Local Existence and Uniqueness of Sobolev Type
Semilinear Initial Value Problems in Banach Spaces

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Abstract
The aim of this paper is to prove the local existence and uniqueness of the mild solution of semilinear initial value control problems in suitable Banach spaces using resolvent operator and Schauder fixed point theorem.

Keywords: Local existence, uniqueness, mild solution, fixed point theorem and resolvent operator of control problems.

Introduction

\[
\frac{d}{dt} x(t) = A x(t) + \int_{0}^{t} F(s,x(s))ds, \quad t > 0, \quad x(0) = x_0 \]

Krishnan Balachandran in 2003 [15], has studied the existence and uniqueness of the mild solution to the semilinear initial value problem:

\[
\frac{d}{dt} x(t) = \int_{0}^{t} A(x(s)) + \int_{0}^{t} F(s,x(s))ds + f(t), \quad x(0) = x_0 \]

Our work is concerned with the semilinear initial value control problem:

\[
\frac{d}{dt} [Ex(t)] = A[x(t)] + \int_{0}^{t} F(s,x(s))ds + f(t), \quad x(0) = x_0 \]

where \( A \) and \( E \) are closed linear operators with domain contained in a suitable Banach space \( X \), \( F(t) \) is a bounded operator for \( 0 \leq t < T \), \( x \) and \( f \) are nonlinear maps defined from \([0, T]; X \rightarrow X \), \( h \) is the real valued continuous function defined from \([0, 1] \rightarrow \mathbb{R} \), where \( R \) is the set of real numbers and \( V \) is a bounded linear operator defined from \( X \) into \( W \), and \( E \) is a Banach space of control functions with \( [0, T]; X \rightarrow W \), \( y \) is a Banach space of control functions with \( [0, T]; X \rightarrow Y \), \( Y \rightarrow Y \) and for \( x(.) \) continuous in \( Y \), \( \text{AF}(x(.) \in L([0,T], Y) \rightarrow Y \}

Manaf in 2005 [14], has studied the local existence and uniqueness of the mild solution to the semilinear initial value control problem:

\[
\frac{d}{dt} x(t) = A x(t) + f(t,x(t)) + \int_{0}^{t} h(t-s)g(s,x(s))ds, \quad t > 0, \quad x(0) = x_0 \]

The local existence and uniqueness of the mild solution to the semilinear initial value control problem given (3) have been developed by using semigroup theory and Schauder fixed point theorem.
Preliminaries

Consider the Sobolev type semilinear initial value control problem:

\[
\frac{d}{dt}\left[\psi(t)\right] = A\left[\psi(t) + \int_{-\infty}^{t} f(t-s,\psi(s))\,ds\right] + g(t,\psi) + \int_{-\infty}^{t} h(t-s)g(t,\psi(s))\,ds + B\theta(t), \quad t > 0;
\]
\[\psi(0) = x_0 \in D(A).\]

**Definition (1):**

A family of bounded linear operators \(R(t) \in \mathcal{B}(X)\) for \(t \in [0, T]\) is called the resolvent operator for:

\[
\frac{d}{dt}\left[\psi(t)\right] = \lambda \psi(t) + \int_{-\infty}^{t} f(t-s,\psi(s))\,ds + g(t,\psi) + \int_{-\infty}^{t} h(t-s)g(t,\psi(s))\,ds + B\theta(t), \quad t > 0;
\]

\[\psi(0) = x_0 \in D(A).\]

(i) \(R(0) = I\), where \(I\) is the identity operator.

(ii) For all \(x \in X\), \(R(t)x\) is continuous for \(t \in J\).

(iii) \(R(t) \in \mathcal{B}(X)\), \(t \in J\), for \(y \in Y\), \(R(t)y \in C([0, T]; X) \cap C([0, T]; Y)\) and

\[
\frac{d}{dt}R(t)y = -\lambda R(t)y - \int_{-\infty}^{t} f(t-s)R(t)y\,ds + \int_{-\infty}^{t} h(t-s)(g(t,\psi(s))\,ds, \quad t \in J.
\]

**Definition (2):**

A function \(\psi(\cdot) \in C([0, T]; X)\) is called a mild solution of equation (3) if it satisfies the integral equation:

\[
\begin{align*}
\psi(t) &= \frac{1}{\lambda} R(t)x_0 + \int_{0}^{t} \left[ f(s,\psi(s)) + g(s,\psi(s))\right] ds + \int_{0}^{t} h(t-s)g(t,\psi(s))\,ds, \quad t \in [0, T].
\end{align*}
\]

The local existencer and uniqueness of a mild solution of problem (3) have been developed, by assuming the following assumptions:

\(A_1\): The operator \(A : D(A) \subset X \rightarrow X\) and \(E : D(E) \subset X \rightarrow X\) are closed linear operators.

\(A_2\): \(D(E) \subset D(A)\) and \(E\) is bijective.

\(A_3\): \(E^{-1} : X \rightarrow D(E)\) is bounded operator and \(E^{-1}F = FE^{-1}\).

\(A_4\): \(A^{-1}\) generates a strongly continuous semigroup of bounded operators in \(X\).

\(A_5\): The resolvent operator \(R(t)\) is compact in \(X\).

\(A_6\): Let \(\rho > 0\), such that \(\psi_0(x_0) = \{x \in X : |x - x_0| < \rho\}\), where \(x_0 \in U\) (open subset of \(X\)), and let the nonlinear maps \(f, g\) define from \([0, T]; U \rightarrow X\) satisfy the locally Lipschitz condition with respect to second argument, i.e.,

\[
\|[f(t, v_1) - f(t, v_2)]\|_X \leq L_1 v_1 - v_2, \quad \text{and,}
\]

\[
|g(t, v_1) - g(t, v_2)| \leq L_2 |v_1 - v_2|_X,
\]

For \(0 \leq t < r\) and \(v_1, v_2 \in \mathcal{B}(x_0)\) and \(L_1, L_2\) are Lipschitz constant.

\(A_7\): \(h\) is continuous function, \(h + L_1^2(0, r, R)\), where \(R\) is the set of real numbers.

\(A_8\): \(\alpha(.)\) be the arbitrary control function is given in \(L^1([0, T]; O)\), a Hamach space of control functions with \(O\) as a Banach space and here \(B\) is a bounded linear operator from \(O\) into \(X\) with \(\|B\|_O \leq L_1\) for \(0 \leq t < r\).

\(A_9\): Let \(t > 0\), such that \(|f(t, v)| \leq N_1\), \(g(t, v)^2 \leq N_2\), for \(0 \leq t < T\) and \(v \leq \bar{v}\), then let \(t^* > 0\), such that \(|R(t^*)^2 R(t^*) x_0|_X < \rho^*\), for \(0 \leq t \leq T\), and \(\chi \in \mathcal{C}(0, t^*)\), where \(\rho^*\) is a positive constant such that \(\rho^* < \rho\).

\(A_{10}\): Let \(\xi > 0\), such that:

\[
\begin{align*}
\left( f_{\xi}(t) = \frac{\rho - \rho^*}{L_0 M (N_1 + K_0 K_1 + h_1 N_2)} \quad \text{and} \quad \left( g_{\xi}(t, v) = \frac{1}{(L_0 + h_1 L_1) L_0 M} \right).
\end{align*}
\]

Main Results

We introduce the following main theorem:

**Theorem (1):**

Assume the hypotheses \((A_1) - (A_5)\) hold. Then, for every \(x_0 \in \mathcal{C}(0, t^*; X)\) such that the semilinear initial value control problem of equation (3) has a unique local mild solution \(\psi_0(x_0) \in \mathcal{C}(0, t^*; X)\) for every control function \(\alpha(.) \in L^1([0, t^*]; O)\).
Proof:

Without loss of generality, we may suppose $c < \infty$, because we are concerned here with the local existence only.

There exist $M > 0$ such that $|R(t)| \leq M$, $0 < r < r$; since $R(t)$ is a bounded linear operator on $X_1$. Assume:

$$h_1(t) = \int_0^t h(s) \, ds$$

we set $Y = C([0, 1], X)$, where $Y$ is a Banach space with sup norm defined as follows:

$$\|y\| = \sup_{0 \leq t \leq 1} \|y(t)\|_{X_1}$$

Define:

$$S_a = \{x_a \in Y : x_a(0) = x_0, x_a(t) \in \mathcal{B}(x_0), \text{ for a given} \ a \in 1, \ R(0, 1, 0)\}$$

Clearly $S_a$ is a bounded, convex and closed set of $Y$.

Define a map $F_a : S_a \to Y$, by:

$$(F_a x_a)(t) = \int_0^t R(t-s)[f(s, x_a(s)) + h(s-t)g(t, x_a(s)) \, dt + \mathcal{B}(s)] \, ds$$

For arbitrary control function $a(t) \in \mathcal{A}(0, 1, 0)

To show that $F_a(S_a) \subset S_a$, let $x_a$ be an arbitrary element in $S_a$, such that $F_a x_a \in F_a(S_a)$. To prove $F_a x_a \in S_a$, notice that $F_a x_a \in Y$ (by the definition of the map $F_a$) and $(F_a x_a)(0) = x_0$ (by equation (6)). To prove $(F_a x_a)(t) \in \mathcal{B}(x_0)$, for any $x_a \in S_a$, from the definition of the closed ball $\mathcal{B}(x_0)$, notice that $(F_a x_a)(t) \in X$ and

$$|F_a x_a(0) - x_0| = \left|\int_0^1 R(t-s)[f(s, x_a(s)) + h(s-t)g(t, x_a(s)) \, dt + \mathcal{B}(s)] \, ds \right|$$

$$\leq \|E^s R(t) x_a - x_a\| \|E^s R(t) x_0 - x_a\|$$

After a series of simplifications and using the conditions $A_A, A_A$, and $A_A$ with equation (5), we get:

$$||F_a x_a - F_a x_0|| \leq (1 + L_x T) \|E^s R(t) x_0 - x_a\|$$

Since $|x_a - x_a| \to 0$, as $n \to \infty$, which implies that:

$$\lim_{n \to \infty} |F_a x_a - F_a x_0| = 0$$

Now, assume that $\tilde{S} = S_a \cup S_\alpha$, and for fixed $t \in [0, 1]$, let $S(t) = \{F_a x_a(0) : x_a \in S_a\}$. To show that $S(t)$ is a precompact set, for every fixed $t \in [0, 1]$. For $t = 0$ we have $S(0) = \{F_a x_a(0) : x_a \in S_a\}$, which is a precompact set in $X$. Now, for $t > 0$, $0 < c < t$, define:
\( (P_x \circ x_k)(t) = E^{-1} R(t) + x_k + \frac{\partial}{\partial t} \int_0^t R(t-s) \left[ f(s, x_k(s)) + \int h(s-t) g(t, x_k(t)) \, dt + B_0(s) \right] \, ds \)

or arbitrary \( x_k \in S_{\alpha} \), then:

\( (P_x \circ x_k)(t) = E^{-1} R(t) + x_k + E^{-1} \int_0^t R(t-s) \left[ f(s, x_k(s)) + \int h(s-t) g(t, x_k(t)) \, dt + B_0(s) \right] \, ds \)

from the compactness of the operator \( R(t) \) and equation (8), which implies that the set

\( S_{\alpha}(t) = \{(P_x \circ x_k)(t) : x_k \in S_{\alpha} \} \)

is precompact for every \( \alpha \in (0, r_1 < t_1) \).

Moreover, for any \( x_k \in S_{\alpha} \) we have:

\( (P_x \circ x_k)(t) - (P_x \circ x_k)(0) = E^{-1} \int_0^t R(t-s) \left[ f(s, x_k(s)) + \int h(s-t) g(t, x_k(t)) \, dt + B_0(s) \right] \, ds \)

After a series of simplifications and using the conditions \( A_3, A_4 \) and \( A_5 \) with equation (3), we get:

\( (P_x \circ x_k)(t) - (P_x \circ x_k)(0) = E^{-1} \int_0^t R(t-s) \left[ f(s, x_k(s)) + \int h(s-t) g(t, x_k(t)) \, dt + B_0(s) \right] \, ds \)

Since \( R(t) \) is compact resolvent operator which implies that \( R(t) \) is continuous in the uniform operator topology for \( t > 0 \), therefore the right hand side of equation (9) tends to zero as \( t_1 - t_2 \) tends to zero. Thus \( \tilde{S} \) is equicontinuous family of functions. It follows from the "Arzela-Ascoli theorem" that \( \tilde{S} = \bar{F}_a(S_a) \) is relatively compact in \( X \) and by applying "Schauder fixed point theorem", which implies \( F_a : S_b \rightarrow S_b \) has a fixed point, i.e., \( F_a x_0 = x_0 \) for arbitrary control function is given in \( L^1(0, t_1, X) \), hence equation (1) has a local mild solution \( x_0 \in C([0, t_1], X) \). To show the uniqueness, let \( \bar{x}_0, \bar{x}_1 \) be two local mild solutions of the semilinear initial value control problem given by equation (1) on the interval \([0, t_1]\), where \( \bar{x}_0, \bar{x}_1 \). Continuous functions depend on \( m \).

We must prove that \( |x_0(t) - \bar{x}_1(t)| = 0 \), assume \( \|x_0(t) - \bar{x}_1(t)\| X = 0 \), and notice that:
\[ \| \tilde{X}_0 - \tilde{X}_0 \| = \| \int_0^t R(t-s) \left[ f(s, \tilde{X}_0(s)) + \int_0^t h(s-t) g(s, \tilde{X}_0(s)) \, ds \right] \, ds \| \]

After a series of simplifications and using the conditions \( A_2 \) and \( A_3 \) with equation (3), we get:

\[ \| \tilde{X}_0 - \tilde{X}_0 \| \leq (L_0 + h_1 L_1) \| \tilde{X}_0 - \tilde{X}_0 \| \]

By using the condition \( (A_0, \tilde{a}) \), we get:

\[ \| \tilde{X}_0 - \tilde{X}_0 \| \leq (L_0 + h_1 L_1) \| \tilde{X}_0 - \tilde{X}_0 \| \]

\[ \| \tilde{X}_0 - \tilde{X}_0 \| \leq \frac{1}{(L_0 + h_1 L_1) L_1 M} \]

Then \( \| \tilde{X}_0(1) - \tilde{X}_0(1) \| \leq \| \tilde{X}_0 - \tilde{X}_0 \| \)

By taking the supremum over \([0, t] \) of the both sides of the above inequality, we get:

\[ \| \tilde{X}_0 - \tilde{X}_0 \| \leq (L_0 + h_1 L_1) \| \tilde{X}_0 - \tilde{X}_0 \| \]

Hence we have a unique mild solution \( x_0 \in C([0, t], X) \), for arbitrary control function \( u(t, \tilde{t}) \in L^2(\tilde{t}, t, X) \).

References


المستند

هدف من هذا البحث هو تقييم وجود وحيوية كامل إطار
(منطق) لمسألة سيطرة شبه حديثة ذات نفسه أمكنية في فضاء
بادر من موضوع باستخدام مؤشر محل ونظرية منغزة الثابتة
المذكورة.