Fibrewise Pairwise bi-Separation Axioms

Y. Y. Yousif\(^1\) and L. A. Hussain\(^2\)

\(^1\) Department of Mathematics, College of Education for Pure Science (Ibn Al-haitham), Baghdad University, Baghdad-Iraq.

\(^2\) Ministry of Education, Directorate of Education, Baghdad, Al-Kark-3

Corresponding author: yoyayousif@yahoo.com\(^1\), Liwaaalhashemy@Gmail.com\(^2\)

Abstract

The main idea of this research is to consider fibrewise pairwise versions of the more important separation axioms of ordinary bitopology named fibrewise pairwise bi-\(T_0\) spaces, fibrewise pairwise bi-\(T_1\) spaces, fibrewise pairwise bi-\(R_0\) spaces, fibrewise pairwise bi-Hausdorff spaces, fibrewise pairwise functionally bi-\(T_0\)-Hausdorff spaces, fibrewise pairwise bi-regular spaces, fibrewise pairwise bi-normal spaces and fibrewise pairwise functionally bi-normal spaces. In addition we offer some results concerning it.

2010 MSC: 55R70, 54C05, 54C08, 54C10, 54D10, 54D15.

Keywords: Fibrewise bitopological spaces, Fibrewise pairwise bi-\(T_0\) spaces, Fibrewise pairwise bi-\(T_1\) spaces, Fibrewise pairwise bi-\(R_0\) spaces, Fibrewise pairwise bi-Hausdorff spaces, Fibrewise pairwise bi-regular spaces and fibrewise pairwise bi-normal spaces.

1. Introduction

In order to begin the category in the classification of fibrewise (briefly F.W.) sets over a given set, named the base set, which say \(B\). A F.W. set over \(B\) consist of a function \(p: M \rightarrow B\), that is named the projection on the set \(M\). The fibre over \(b\) for every point \(b\) of \(B\) is the subset \(M_b = p^{-1}(b)\) of \(M\). Since we do not require \(p\) is surjective, the fibre Perhaps, will be empty, also, for every \(B^*\) subset of \(B\) we considered \(M_{B^*} = p^{-1}(B^*)\) like a F.W. set with the projection determined by \(p\) over \(B^*\).

The another notation \(M \upharpoonright B^*\) is some time fitting. We considered for every set \(T\), the Cartesian product \(B \times T\), by the first projection like a F.W. set \(B\).

Definition 1.1, [4]:

If \(M\) and \(N\) with projections \(p_M\) and \(p_N\), respectively, are F.W. sets over \(B\), a function \(\varphi: M \rightarrow N\) is named F.W. function if \(p_N \circ \varphi = p_M\), or \(\varphi(M_b) \subset N_b\) for every \(b \in B\).

Observe that a F.W. function \(\varphi: M \rightarrow N\) over \(B\) limited by restriction, a F.W. function \(\varphi_{B^*}: M_{B^*} \rightarrow N_{B^*}\) over \(B^*\) for every subset \(B^*\) of \(B\).

Definition 1.2, [4]:

Let \((B, \Lambda)\) be a topological space. The F.W. topology on a F.W. set \(M\) over \(B\) mean any topology on \(M\) makes the projection \(p\) is continuous.

Remark 1.3, [4]:

(a) The coarsest such topology is the topology made by \(p\), in which the open sets of \(M\) are exactly the inverse image of the open sets of \(B\); this is named F.W. indiscrete topology.

(b) The F.W. topological space over \(B\) is defined to be a F.W. set over \(B\) with a F.W. topology.

We consider the topology product \(B \times_B T\), for every topological space \(T\), like a F.W. topological spaces over \(B\) by the first projection. The equivalences in the type of F.W. topological spaces are named F.W. topological equivalences. We say that \(M\) is trivial, as a F.W. topological spaces over \(B\), if for some topological space \(T\), \(M\) is F.W. topologically equivalent to \(B \times_B T\). In F.W. topology the word neighborhood (briefly nbd) is used in exactly in the similar sense like it is in ordinary topology, but the words F.W. basic may want some details, hence let \(M\) be F.W. topological space over \(B\), if \(x\) is a point of \(M_b\); \(b \in B\), describe a family \(N(x)\) of nbds of \(x\) in \(M\) as F.W. basic if for each nbd \(U\) of \(x\) where \(M_W \cap V \subset U\), for a few member \(V\) of \(N(x)\) and nbd \(W\) of \(b\) in \(B\). For example, as in the topological product \(B \times T\), where \(T\) is a topological space, the family of Cartesian products \(B \times N(t)\), where \(N(t)\) runs through
the nbds of \( t \), is F.W. basic for \((b, t)\). Otherwise we follow closely James [4], Engelking [3] and Bourbaki [2].

**Definition 1.4, [4]:**
The F.W. function \( \varphi : M \to N \), where \( M \) and \( N \) are F.W. topological spaces over \( B \) is named:
(a) Continuous if for every \( x \in M_B; \ b \in B \), the inverse image of every open set of \( \varphi(x) \) is an open set of \( x \).
(b) Open if for every \( x \in M_B; \ b \in B \), the direct image of every open set of \( x \) is an open set of \( \varphi(x) \).

**Definition 1.5, [3]:**
Assume we are given a topological space \( M \), a family \( \{\varphi_s\}_{s \in S} \) of continuous functions, and a family \( \{N_s\}_{s \in S} \) of topological spaces where \( \varphi_s : M \to N_s \) the function transfer \( x \in M \) to the point \( \{\varphi_s(x)\} \in \prod_{s \in S} N_s \) is continuous, it is named the diagonal of the functions \( \{\varphi_s\}_{s \in S} \) and is denoted by \( \Delta_{s \in S} \varphi_s \) or \( \Delta \varphi_1 \Delta \varphi_2 \Delta... \Delta \varphi_k \) if \( S = \{1, 2, ..., k\} \).

**Definition 1.6, [4]:**
The F.W. topological space \((M, \tau)\) over \((B, A)\) is named F.W. closed, (resp. F.W. open) if the projection \( p \) is closed (resp. open).

The bitopological spaces study was first created by Kelly [5] in 1963 and after that a large number of researches have been completed to generalize the topological ideas to bitopological setting. In this research \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) (or briefly, \( M \) and \( N \)) always mean bitopological spaces on which no separation axioms are supposed unless clearly stated. By \( \tau_i\)-open (resp., \( \tau_i\)-closed), we shall mean the open (resp., closed) set with respect to \( \tau_i \) in \( M \), where \( i = 1, 2 \). \( A \) is open (resp., closed) if it is both \( \tau_1\)-open (resp., \( \tau_1\)-closed), \( \tau_2\)-open (resp., \( \tau_2\)-closed) in \( M \). As well as, we built on some of the result in [1, 6, 9, 10, 11]. Otherwise we go behind closely I. M. James [4], R. Engelking [3] and N. Bourbaki [2].

**Definition 1.7, [5]:**
The triple \((M, \tau_1, \tau_2)\) where \( M \) is a non-empty set and \( \tau_1 \) and \( \tau_2 \) are topologies on \( M \) is named bitopological spaces.

**Definition 1.8, [5]:**
A function \( \varphi : (M, \tau_1, \tau_2) \to (N, \sigma_1, \sigma_2) \) is said to be \( \tau_i\)-continuous (resp. \( \tau_i\)-closed), if the functions \( \varphi : (M, \tau_i) \to (N, \sigma_i) \) are continuous (resp. open and closed), \( \varphi \) is named continuous (resp. open and closed) if it is \( \tau_i\)-continuous (resp. \( \tau_i\)-open and \( \tau_i\)-closed) for every \( i = 1, 2 \).

**Definition 1.9, [8]:**
Let \((B, A_1, A_2)\) be a bitopological space. The F.W. bitopology on a F.W. set \( M \) over \( B \) mean any bitopology on \( M \) makes the projection \( p \) is continuous.

**Definition 1.10, [7]:**
A bitopological space \((M, \tau_1, \tau_2)\) is said to be pairwise \( T_0 \) space if for every pair of points \( x \) and \( y \) such that \( x \neq y \) there exists a \( \tau_i\)-open set containing \( x \) but not containing \( y \) or a \( \tau_j\)-open set containing \( y \) but not containing \( x \), where \( i, j = 1, 2, i \neq j \).

2. Fibrewise pairwise \( bi-T_0 \), pairwise \( bi-T_1 \), pairwise \( bi-R_0 \), and pairwise \( bi-Hausdorff \) spaces.

The concepts of pairwise open sets have an important role in F.W. separation axioms. By using these concepts we can construct many several F.W. separation axioms. Now we introduce the versions of F.W. pairwise \( bi-T_0 \), F.W. pairwise \( bi-T_1 \), F.W. pairwise \( bi-R_0 \), and F.W. pairwise \( bi-Hausdorff \) spaces as follows.

**Definition 2.1:**
Let \((M, \tau_1, \tau_2)\) be F.W. bitopological space over \((B, A_1, A_2)\). Then \( M \) is named F.W. pairwise \( bi-T_0 \) if whenever \( x, y \in M_B; \ b \in B \) and \( x \neq y \), either there exists a \( \tau_i\)-open set \( U \) of \( x \) which does not contain \( y \) in \( M \) or \( \tau_j\)-open set \( V \) of \( y \) which does not contain \( x \) in \( M \), where \( i, j = 1, 2, i \neq j \).

**Remark 2.2:**
(a) \((M, \tau_1, \tau_2)\) is F.W. pairwise \( bi-T_0 \) space iff each fiber \( M_b \) is pairwise \( bi-T_0 \) space.
(b) Subspaces of F.W. pairwise \( bi-T_0 \) spaces are F.W. pairwise \( bi-T_0 \) spaces.
(c) The F.W. bitopological products of F.W. pairwise \( bi-T_0 \) spaces with the family of F.W. pairwise projections are F.W. pairwise \( bi-T_0 \) spaces.
For sure anyone can makes a F.W. version of the pairwise bi-$T_1$ space in a similar way. Let $(M, \tau_{1}, \tau_{2})$ be F.W. bitopological space over $(B, A_1, A_2)$. Then $M$ is named F.W. pairwise bi-$T_1$ if whenever $x, y \in M_b; b \in B$ and $x \neq y$, there exist a $\tau_i$-open sets $U_i$ and a $\tau_j$-open set $V$ in $M$ such that $x \in U_i, y \notin U_i$ and $x \notin V, y \in V, i, j = 1, 2, i \neq j$. But it turns out that there is no real use for this in what we are going to do. In its place we formulate some use of a new axiom “The axiom is that every $\tau_i$-open set contains the $\tau_j$-closure of each of its points”, and use the word pairwise bi-$R_0$ space. This is correct for pairwise bi-$T_1$ spaces and for pairwise bi-$T_{1\frac{3}{4}}$ regular spaces. Thinking of it like a weak structure of pairwise bi-regularity. For example, indiscrete spaces are pairwise bi-$R_0$ spaces. The F.W. version of the pairwise bi-$R_0$ axiom as the following.

**Definition 2.3:**
A F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, A_1, A_2)$ is named F.W. pairwise bi-$R_0$ if for every $x \in M_b; b \in B$, and every $\tau_i$-open set $V$ in $M$, there exists a nbd $W$ of $b$ in $V$ where $W$ is containing the $\tau_i$-closure of $\{x\} \in M_b$ is (i.e., $M_b \cap \tau_i - \text{Cl}(x) = \varnothing$) where $i, j = 1, 2, i \neq j$.

For example, $(B, A_1, A_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise bi-$R_0$ space for all pairwise bi-$R_0$ spaces $T$.

**Remark 2.4:**
(a) The nbd of $x$ are given by a F.W. basis it is enough if the condition in Definition (2.3) is satisfied for every F.W. basic nbd.
(b) If $(M, \tau_1, \tau_2)$ is F.W. pairwise bi-$R_0$ space over $(B, A_1, A_2)$, then for each subspace $(B, A_1, A_2) \times_B (T, \tau_1, \tau_2)$ is F.W. pairwise bi-$R_0$ space over $B^*$.

**Proposition 2.5:**
Let $\varphi : M \rightarrow M^*$ be a continuous F.W. embedding function, where $(M, \tau_1, \tau_2)$ and $(M^*, \tau_1^*, \tau_2^*)$ are F.W. bitopological spaces over $(B, A_1, A_2)$. If $M^*$ is F.W. pairwise bi-$R_0$ then so is $M$.

**Proof:**
Let $V$ be a $\tau_1$-open set of $x$ in $M$, where $x \in M_b; b \in B$. Then $V = \varphi^{-1}(V^*)$, where $V^*$ is a $\tau^*_1$-open set of $x^* = \varphi(x)$ in $M^*$. Because $M^*$ is F.W. pairwise bi-$R_0$ then we have a nbd. $W$ of $b$ in $B$, where $M^*_W \cap \tau_j^* - \text{Cl}(x^*) \subset V^*$. Hence, $M^*_W \cap \tau_j - \text{Cl}(x) \subset \varphi^{-1}(M^*_W \cap \tau_j^* - \text{Cl}(x^*)) \subset \varphi^{-1}(V^*) = V$ and hence $M$ is F.W. pairwise bi-$R_0$ where $i, j = 1, 2, i \neq j$.

The class of F.W. pairwise bi-$R_0$ spaces is finitely multiplicative, like in the following.

**Proposition 2.6:**
If $(M_1, \tau_{1i}, \tau_{1j})$ is a finite family of F.W. pairwise bi-$R_0$ spaces over $B$. Then the F.W. bitopological product $M = \prod_b M_b$ is F.W. pairwise bi-$R_0$.

**Proof:**
Let $x \in M_b; b \in B$. Consider a $\tau_i$-open set $V = \prod_b V_b$ of $x$ in $M$, where $V_b$ is a $\tau_i$-open set of $\pi_b(x) = x_b$ in $M_b$ for each index $r$. Since $M_r$ is F.W. pairwise bi-$R_0$ then, we have a nbd $W_r$ of $b$ in $V$ where $(M_r \mid \tau_r) \cap \tau_r - \text{Cl}(x_r) \subset V_r$. Then we regard $W$ as a nbd of $b$ where $W$ is an intersection of $W_r$ and $M_r$ is a finite family of F.W. pairwise bi-$R_0$ where $i, j = 1, 2, i \neq j$.

The similar conclusion holds for infinite F.W. products provided all of the factors is F.W. nonempty.

**Proposition 2.7:**
Assume that $\varphi : M \rightarrow N$ is closed, continuous F.W. surjection function, where $(M, \tau_1, \tau_2)$ and $(N, \sigma_1, \sigma_2)$ are F.W. bitopological spaces over $B$. If $M$ is F.W. pairwise bi-$R_0$ then so is $N$.

**Proof:**
Assume that $V$ is a $\sigma_1$-open set of $y$ in $N$, where $y \in N_b; b \in B$. Choose $x \in \varphi^{-1}(y)$.

Then $U = \varphi^{-1}(V)$ is a $\tau_1$-open set of $x$ in $M$. Because $M$ is F.W. pairwise bi-$R_0$, then we have a nbd $W$ of $b$ in $V$, where $M_b \cap \tau_j - \text{Cl}(x) \subset U$. Therefore $N_W \cap \varphi(\tau_j - \text{Cl}(x)) \subset \varphi(U) = V$. Because $\varphi$ is closed, $\varphi(\tau_j - \text{Cl}(x)) = \sigma_j - \text{Cl}(\varphi(x))$. Hence $NW \cap \sigma_j - \text{Cl}(\varphi(x)) \subset V$ and $N$ is F.W. pairwise bi-$R_0$ where $i, j = 1, 2, i \neq j$.

Now we introduce the version of F.W. pairwise bi-Hausdorff spaces like the following.

**Definition 2.8:**
A F.W. bitopological space $(M, \tau_1, \tau_2)$ over $(B, A_1, A_2)$ is named F.W. pairwise bi-
Hausdorff if whenever \( x, y \in M_b \); \( b \in B \) and \( x \neq y \) there exist a disjoint pair of \( \tau_i \)-open set \( U \) of \( x \) and \( \tau_j \)-open set \( V \) of \( y \) in \( M \), where \( i, j = 1, 2, i \neq j \).

For example, \((B,A_1,A_2) \times_B (T,\tau_1,\tau_2)\) is F.W. pairwise \( bi \)-Hausdorff space for all pairwise \( bi \)-Hausdorff spaces \( T \).

**Remark 2.9:**

If \((M,\tau_1,\tau_2)\) is F.W. pairwise \( bi \)-Hausdorff space over \((B,A_1,A_2)\) then \(M_b^*\) is F.W. pairwise \( bi \)-Hausdorff over \( B^* \) for every subspace \( B^* \) of \( B \). Specially the fibers of \((M,\tau_1,\tau_2)\) are pairwise \( bi \)-Hausdorff spaces. On the other hand a F.W. bitopological space with pairwise \( bi \)-Hausdorff fibres is not necessarily pairwise \( bi \)-Hausdorff.

**Example 2.10:**

Let \( M=\{1, 2, 3\}, \tau_1 = \{M,\varphi,\{1\},\{1,2\}\}, \tau_2 = \{M,\varphi,\{1\},\{1,3\}\} \). Let \( B=\{a, b\}, A_1 = \{B,\varphi,\{a\}\}, A_2 = I \). Let \( p: M \rightarrow B \) where: \( p(1) = a, p(2) = b = p(3) \). Then, we have \( M_b = \{2, 3\}, \tau_1 \mid_{M_b} = \{M_b,\varphi,\{2\}\}, \tau_2 \mid_{M_b} = \{M_b,\varphi,\{3\}\} \). Then \( \exists \tau_1 \mid_{M_b} \) open set \( U = \{2\} \) where \( 2 \in U, 3 \in U \) and there exist \( \tau_2 \mid_{M_b} \) open set \( V = \{3\} \) where \( 3 \in V, 2 \notin V \), where \( U \cap V = \varnothing \). But \( M \) is not pairwise \( bi \)-Hausdorff since: \( 2 \) and \( 3 \in M \) and \( 2 \neq 3 \), and there is no disjoint pair of open sets of \( 2 \) and \( 3 \).

**Proposition 2.11:**

The F.W. bitopological space \((M,\tau_1,\tau_2)\) over \((B,A_1,A_2)\) is F.W. pairwise \( bi \)-Hausdorff iff the diagonal embedding \( \Delta: M \rightarrow M \times_B M \) is \( \tau_i \times_B \tau_i \)-closed.

**Proof:**

(\(\Rightarrow\)) Let \( x, y \in M_b \); \( b \in B \) and \( x \neq y \). Since \( \Delta(M) \) is \( \tau_i \times_B \tau_i \)-closed in \( M \times_B M \), then \((x,y)\) a point of the complement, admits a F.W. product \( \tau_i \times_B \tau_j \)-open set \( U \times_B V \) which does not meet \( \Delta(M) \), and then \( U \), \( V \) are disjoint pair of \( x, y \). Where \( U \) is \( \tau_i \)-open set of \( x \), and \( V \) is \( \tau_j \)-open set of \( y \), where \( i, j = 1, 2, i \neq j \).

(\(\Leftarrow\)) The reverse direction is similar.

Subspaces of F.W. pairwise \( bi \)-Hausdorff spaces are F.W. pairwise \( bi \)-Hausdorff spaces. Actuality we have.

**Proposition 2.12:**

Assume that \( \varphi: M \rightarrow M^* \) is a continuous embedding F.W. function, where \((M,\tau_1,\tau_2)\) and \((M^*,\tau'_1,\tau'_2)\) are F.W. bitopological spaces over \((B,A_1,A_2)\). If \( M^* \) is F.W. pairwise \( bi \)-Hausdorff then so is \( M \).

**Proof:**

Let \( x, y \in M_b \); \( b \in B \) and \( x \neq y \). Then \( \varphi(x), \varphi(y) \in M_{b^*} \) are distinct, since \( M^* \) is F.W. pairwise \( bi \)-Hausdorff, then we have a \( \tau_i \)-open set \( U^* \) of \( \varphi(x) \) and \( \tau_j \)-open set \( V^* \) of \( \varphi(y) \) in \( M^* \) which are disjoint. Because \( \varphi \) is continuous, the inverse images \( \varphi^{-1}(U^*) = U, \varphi^{-1}(V^*) = V \), such that \( U \) is \( \tau_i \)-open set of \( x \) and \( V \) is \( \tau_j \)-open set of \( y \) in \( M \) which are disjoint and so \( M \) is F.W. pairwise \( bi \)-Hausdorff where \( i, j = 1, 2, i \neq j \).

**Proposition 2.13:**

Let \( \varphi: M \rightarrow N \) be a continuous F.W. function, where \((M,\tau_1,\tau_2)\) and \((N,\sigma_1,\sigma_2)\) are F.W. bitopological spaces over \((B,A_1,A_2)\). If \( N \) is F.W. pairwise \( bi \)-Hausdorff then the F.W. graph \( \Gamma: M \rightarrow M \times_B N \) of \( \varphi \) is a \( \tau_i \times_B \sigma_i \)-closed embedding.

**Proof:**

The F.W. graph is defined in the similar method like the ordinary graph, but with values in the F.W. product, hence the figure shown below is commutative.

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & M \times_B N \\
\downarrow{\Delta} & & \downarrow{\varphi \times_B \text{id}_N} \\
N & \xrightarrow{\Gamma} & N \times_B N \\
\end{array}
\]

**Fig. (1.1) Diagram of Proposition 2.13.**

Since \( \Delta(N) \) is \( \sigma_i \times_B \sigma_i \)-closed in \( N \times_B N \), by Proposition (2.11), so \( \Gamma(M) = (\varphi \times_B \text{id}_N)^{-1}(\Delta(N)) \) is \( \tau_i \times_B \sigma_i \)-closed in \( M \times_B N \), as asserted, where \( i,j = 1, 2, i \neq j \).

The category of F.W. pairwise \( bi \)-Hausdorff spaces is multiplicative, like the following sense.

**Proposition 2.14:**

Assume that \( \{(M_r,\tau_{1r},\tau_{2r})\} \) is a family of F.W. pairwise \( bi \)-Hausdorff spaces over
The F.W. bitopological product \( M = \prod_{r} M_{r} \) with the family of F.W. projection \( \pi_{r} : M = \prod_{r} M_{r} \rightarrow M_{r} \) is F.W. pairwise bi-Hausdorff.

**Proof:**

Let \( x, y \in M_{b} \); \( b \in B \) and \( x \neq y \). Then \( \pi_{r}(x) = x_{r} \neq \pi_{r}(y) = y_{r} \) for some index \( r \). Because \( M_{r} \) is F.W. pairwise bi-Hausdorff, then we have an \( \tau_{ir} \)-open set \( U_{r} \) of \( x_{r} \) and \( \tau_{jr} \)-open set \( V_{r} \) of \( y_{r} \) in \( M_{r} \) which are disjoint. Because \( \pi_{r} \) is continuous, the inverse images \( U, V \) are disjoint \( \tau_{i} \)-open and \( \tau_{j} \)-open sets respectively of \( x, y \) in \( M \), where \( i, j = 1, 2, i \neq j \).

The pairwise functionally version of the F.W. pairwise bi-Hausdorff axiom is stronger than the non pairwise functional version but its properties are quite like. Here and in another place we use \( I \) to mean the closed unit interval \([0,1]\) in the real line \( \mathbb{R} \).

**Definition 2.15:**

A F.W. bitopological space \( (M, \tau_{1}, \tau_{2}) \) over \( (B, A_{1}, A_{2}) \) is F.W. pairwise functionally bi-Hausdorff if for every \( x, y \in M_{b}; b \in B \) and \( x \neq y \), there exists a nbd \( W \) of \( b \) in \( B \) and disjoint pair \( \tau_{i} \)-open sets \( U \) of \( x \) and \( \tau_{j} \)-open set \( V \) of \( y \) in \( M \) and a continuous function \( \lambda: M_{W} \rightarrow I \) where \( M_{b} \cap U \subset \lambda^{-1}(0) \) and \( M_{b} \cap V \subset \lambda^{-1}(1) \) where \( i, j = 1, 2, i \neq j \).

For example, \( (B, A_{1}, A_{2}) \times_{B} (T, \tau_{1}, \tau_{2}) \) is F.W. pairwise functionally bi-Hausdorff space for each pairwise functionally bi-Hausdorff spaces \( T \).

**Remark 2.16:**

If \( (M, \tau_{1}, \tau_{2}) \) is F.W. pairwise functionally bi-Hausdorff space over \( (B, A_{1}, A_{2}) \) then \( M_{b}^{*} \) is F.W. pairwise functionally bi-Hausdorff over \( B^{*} \) for every subspace \( B^{*} \) of \( B \). In particular the fibres of \( M \) are pairwise functionally bi-Hausdorff spaces.

Subspaces of F.W. pairwise functionally bi-Hausdorff spaces are F.W. pairwise functionally bi-Hausdorff spaces. Actually we have.

**Proposition 2.17:**

Assume that \( \varphi : M \rightarrow M^{*} \) is a continuous embedding F.W. function, where \( (M, \tau_{1}, \tau_{2}) \) and \( (M^{*}, \tau_{1}^{*}, \tau_{2}^{*}) \) are F.W. bitopological spaces over \( (B, A_{1}, A_{2}) \). If \( M^{*} \) is F.W. pairwise functionally bi-Hausdorff then so is \( M \).

**Proof:**

Let \( x, y \in M_{b} \) and \( x \neq y; b \in B \). Then \( \varphi(x) = x^{*}, \varphi(y) = y^{*} \in M_{b}^{*} \). \( x^{*} \neq y^{*} \). Since \( M^{*} \) is F.W. pairwise functionally bi-Hausdorff, then we have a nbd \( W \) of \( b \) in \( B \) and disjoint pair \( \tau_{i}^{*} \)-open set \( U^{*} \) of \( x^{*} \) and \( \tau_{j}^{*} \)-open set \( V^{*} \) of \( y^{*} \) and a continuous function \( \lambda^{*}: M^{*} \rightarrow I \) where \( M_{b}^{*} \cap U^{*} \subset (\lambda^{*})^{-1}(0) \) and \( M_{b}^{*} \cap V^{*} \subset (\lambda^{*})^{-1}(1) \). Now, since \( \varphi \) is continuous, \( \varphi^{-1}(U^{*}) = U \) and \( \varphi^{-1}(V^{*}) = V \) are disjoint pair \( \tau_{i} \)-open set of \( x \) and \( \tau_{j} \)-open set of \( y \) respectively and the continuous function \( \lambda \) where \( \lambda = \lambda^{*} \circ \varphi: M_{W} \rightarrow I \) such that \( M_{b} \cap U \subset \lambda^{-1}(0) \) and \( M_{b} \cap V \subset \lambda^{-1}(1) \), where \( i, j = 1, 2, i \neq j \).

Furthermore the category of F.W. pairwise functionally bi-Hausdorff spaces is multiplicative, as the following.

**Proposition 2.18:**

Assume that \{\( (M_{r}, \tau_{1r}, \tau_{2r}) \)\} is a family of F.W. pairwise functionally bi-Hausdorff spaces over \( (B, A_{1}, A_{2}) \). The F.W. bitopological product \( M = \prod_{r} M_{r} \) with the family of F.W. projection \( \pi_{r} : M = \prod_{r} M_{r} \rightarrow M_{r} \) is F.W. pairwise functionally bi-Hausdorff.

**Proof:**

Let \( x, y \in M_{b} \); \( b \in B \), and \( x \neq y \). Then \( \pi_{r}(x) = x_{r}, \pi_{r}(y) = y_{r} \in (M_{r})_{b} \) for some index \( r \) where \( x_{r} \neq y_{r} \). Since \( M_{r} \) is F.W. pairwise functionally bi-Hausdorff, then we have a nbd \( W_{r} \) of \( b \) in \( B \) and disjoint pair \( \tau_{ir} \)-open set \( U_{r} \) of \( x_{r} \) and \( \tau_{jr} \)-open set \( V_{r} \) of \( y_{r} \) and a continuous function \( \lambda: M_{W} \rightarrow I \) such that \( (M_{r})_{b} \cap U_{r} \subset \lambda^{-1}(0) \) and \( (M_{r})_{b} \cap V_{r} \subset \lambda^{-1}(1) \). Now the intersection of \( W_{r} \) is a nbd \( W \) of \( b \) in \( B \), and since \( \pi_{r} \) is continuous, then \( \pi_{r}^{-1}(U_{r}) = U \) and \( \pi_{r}^{-1}(V_{r}) = V \) are disjoint pair \( \tau_{i} \)-open set of \( x \) and \( \tau_{j} \)-open set of \( y \) respectively and the continuous function \( \Omega \) where \( \Omega = \lambda \circ \pi_{r}: M_{W} \rightarrow I \) such that \( M_{b} \cap U \subset \Omega^{-1}(0) \) and \( M_{b} \cap V \subset \Omega^{-1}(1) \) where \( i, j = 1, 2, i \neq j \).

3. Fibrewise pairwise bi-regular and pairwise bi-normal spaces

We at this time go on to consider the F.W. versions of the advanced pairwise separation
axioms, first with F.W. pairwise bi-regularity and F.W. pairwise completely bi-regularity.

**Definition: 3.1.**

The F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, \Lambda_1, \Lambda_2)\) is named F.W. pairwise bi-regular if for every \(x \in M_b; b \in B\), and for every \(\tau_1\)-open set \(V\) of \(x\) in \(M\), there exists a nbhd. \(W\) of \(b\) in \(B\), and a \(\tau_1\)-pen set \(U\) of \(x\) in \(MW\) such that \(V\) is containing the \(\tau_1\)-closure of \(U\) in \(MW\) (i.e., \(M_W \cap \tau_1 - Cl(U) \subset V\)), where \(i, j = 1, 2, i \neq j\).

For example, trivial F.W. spaces with pairwise bi-regular fibre are F.W. pairwise bi-regular.

**Remark 3.2:**

(a) The nbds of \(x\) are given by a F.W. basis it is enough if the condition in Definition (3.1) is satisfied for every F.W. basic nbds.
(b) If \((M, \tau_1, \tau_2)\) is F.W. pairwise bi-regular space over \((B, \Lambda_1, \Lambda_2)\) then \((M^*, \tau_1, \tau_2)\) is F.W. pairwise bi-regular space over \((B^*, \Lambda_1, \Lambda_2)\) for every subspace \(B^*\) of \(B\).

Subspaces of F.W. pairwise bi-regular spaces are F.W. pairwise bi-regular spaces. Actuality we have.

**Proposition 3.3:**

Assume that \(\varphi : M \rightarrow M^*\) is a continuous embedding F.W. function, where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1, \tau_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \(M^*\) is F.W. pairwise bi-regular then so is \(M\).

**Proof:**

Let \(V\) be a \(\tau_1\)-open set of \(x\) in \(M\), where \(x \in M_b; b \in B\). Then \(V = \varphi^{-1}(V^*)\), where \(V^*\) is a \(\tau_1\)-open set of \(x^* = \varphi(x)\) in \(M_b^*\). Because \(M^*\) is F.W. pairwise bi-regular then, we have a nbhd \(W\) of \(b\) in \(B\) and a \(\tau_1\)-open set \(U^*\) of \(x^*\) in \(M_b^*\) where \(M_W \cap \tau_1 - Cl(U^*) \subset V^*\). Then \(U = \varphi^{-1}(U^*)\) is a \(\tau_1\)-open set of \(x\) in \(M\) such that \(M_W \cap \tau_1 - Cl(U) \subset V\), hence \(M\) is F.W. pairwise bi-regular, where \(i, j = 1, 2, i \neq j\) as required.

The class of F.W. pairwise bi-regular spaces is F.W. multiplicative, like in the following.

**Proposition 3.4:**

Assume that \(\{(M_r, \tau_1r, \tau_2r)\}\) is a finite family of F.W. pairwise bi-regular spaces over B. The F.W. bitopological product \(M = \prod_{B} M_r\) is F.W. pairwise bi-regular.

**Proof:**

Consider a \(\tau_1\)-open set \(V = \prod_{B} V_r\) of \(x\) in \(M\), where \(x \in M_b; b \in B\) and \(V_r\) is a \(\tau_1\)-open set of \(\pi_r(x) = x_r\) in \(M_r\) for each index \(r\). Since \(M_r\) is F.W. pairwise bi-regular we have a nbhd. \(W_r\) of \(b\) in \(B\), and a \(\tau_1\)-open set \(U_r\) of \(x_r\) in \(M_r \cap W_r\) such that the \(\tau_1\)-closure of \(U_r\) in \(M_r \cap W_r\) is contained in \(V_r\). Then we regard \(W_r\) as a \(\tau_1\)-open set of \(x\) in \(MW\) where the \(\tau_1\)-closure of \(U\) in \(M_W\) is contained in \(V\). (i.e. \(M_W \cap \tau_1 - Cl(U) \subset V\)).

**Proposition 3.5:**

Assume that \(\varphi : M \rightarrow N\) is a closed, open and continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \(B\). Then \(M\) is F.W. pairwise bi-regular iff \(N\) is F.W. pairwise bi-regular.

**Proof:**

\((\Rightarrow)\) Let \(V\) be a \(\sigma_i\)-open set of \(y\) in \(N\) where \(y \in N_b; b \in \Box B\), choose \(x \in \varphi^{-1}(y)\). Then \(U = \varphi^{-1}(V)\) is a \(\tau_1\)-open set of \(x\) in \(M\). Because \(M\) is F.W. pairwise bi-regular, we have a nbhd \(W\) of \(b\) in \(B\), and a \(\tau_1\)-open set \(U^*\) of \(x^*\) such that \(M_W \cap \tau_1 - Cl(U^*) \subset U\). Then \(M_W \cap \tau_1 - Cl(U) \subset V\). Hence \(\sigma_i\) is open, then \(\varphi(U^*)\) is a \(\sigma_i\)-open set of \(y\). Hence \(N\) is F.W. pairwise bi-regular, where \(i, j = 1, 2, i \neq j\) as asserted.

\((\Leftarrow)\) By similar way of first direction.

The pairwise functionally version of the F.W. pairwise bi-regularity axiom is stronger than the non-pairwise functionally version but its properties are quite like. In the ordinary theory the word completely bi-regular is all the time used instead of functionally bi-regular and we widen this usage to the F.W. theory.
Definition 3.6:

A F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, A_1, A_2)\) is named F.W. pairwise completely bi-regular if for every \(x \in M_b\); \(b \in B\), and for every \(\tau_i\)-open set \(V\) of \(x\) there exists a nbd \(W\) of \(b\) in \(B\) and a \(\tau_j\)-open set \(U\) of \(x\) in \(M_w\) and a continuous function \(\lambda: (M_w, \tau_1 w, \tau_2 w) \to I\) such that \(M_b \cap U \subset \lambda^{-1}(0)\) and \(M_w \cap (M_w - V) \subset \lambda^{-1}(1)\), where \(i, j = 1, 2, i \neq j\).

For example, \((B, A_1, A_2) \times_B (T, \tau_1, \tau_2)\) is F.W. pairwise completely bi-regular space for every pairwise completely bi-regular spaces \(T\).

Remark: 3.7.

(a) The nbds of \(x\) are given by a F.W. basis it is enough if the condition in Definition (3.6) is satisfied for every F.W. basic nbds.

(b) If \((M, \tau_1, \tau_2)\) is F.W. pairwise completely bi-regular space over \((B, A_1, A_2)\) then \((M^*, \tau_1^*, \tau_2^*)\) is F.W. Pairwise completely bi-regular space over \((B^*, A_1^*, A_2^*)\) for every subspace \(B^*\) of \(B\).

Subspaces of F.W. pairwise completely bi-regular spaces are F.W. pairwise completely bi-regular spaces. In fact we have.

Proposition 3.8:

Assume that \(\varphi: M \to M^*\) is a continuous embedding F.W. function, where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \((B, A_1, A_2)\). If \(M^*\) is F.W. pairwise completely bi-regular then so is \(M\).

Proof:

Let \(V\) be a \(\tau_i\)-open set of \(x\) in \(M\) where \(x \in M_b\); \(b \in B\), then \(\varphi(x) = x^* \in M_b^*\) and \(V = \varphi^{-1}(V^*)\) is a \(\tau_i^*\)-open set of \(x^*\). Because \(M^*\) is F.W. pairwise completely bi-regular, then we have a nbd. \(W\) of \(b\) in \(B\) and \(\tau_j\)-open set \(U^*\) of \(x^*\) and a continuous function \(\lambda: M_w^* \to I\) such that \(M_b^* \cap U^* \subset \lambda^{-1}(0)\) and \(M_w^* \cap (M_w^* - U^*) \subset \lambda^{-1}(1)\). Now, because \(\varphi\) is continuous, then \(\varphi^{-1}(U^*) = U\) is \(\tau_i\)-open set of \(x\) in \(M_w\) and the continuous function \(\Omega = \lambda \varphi\) such that \(\Omega: M_w \to I\) and \(M_b \cap U \subset \Omega^{-1}(0)\) and \(M_w \cap (M_w - V) \subset \Omega^{-1}(1)\), where \(i, j = 1, 2, i \neq j\).

The class of F.W. pairwise completely bi-regular spaces is finitely multiplicative, like in the following.

Proposition 3.9:

Assume that \(\{M_r, \tau_1 r, \tau_2 r\}\) is a finite family of F.W. pairwise completely bi-regular spaces over \((B, A_1, A_2)\). The F.W. bitopological product \(M = \prod M_r\) is F.W. pairwise completely bi-regular.

Proof:

Let \(x \in M_b\); \(b \in B\). Consider a F.W. \(\tau_i\)-open set \(\prod_b V_r\) of \(x\) in \(M\), where \(V_r\) is a \(\tau_{ir}\)-open set of \(\pi_r(x) = x_r\) in \(M_r\) for all index \(r\). Because \(M_r\) is F.W. pairwise completely bi-regular, we have a nbd. \(W_r\) of \(b\) in \(B\), and a \(\tau_{jr}\)-open set \(U\) of \(x_r\) in \(M_r\) and a continuous function \(\lambda_r: (M_r)_w \to I\) where \((M_r)_b \cap U \subset \lambda_r^{-1}(0)\) and \((M_r)_w \cap ((M_r)_w - V_r) \subset \lambda_r^{-1}(1)\). Then we regard \(W\) like a nbd of \(b\) in \(B\) where \(W\) is the intersection of \(W_r\) and \(\lambda: M_w \to I\) is a continuous function where:

\[\lambda(\xi) = \inf_{r=1,2,3,...,n} (\lambda_r \xi_r)\] for \(\xi = (\xi_r) \in M_w\).

Since:

\[(M_r)_b \cap \pi_r^{-1}(U) \subset \pi_r^{-1}((M_r)_b \cap U) \subset \pi_r^{-1}(\lambda_r^{-1}(0)) = (\lambda_r \circ \pi_r)^{-1}(0)\]

and

\[(M_r)_w \cap \pi_r^{-1}((M_r)_w - V_r) \subset \pi_r^{-1}((M_r)_w \cap ((M_r)_w - V_r)) \subset \pi_r^{-1}(\lambda_r^{-1}(1)) = (\lambda_r \circ \pi_r)^{-1}(1)\]

where \(i, j = 1, 2, i \neq j\).

The similar conclusion holds for infinite F.W. products provided that all of the factors is F.W. non-empty.

Lemma 3.10:

Assume that \(\varphi: M \to N\) is a closed, open F.W. surjection function, where \(M\) and \(N\) are F.W. bitopological spaces over \(B\). Let \(\alpha: M \to \mathbb{R}\) be a continuous real-valued function which is F.W. bounded above, in the sense that \(\alpha\) is bounded above on each fibre of \(M\). Then \(\beta: N \to \mathbb{R}\) is continuous, where:

\[\beta(\eta) = \sup_{\xi \in \varphi^{-1}(\eta)} \alpha(\xi)\]

Proposition 3.11:

Assume that \(\varphi: M \to N\) is a closed, open and continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, A_1, A_2)\). If \(M\) is F.W. pairwise completely bi-regular then so is \(N\).
Proof:

Let \( V_y \) be a \( \sigma_l \)-open set of \( y \) in \( N \), where \( y \in N_b; b \in B \). Choose \( x \in \varphi^{-1}(y) \), so that \( V_x = \varphi^{-1}(V_y) \) is a \( \tau_l \)-open set of \( x \). Because \( M \) is F.W. pairwise completely bi-regular, we have a nbd. \( W \) of \( b \) in \( B \) and a \( \tau_l \)-open set \( U_x \) of \( x \) in \( M_W \) and a continuous function \( \lambda: M_W \rightarrow I \) such that \( M_b \cap U_x \subset \lambda^{-1}(0) \) and \( M_W \cap (M_W - V_x) \subset \lambda^{-1}(1) \). Using Proposition lemma (3.10), we get a continuous function \( \Omega: N_W \rightarrow I \) such that \( N_b \cap U_y \subset \Omega^{-1}(0) \) and \( N_W \cap (N_W - V_y) \subset \Omega^{-1}(1) \), where \( i, j = 1, 2, i \neq j \).

Now we define the version of F.W. pairwise normal space like in the following.

Definition 3.12:

A F.W. bitopological space \((M, \tau_1, \tau_2)\) over \((B, A_1, A_2)\) is named F.W. pairwise bi-normal if for every \( b \in B \) and every disjoint pair of \( \tau_l \)-closed set \( H \) and \( \tau_j \)-closed set \( K \) of \( M \), there exists a nbd. \( W \) of \( b \) in \( B \) and a disjoint pair of \( \tau_l \)-open set \( U \) and \( \tau_j \)-open set \( V \) of \( M \) in \( M_W \), where \( i, j = 1, 2, i \neq j \).

Remark 3.13:

If \((M, \tau_1, \tau_2)\) is F.W. pairwise bi-normal space over \((B, A_1, A_2)\), then for each subspace \( B^* \) of \( B \), \((M^*_B, \tau_1^*, \tau_2^*)\) is F.W. pairwise bi-normal space over \((B^*, A_1^*, A_2^*)\).

Closed subspaces of F.W. pairwise bi-normal spaces are F.W. pairwise bi-normal. Actually we have.

Proposition 3.14:

Assume that \( \varphi: M \rightarrow M^* \) is a closed, continuous embedding F.W. function where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \( B \). If \((M^*, \tau_1^*, \tau_2^*)\) is F.W. pairwise bi-normal then so is \((M, \tau_1, \tau_2)\).

Proof:

Let \( H, K \) be disjoint pair of \( \tau_l \)-closed, and \( \tau_j \)-closed sets of \( M \) and let \( b \in B \). Then \( \varphi(H), \varphi(K) \) are disjoint pair of \( \tau_l^* \)-closed set and \( \tau_j^* \)-closed set of \( M^* \). Since \( M^* \) is F.W. pairwise bi-normal then, we have a nbd \( W \) of \( b \) in \( B \) and a \( \tau_l^* \)-open set \( U \) and \( \tau_j^* \)-open set \( V \) of \( M^*_b \cap \varphi(H), M^*_W \cap \varphi(K) \), in \( M^*_W \).

\( \varphi^{-1}(U^*) = U \) and \( \varphi^{-1}(V^*) = V \) are disjoint pair of \( \tau_l \)-open and \( \tau_j \)-open sets of \( M_W \cap H, M_W \cap K \) in \( M_W \), where \( i, j = 1, 2, i \neq j \).

Proposition: 3.15.

Let \( \varphi: M \rightarrow N \) be a closed continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, A_1, A_2)\). Then \((M, \tau_1, \tau_2)\) is F.W. pairwise bi-normal iff \((N, \sigma_1, \sigma_2)\) is F.W. pairwise bi-normal.

Proof:

(\(\Rightarrow\)) Let \( H, K \) be disjoint pair of \( \sigma_l \)-closed, and \( \sigma_j \)-closed sets of \( N \) and let \( b \in B \). \( \varphi^{-1}(H), \varphi^{-1}(K) \) are disjoint pair of \( \tau_l \)-closed and \( \tau_j \)-closed sets of \( M \). Because \( M \) is F.W. pairwise bi-normal then, we have a nbd. \( W \) of \( b \) in \( B \) and a disjoint pair of \( \tau_l \)-open set and \( \tau_j \)-open set \( U, V \) of \( M_W \cap \varphi^{-1}(H) \) and \( M_W \cap \varphi^{-1}(K) \). Because \( \varphi \) is closed then, the sets \( N_W \cap \varphi(M_W - U), N_W \cap \varphi(M_W - V) \) are open in \( N_W \), and structure a disjoint pair of \( \sigma_l \)-open, \( \sigma_j \)-open sets of \( N_W \cap H, N_W \cap K \) in \( N_W \), as required, where \( i, j = 1, 2, i \neq j \).

(\(\Leftarrow\)) By similar way of first direction.

Lastly, we define the version of F.W. pairwise functionally bi-normal space like in the following.

Definition: 3.16.

A F.W. bi-topological space \((M, \tau_1, \tau_2)\) over \((B, A_1, A_2)\) is named F.W. pairwise functionally bi-normal if for every \( b \in B \) and every disjoint pair of \( \tau_l \)-open set \( H \) and \( \tau_j \)-open set \( K \) of \( M \), there exists a nbd. \( W \) of \( b \) in \( B \) and a disjoint pair of \( \tau_l \)-open set \( U \) and \( \tau_j \)-open set \( V \) in \( M_W \), where \( i, j = 1, 2, i \neq j \).

Example, \((B, A_1, A_2) \times_B (T, \tau_1, \tau_2)\) is F.W. pairwise functionally bi-normal space when \( T \) is pairwise functionally bi-normal space.

Remark 3.17:

If \((M, \tau_1, \tau_2)\) is F.W. pairwise functionally bi-normal space over \((B, A_1, A_2)\) then for every subspace \( B^* \) of \( B \) we have \((M^*_B, \tau_1^*, \tau_2^*)\) is F.W. pairwise functionally bi-normal space over \((B^*, A_1^*, A_2^*)\).
Closed subspaces of F.W. pairwise functionally bi-normal spaces are F.W. pairwise functionally bi-normal. Actuality we have.

**Proposition 3.18:**
Assume that \( \varphi: M \to M' \) is a closed, continuous embedding F.W. function where \((M, \tau_1, \tau_2)\) and \((M^*, \tau_1^*, \tau_2^*)\) are F.W. bitopological spaces over \( B \). If \( M^* \) is F.W. pairwise functionally bi-normal then \( M \) is so.

**Proof:**
Let \( H, K \) be disjoint pair of \( \tau_i \)-closed and \( \tau_j \)-closed sets of \( M \) and let \( b \in B \). Then \( \varphi(H), \varphi(K) \) are disjoint pair of \( \tau_i^* \)-closed set and \( \tau_j^* \)-closed set of \( M^* \). Since \( M^* \) is F.W. pairwise functionally bi-normal we have a nbd \( W \) of \( b \) in \( B \) and a disjoint pair of \( \tau_i^* \)-open set \( U \) and \( \tau_j^* \)-open set \( V \) and a continuous function \( \lambda : M^*_W \to I \) such that \( M^*_W \cap \varphi(H) \cap U \subseteq \lambda^{-1}(0) \) and \( M^*_W \cap \varphi(K) \cap V \subseteq \lambda^{-1}(1) \) in \( M^*_W \).

Since \( \varphi \) is continuous, then \( \varphi^{-1}(U), \varphi^{-1}(V) \) are \( \tau_j \)-open set, \( \tau_i \)-open set and the function, \( \Omega = \lambda \circ \varphi \) is a continuous, \( \Omega: M^*_W \to I \) such that \( M^*_W \cap H \cap \varphi^{-1}(U) \subseteq \Omega^{-1}(0) \) and \( M^*_W \cap K \cap \varphi^{-1}(V) \subseteq \Omega^{-1}(1) \) in \( M^*_W \) as required where \( i, j = 1, 2, i \neq j \).

**Proposition 3.19:**
Assume that \( \varphi: M \to N \) is a closed, open and continuous F.W. surjection function, where \((M, \tau_1, \tau_2)\) and \((N, \sigma_1, \sigma_2)\) are F.W. bitopological spaces over \((B, \Lambda_1, \Lambda_2)\). If \((M, \tau_1, \tau_2)\) is F.W. pairwise functionally bi-normal then so is \((N, \sigma_1, \sigma_2)\).

**Proof:**
Let \( H, K \) be disjoint pair of \( \sigma_i \)-closed, and \( \sigma_j \)-closed sets of \( N \) and let \( b \in B \). Then \( \varphi^{-1}(H), \varphi^{-1}(K) \) are disjoint pair of \( \tau_i \)-closed and \( \tau_j \)-closed sets of \( M \). Because \( M \) is F.W. pairwise functionally bi-normal, then we have a nbd \( W \) of \( b \) in \( B \) and a disjoint pair of \( \tau_i \)-open set and \( \tau_j \)-open set \( U, V \) and a continuous function \( \lambda: M^*_W \to I \) such that \( M^*_W \cap \varphi^{-1}(H) \cap U \subseteq \lambda^{-1}(0) \) and \( M^*_W \cap \varphi^{-1}(K) \cap V \subseteq \lambda^{-1}(1) \) in \( M^*_W \).

Hence \( N_W \cap H \cap \varphi(U) \subseteq \Omega^{-1}(0) \) and \( N_W \cap K \cap \varphi(V) \subseteq \Omega^{-1}(1) \) in \( M^*_W \) where \( i, j = 1, 2, i \neq j \).

**References**