On Lie's Reduction Theorem with an Application to Isentropic Fluid Spheres

By

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Abstract

Assuming that the interior of the star is filled up with perfect fluid the corresponding Einstein's equations can represent ordinary differential equation of 2nd order involving a parameter k. Different values of k are responsible for different models of the fluid sphere.

There exist many papers [2], [3], [4], [5] and [6] study this problem by seek a series solution \( (u = \sum A_k e^{k}) \) and found special cases for parameter k include particular solutions with physical analysis.

In the present paper, the general solution (optimal solution) have been found by use similarity method of differential equations and specially Lie's reduction theorem. the solution that we obtained was in terms of special functions, namely the confluent hypergeometric functions.

\[
\begin{align*}
\frac{ds^2}{ds^2} &= -\frac{1}{2} \frac{2}{k^2} \left[ 1 - \frac{r^2}{k^2} \right] \frac{dr^2}{r^2} - \left( \frac{\sigma}{k^2} \right) \frac{d\sigma^2}{\rho^2} \\
&= r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 - y^2 \phi^2 \right) \\
\end{align*}
\]

where \( k = 1 - \frac{k^2}{R^2} \), the metric (2.1) is regular and positive definite at all points \( r^2 \leq R^2 \) and its describes the fluid filled star the Einstein's field equations.

\[
\begin{align*}
n_{ij} &= \frac{1}{2} \phi \gamma_{ij} - s \left[ \delta_{ij} + \rho \gamma_{ij} - p \gamma_{ij} \right] \\
&= 1, 2, 3 \quad j = 1, 2, 3
\end{align*}
\]

Supply the following expression for every density \( \rho \), Pressure \( P \), flow vector \( \mathbf{V} \) with isotropy conditions \( (1, i) = (T_i^2 + T_j^2) \)

\[
\begin{align*}
\delta_{ij} &= \frac{3 (1 + 1)}{R^2} \left[ \frac{1}{3 R^2} \left( 1 - \frac{k^2}{R^2} \right) \right] \left[ \frac{1}{R^2} \right] \left[ \frac{1}{R^2} \right] \\
&= \frac{3 (1 + 1)}{R^2} \left[ \frac{1}{3 R^2} \left( 1 - \frac{k^2}{R^2} \right) \right] \left[ \frac{1}{R^2} \right] \left[ \frac{1}{R^2} \right] \\
\end{align*}
\]

and the isotropy condition

\[
\begin{align*}
\mathbf{V}^j &= \left( 0, 0, 0, \frac{1}{R^2} \right) \\
&= \left( 0, 0, 0, \frac{1}{R^2} \right)
\end{align*}
\]

2. Basic equations

The Vaidya Timelike space-time have hypersurfaces \( t \) constant as 3 - spheroid and can be expressed by the metric

\[
\left[ 1 - k + k (1 - k) \left( 0 - R \right) \frac{k^2}{R^2} \frac{1}{R^2} \right] = 0
\]
with \( c = \frac{k}{\sqrt{k-1}} \) and the equation (2.6)

\[
(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + (1 - k) y = 0 \quad (2.7)
\]

Assume the form

\[
(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + (1 - k) y = 0 \quad (2.7)
\]

for \( k < 0 \) or \( k > 1 \) with

\[
x = \sqrt{k-1} \int \frac{dx}{\sqrt{1 - x^2}}
\]

Once we solve the linear equation (2.7) the expression for pressure and flow vector can be obtained easily.

Vaidya–Tikekar have solved the equation (2.7) for \( k = -2 \) and . Tikekar discussed the problem with \( k = -7 \).

In the next section the authors obtained a new general solution (optimal solution) of equation (2.7) and construct a most general model. The method start with the usual symmetry solution but a useful manipulation sends the whole solution into closed form with the help of Matlab v 6.

### 3. Mathematical Formulation

We consider a one–parameter Lie group of transformation

\[
x^* = x + \varepsilon x (x, y) - O (\varepsilon^2) \quad \ldots (3.1)
\]

\[
y^* = y - \varepsilon y (x, y) + O (\varepsilon^2)
\]

Then the extended transformation, will be

\[
P^* = P + \zeta (x, y; p, q) + O (\varepsilon^2) \quad \ldots (3.2)
\]

\[
q^* = q + \xi (x, y; p, q) + O (\varepsilon^2)
\]

Where

\[
p = \frac{dy}{dx}, q = \frac{d^2 y}{dx^2}, P^* = \frac{dy^*}{dx^*}, q^* = \frac{d^2 y^*}{dx^2} \ldots (3.3)
\]

We derive \( \zeta \) and \( \xi \) as follows:

\[
\zeta = k + pY_x + pX_y + P_X - P_Y
\]

\[
\xi = q + qX_x + qY_y + qP_X - qP_Y
\]

The infinitesimal generators \( D^{(1)} \) and \( D^{(2)} \) are:

\[
D^{(1)} = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial p}
\]

\[
D^{(2)} = D^{(1)} + \xi \frac{\partial}{\partial q}
\]

### 4. Group – Invariant solution

If we write equation (2.7) in the primed form and substituted from equations (3.1) and (3.2) for the primed variables and simplification, we find equations coefficients of \( p^n q^m \) give the infinitesimal elements \( (X, Y) \) leaving invariant the equation (2.7).

Now, we find the determining equations for \( X \) and \( Y \) as follows:

\[
q + \left[ Y_{xx} - x^2 Y_{xx} + x^2 Y_{xy} - 2xY_{x} + p X_{x} + 2p^2 X_{xx} - 2p^2 X_{xy} + q X_{y} - qp Y_{x} \right] Y_{y} - \left[ 2x^2 Y_{xy} - 2x^2 Y_{xx} + 2x^2 Y_{yy} \right] Y_{x} - 2x^2 Y_{y} - 2x^2 Y_{xy} + 2x^2 Y_{yy} = 0 \ldots (4.1)
\]

we find a primary set of determining equations is the following:

<table>
<thead>
<tr>
<th>Monomials</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>( p Y_{xx} + 2p^2 Y_{xx} - 2p^3 Y_{x} + pX_{x} + 2p^2 X_{xx} - 2p^3 X_{xy} - q X_{y} + qp Y_{x} )</td>
</tr>
<tr>
<td>( p^2 )</td>
<td>( p Y_{x} + 2p^2 X_{xx} - 2p^3 X_{xy} + q X_{y} - qp Y_{x} )</td>
</tr>
<tr>
<td>( p^3 )</td>
<td>( p Y_{xx} + 2p^2 X_{xx} - 2p^3 X_{xy} + q X_{y} - qp Y_{x} )</td>
</tr>
<tr>
<td>( q )</td>
<td>( q Y_{xx} + q^2 Y_{xy} - 2q^2 Y_{x} + q^3 Y_{y} - q^2 p X_{x} - q^3 Y_{xy} + p X_{y} - p^2 X_{x} + p^2 X_{xy} - p^3 X_{yy} )</td>
</tr>
<tr>
<td>( qp )</td>
<td>( q Y_{x} - q^3 Y_{xy} - p X_{y} + p^2 X_{x} - p^2 X_{xy} + p^3 X_{yy} )</td>
</tr>
</tbody>
</table>

and we found terms for \( \varepsilon \), empty from the derivatives in extended equation (4.1)

\[
Y_{xx} - x^2 Y_{xx} + x X_{y} - Y - k Y' = 0
\]

This implies

\[
\left( 1 - x^2 \right) Y_{xx} + x Y_{x} - (1 - k) Y = 0 \ldots (4.7)
\]

Now, from equation (4.6)

\[
(2x^2 - 1) x X_{y} - 0 \Rightarrow x X_{y} = 0
\]

\[
x (x, y) = \alpha + H (x)
\]

\[
X_{y} - H' (x) \ldots (4.8)
\]
substitute (4.8) in equation (4.5), we have

$$ H(x) = \frac{x^2 - 1}{x - 1} = 1 $$

we know

$$ X(x) = \frac{x^2 - 1}{x - 1}(x) \quad \text{in} \quad 2x $$

equation (4.5), this implies $X'(x,y) = 0$

Now, rewrite equation (4.1) using the result obtained above, the second set of the determining equations is the following:

<table>
<thead>
<tr>
<th>Monomial</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>$y^2 + 2y + 2 - 2x^2 + 2y + 4x - 3$</td>
</tr>
</tbody>
</table>

From equation (4.10), we have

$$ (1 - x^2)K + 0 = 0 \quad \text{or} \quad Y = 0 $$

This implies

$$ Y(x, y) = c_1 + c_2 $$

But we find $c_1 = 0$ from equation (4.11)

$$ Y(x, y) = R(x) $$

i.e. $Y(x, y)$ is a function of $x$ only we find the second order differential equation (2.6) is invariant in the twice-extended group

$$ x^2 - x $$

$$ y^2 = y + Y(x) $$

Where

$$ Y(x) = \frac{\Gamma(2 - x) + \Gamma(2 + x)}{2.2 + 2} $$

by

$$ c_1(x) = \frac{\Gamma'(x)}{\Gamma(x)} $$

Matlab program.

In [51, 8] the series of confluent hypergeometric

$$ P(\alpha, \beta, \gamma) = \lim_{\alpha \to 1} P(\alpha, \beta, \gamma, 1) = 1 + \frac{\alpha}{\beta} + \frac{\alpha(\alpha + 1)}{\beta(\beta + 1)} + \frac{\alpha(\alpha + 1)(\alpha + 2)}{\beta(\beta + 1)(\beta + 2)} + $$

is well defined provided $\alpha$ is not a negative integer and converges for $|x| < 1$ and any solutions in other regions are obtained by analytic continuation of these solutions. Therefore we find

$$ P(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

when $1 = \frac{3}{2}$

$$ P(\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

when $k = -\frac{1}{2}$

$$ P(\frac{7}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

when $k = \frac{1}{2}$

Now, there exist a simple relations between the confluent hypergeometric with different parameters by $\pm 1$ 81.

In functions which obtained it, we find

$$ \frac{3}{2} - c $$ is constant value, but a variable value

dependent of value $k$ such that

$$ \alpha - \frac{1}{2}, P_1(\alpha, \beta, \gamma, 1) = P(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

This implies

$$ -P_1(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$$$ -P_1(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

when $u = 5/2, v = 3/2, k = \frac{3}{2}$

$$ P_1(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

as the same as when $u = 5/2$ and $v = 3/2, \quad k = 14$

therefore, we consequently

$$ P_1(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) $$

by using Matlab program V.6.

5. Lie's Reduction Theorem [4]

Let the general form of Ordinary Differential Equation of the second order is

$$ w(x, y, y', y'') = 0 $$

We always be written as a pair coupled DR of the first order as follow

$$ y' = u $$

$$ w(x, y, u, u') = 0 $$

Equation (5.2) determine a two-parameter family of curves in 3-dimensional space. Equation (5.2) are invariant to the once-extended group $(X, Y, \zeta)$ the transformations of the group carry each of these curves into other curves of the family.

Each one-parameter family of curves defines a surface in $(x, y, u, c)$ - space and denoted by equation $\phi(x, y, u, c) = 0$ (5.3)

Equation (5.3) is invariant

$\phi(x, y, u, c) = \phi(X, Y, U, c)$ (5.4)

and satisfy $X \phi_y + Y \phi_u + \zeta \phi_c = 0$ (5.5)

The characteristics equations of which are
If \( p(x, y) \) and \( q(x,y) \) are two integrals of equation (5.4), the general solution for \( \phi \) is an arbitrary function \( \xi \) of \( P \) and \( q \), the function \( p(x, y) \) being an integral of the first pair of equation (5.4) is a group invariant, the function \( q(x, y, u) = q(x, y, x) \) which is an invariant of the once extended group called a first differential invariant.

(If we adopt the invariant \( P \) and first differential invariant \( q \) as new variables, the second order differential equation \( w(x, y, x, y) = 0 \) will reduce to a first order differential equation in \( P \) and \( q \)).

6. Application of Lie’s Reduction Theorem

If we write a differential equation (2.7) in the form \( u(x, y, x, y) = 0 \) where

\[
 u(x, y, x, y) = \frac{1}{1-x^2} y + \frac{1}{1-x^2} y = 0
\]

then \( u \) satisfy the condition

\[
 X u_x + Y u_y + \xi u_x + \eta u_y = 0
\]

The infinitesimal coefficient of the group are:

\[
 X(x, y) = 0, \quad Y(x, y) = Y(x), \quad \zeta = y(x), \quad \xi = y(x)
\]

direct substitution now shows that

\[
 X u_x + Y u_x + \xi u_x + \eta u_y = 0
\]

as required.

Now, by using the characteristic equation of which are

\[
 \frac{dx}{X(x, y)} = \frac{dy}{Y(x)} = \frac{dy}{Y(x)} \Rightarrow dx = dy = \frac{dy}{Y(x)}
\]

from the first equality that

\[
 \frac{2}{3} x e^x \left( x^2 - 1 \right)^{3/4} + \frac{3}{5} e^{x^2} \left( 1 \right)^{3/4} - n = \text{where a constant}
\]

\[
 \left[ \begin{array}{c} 1 \\ 2 \\ 2 \\ 2 \\ \end{array} \right] \text{and}
\]

\[
 \gamma(x) = \left( x^2 - 1 \right)^{3/4} e^x
\]

The greater generality of solution in terms of hypergeometric functions outweighs the tractability of the solution presented [4].

Substituting this value of \( x \) in the second and third term tells us that we can treat \( Y(x) \) for a constant when integrating the second equality

\[
 \frac{dy}{Y(x)} = \frac{d\gamma}{Y(x)} = \frac{3}{2} e^{x^2} \left( 1 \right)^{3/4}
\]

\[
 \gamma(x) = \left( x^2 - 1 \right)^{3/4} e^x
\]

and

\[
 \left[ \begin{array}{c} 1 \\ 2 \\ 2 \\ 2 \\ \end{array} \right] \text{and}
\]

\[
 \gamma(x) = \left( x^2 - 1 \right)^{3/4} e^x
\]

This implies \( Y(x) = y(x) = e^{x^2} \) where \( b \) is a second constant.

Thus \( x^2 + y(x) = \left( x^2 - 1 \right)^{3/4} + \frac{3}{5} e^x \left( 1 \right)^{3/4} \) is an invariant. -----(6.1)

and

\[
 \left( e^{x^2} \left( x^2 - 1 \right)^{3/4} \right) y = \text{is a first}
\]

differential equation -----(6.2)

Differentiating we find

\[
 \frac{dx}{dy} = y - y \frac{dy}{dy} = R(x) Y(x) - P(x) q
\]

\[
 \text{where}\ Y(x) = y(x) = \frac{dx}{dy} \quad \text{...(Linear equation)}
\]

\[
 \Rightarrow \ q = D \sqrt{x^2 - 1} \quad \text{...(6.3)}
\]

by substituting (6.3) in (6.3)

we

\[
 \frac{dy}{Y(x)} = \frac{dx}{Y(x)} = \frac{D \sqrt{x^2 - 1}}{y(x)} \quad \text{...(Linear equation)}
\]

This implies

\[
 \gamma(x) = \left( x^2 - 1 \right)^{3/4} e^{x^2} \left( 1 \right)^{3/4} \text{...(6.4)}
\]

Such that \( E(x) \) is called (Cauchy Newton function) and defined by

\[
 E(x) = \int_{0}^{x} e^t dt
\]
References


8. يشير بند بند وفقًا للقواعد التقنية، "خطوات في الورشات العلمية، جامعة البصرة، المكاسب المستفادة من البحث العلمي (143-155) (1980).

المستند

أقترح أن يكون الفهيم والمملكة ببناءة تابع للدكتور K دور كبير في تطوير النموذج.

عند سيفا الإلهام [7], [8], [9], [10], [11], [12], [13] في دراسة هذا النموذج. باستخدام الحل المشتق الذي أنتج حالتين خاصة للنموذج (k= -14, k= -2) ضمن حلول خاصة للساحة مع تطبيق قياسات تلك الحلول.

في بحث هذا نموذج مستخدم طريقة تمثيل المعادلات الفيزيائية على وجه التعميم، نظرًا للاختلاف.

ولذا، فإن هذه الطريقة بخصوصًا على الحل الأمثل والمجمع

بالاعتماد على الدور الفعال لـ principio الطاقة الانجمادية التي يدورها

لم يتم حل جميع الحالات الخاصة، حيث أن قيمة الرمي k= -2

تولد كتلة الحلال الأخرى (k= -14, k= -16) تمثل الحالة

المماثلة (الكلية) للساحة.