending chain \( A_1 \subseteq A_2 \subseteq \ldots \) of St-closed submodules in M is finite. That is there exists \( k \in \mathbb{Z}^+ \) such that \( A_n = A_k \) for all \( n \geq k \).

**Definition (3.2):**

An R-module M is said to have the descending chain condition of St-closed submodules (briefly DCC on St-closed submodules), if every descending chain \( A_1 \supseteq A_2 \supseteq \ldots \) of St-closed submodules in M is finite. That is there exists \( k \in \mathbb{Z}^+ \) such that \( A_n = A_k \) for all \( n \geq k \).

**Examples and Remarks (3.3):**

1) Every Noetherian (respectively Artinian) module satisfies ACC (DCC) on St-closed submodules.
2) Every uniform modules satisfies ACC on St-closed submodules. In fact in a uniform module, the only St-closed submodules are only M and sometime (0).
3) If M satisfies ACC on closed submodules, then M satisfies ACC on St-closed submodules.

**Proof:**

Let \( A_1 \subseteq A_2 \subseteq \ldots \) be an ascending chain of St-closed submodules of M. Since every St-closed submodule is closed submodule, then \( A_i \) is a closed submodule \( \forall \ i = 1, 2, \ldots \). By assumption M is satisfies ACC on closed submodule, so that \( \exists k \in \mathbb{Z}^+ \) such that \( A_n = A_k \) \( \forall \ n \geq k \). That is M satisfies ACC on St-closed submodules.

**Proposition (3.4):**

Let M be a finitely generated, faithful and multiplication R-module. Assume that M satisfies the condition (*), then M satisfies ACC on St-closed submodules, if and only if R satisfies ACC on St-closed ideals.

**Proof:**

\( \Rightarrow \): Let \( J_1 \subseteq J_2 \subseteq \ldots \) be an ascending chain of St-closed ideals in R. Since \( J_i \) is an St-closed ideal in R, then by Theorem (2.6), \( J_i \) is an St-closed submodule in M. But M satisfies ACC on St-closed submodules, so \( \exists k \in \mathbb{Z}^+ \) such that \( J_n = J_k \) \( \forall \ n \geq k \). But M is a finitely generated, faithful and multiplication module, therefore there exists \( k \in \mathbb{Z}^+ \) such that \( J_n = J_k \) \( \forall \ n \geq k \). Hence M satisfies ACC on St-closed ideals.

\( \Leftarrow \): Let \( A_1 \subseteq A_2 \subseteq \ldots \) be an ascending chain of St-closed submodules in M. Since M is a multiplication module, then \( A_i = J_i \) for some ideal \( J_i \) of R \( \forall i = 1, 2, \ldots \). It is clear that \( J_1 \subseteq J_2 \subseteq \ldots \), since \( A_i \) is an St-closed submodule in M, so that \( M \subseteq \ldots \), and M is a finitely generated, faithful and multiplication module and satisfying the condition (*), so by Theorem (2.6), J is an St-closed ideal in R. By [5, Theorem (3.1)], \( J_1 \subseteq J_2 \subseteq \ldots \), but R satisfies ACC on St-closed ideals, therefore there exists \( k \in \mathbb{Z}^+ \) such that \( J_n = J_k \forall n \geq k \), so that \( J_n M = J_k M \), for each \( n \geq k \), thus \( A_n = A_k \forall n \geq k \). That is M satisfies ACC on St-closed submodules.

**Proposition (3.5):**

Let M be a chained R-module, and let \( A \) be an St-closed submodule of M. If M satisfies ACC on St-closed submodules, then A satisfies ACC on St-closed submodules.

**Proof:**

Assume that M satisfies ACC on St-closed submodules and \( A_1 \subseteq A_2 \subseteq \ldots \), be ascending chain of St-closed submodules of A. Since \( A_1 \) is an St-closed submodule of M, and M is a chained module, so by Corollary (1.6), \( A_i \) is an St-closed submodule of M. Hence \( A_1 \subseteq A_2 \subseteq \ldots \), be ascending chain of St-closed submodules of M. By assumption there exists \( k \in \mathbb{Z}^+ \) such that \( A_n = A_k \forall n \geq k \), and we are done.

**Proposition (3.6):**

Let M be an R-module, and let \( N \) be a submodule of M such that \( N \subseteq \text{rad}(M) \cap H \), where H is any St-closed submodule in M. If \( \frac{M}{N} \) satisfies ACC on St-closed submodules, then M is satisfies ACC on St-closed submodules.
Let $A_1 \subseteq A_2 \subseteq \ldots$ be an ascending chain of St-closed submodules in $M$. Since $A_i$ is an St-closed submodule in $M$, and by assumption $N \subseteq \text{rad}(M) \cap A_i$, for each $i; i = 1, 2, \ldots$ so by Corollary (1.22), we get $\frac{A_i}{N}$ is an St-closed submodule in $\frac{M}{N}$ for each $i; i = 1, 2, \ldots$. Hence $\frac{A_1}{N} \subseteq \frac{A_2}{N} \subseteq \ldots$ be ascending chain of St-closed submodules in $\frac{M}{N}$. Since $\frac{M}{N}$ is satisfied ACC on St-closed submodules, so there exists $k \in Z^+$ such that $\frac{A_n}{N} = \frac{A_k}{N} \forall n \geq k$. So that $A_n = A_k$ and we get the result.

**Proposition (3.7):**
Let $M = M_1 \oplus M_2$ be a fully essential R-module, where $M_1$ and $M_2$ are submodules. If $M$ satisfies ACC on St-closed submodules, then $M_1$ (or $M_2$) satisfies ACC on nonzero St-closed submodules, provided that $\text{ann} M_1 + \text{ann} M_2 = R$.

**Proof:**
Let $A_1 \subseteq A_2 \subseteq \ldots$, be ascending chain of nonzero St-closed submodules of $M_1$. If $M_2$ is equal to zero then $M = M_1$, and this implies that $M_1$ satisfies ACC on nonzero St-closed submodule. Otherwise, since $A_i$ is a nonzero St-closed submodule in $M_1$, and $M_2$ is an St-closed submodule in $M_2$. So by Proposition (1.18), $A_i \oplus M_2$ is an St-closed submodule in $M \forall i = 1, 2, \ldots$. Since $M$ satisfies ACC on St-closed submodules, then there exists $k \in Z^+$ such that $A_n \oplus M_2 = A_k \oplus M_2 \forall n \geq k$. Thus $A_n = A_k, \forall n \geq k$. Similarity for $M_2$.

The converse of Proposition (3.7) is true when every closed submodule of $M$ is fully invariant as the following proposition shows.

**Proposition (3.8):**
Let $M = M_1 \oplus M_2$ be an R-module, where $M_1$ and $M_2$ are St-closed submodules in $M$. If $M_1$ satisfies ACC on nonzero St-closed submodules, for each $i; i = 1, 2$. Then $M$ satisfies ACC on nonzero St-closed submodules, provided that every St-closed submodule of $M$ is a fully invariant.

**Proof:**
Assume that $A_1 \subseteq A_2 \subseteq \ldots$ is an ascending chain of nonzero St-closed submodules in $M$, and let $\pi_j : M \rightarrow M_i$ be the projection maps for each $j \in J$ where $J = 1, 2, \ldots$. We claim that $A_j = (A_j \cap M_1) \oplus (A_j \cap M_2)$. To verify that, let $x \in A_j$ then $x = m_1 + m_2$, where $m_1 \in M_1$ and $m_2 \in M_2$. Since $A_j$ is an St-closed submodule of $M$ for each $j \in J$, and by our assumption, $A_j$ is a fully invariant which implies that $\pi_j(x) = m_1 \in A_j \cap M_1$ and $\pi_j(x) = m_2 \in A_j \cap M_2$. So $x \in (A_j \cap M_1) \oplus (A_j \cap M_2)$. Thus $A_j \subseteq (A_j \cap M_1) \oplus (A_j \cap M_2)$. But $(A_j \cap M_1) \oplus (A_j \cap M_2) \subseteq A_j$, therefore $A_j = (A_j \cap M_1) \oplus (A_j \cap M_2)$.

Note that $A_j$ and $M_i$ are St-closed submodule in $M$, so by Proposition (1.14), $A_j \cap M_i$ is an St-closed submodule in $M$. Since $A_j \cap M_i \subseteq M_i \subseteq M$, then by Proposition (1.15), $A_j \cap M_i$ is an St-closed submodules in $M_i$ for each $i = 1, 2$ and $j = 1, 2, \ldots$. We can easily show that $(A_j \cap M_i) \neq (0)$ for each $j = 1, 2, \ldots$, and $i = 1, 2$. In fact if $A_j \cap M_i = (0)$ for each $i = 1, 2$ and $j = 1, 2, \ldots$, then by using $A_j = (A_j \cap M_1) \oplus (A_j \cap M_2)$, we get $A_j = (0)$, which is contradicts with our assumption. That is $A_j \cap M_i$ are nonzero St-closed submodules in $M$ for all $i, j$. We have the following ascending chain of St-closed submodules in $M_i$, $(A_j \cap M_i) \subseteq (A_2 \cap M_i) \subseteq \ldots, \forall i = 1, 2$. But $M_i$ satisfies ACC on nonzero St-closed submodules, then for each $i = 1, 2$, there exists $k_i \in Z^+$ such that $A_n \cap M_i = A_{k_i} \cap M_i \forall n \geq k_i$. Let $k = \max\{k_1, k_2\}$. So $A_n = (A_n \cap M_1 \oplus (A_n \cap M_2) = (A_k \cap M_1) \oplus (A_k \cap M_2) = A_k$ for each $n \geq k$. Thus $M$ satisfies ACC on nonzero St-closed submodules.

**Remark (3.9):**
We can generalize Proposition (3.8) for finite index I of the direct sum of R-modules.

**Reference**


