EXISTENCE AND UNIQUENESS FOR A CLASS OF SEMILINEAR INITIAL VALUE CONTROL PROBLEMS IN SUITABLE BANACH SPACES

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Abstract
The local existence and uniqueness of S-classical solution (semi-classical solution) for a class of semilinear initial value control problems in suitable Banach spaces have been discussed and proved. The theoretical results are depending on the theory of analytic semigroup and Banach contraction principle.

Keywords: S-classical solution (semi-classical solution), control problem in infinite dimensional spaces, fixed point theorem and analytic semigroup theory.

Introduction
Byszewski in 1991 [1], has study the local existence and uniqueness of the mild solution to the semilinear initial value problem:

\[ \frac{dv}{dt} + Av(t) f(t,v(t)) = v(0) = v_0 \] ..........................(1)

Where A is the infinitesimal generator of a C_0 semigroup (strongly continuous semigroup) defined from D(A)⊂X into X (X is suitable Banach space) and f is a nonlinear continuous map define from [0,r)×X into X. Eduardo [2] in 2001, has study the local existence and uniqueness of the of S-classical solution (Semiclassical Solution) to the problem defined in (1).

Definition:
A continuous function \( x_w \) is said to be a mild solution to the semilinear initial value problem defined in (1), if \( x(t) \) has the following form:

\[ x(t)=T(t)x_0 + \int_0^t T(t-s)f(s,x(s))ds \] ..........................(2)

Satisfies the following conditions: \( x(0) = x_0 \), \( \frac{d}{dt} x(t) \) is continuous on (0, r), \( x(t) \in D(A) \) for all \( t \in (0, r) \) and \( x(.) \) satisfies equation (1) on (0, r).

Manaf [3] in 2005, has study the local existence and uniqueness of the mild solution to the semilinear initial value control problem:

\[ \frac{dx}{dt} + Ax(t) = f(t, x(t)) + \int_0^t h(t-s)g(s, x(s))ds + (Bw)(t), t > 0 \]

\[ x(0) = x_0 \] ..........................(3)

where A is the infinitesimal generator of a C_0 semigroup defined from D(A)c⊂X into X and f and g are a nonlinear continuous maps defined from [0,r)×X into X. h is the real valued continuous function defined from [0,r) into \( \mathbb{R} \) where \( \mathbb{R} \) is the real number and B is a bounded linear operator define from O into X. Where O is a Banach space and \( w(.) \) be the arbitrary control function is given in \( L^p([0, r): O) \), a Banach space of control functions with \( ||w(t)||_O \leq k_1 \), for \( 0 \leq t < r \).

Definition:
A continuous function \( x_w \) is said to be a mild solution to the semilinear initial value problem defined in (3) given by:

\[ x_w(t) = T(t)u_0 + \int_0^t T(t-s) \left[ (Bw)(s) + f(s, x_w(s)) + \int_{s=0}^t h(s-\tau)g(s, x_w(\tau))d\tau \right] ds \]

\( \forall w \in L^p([0, r): O) \)

In the present paper, the S-classical solution (semi-classical solution) of the semilinear initial value control problem defined in (3) will be developed by the following definition:
Definition:
A function \( v \in C\left([0, r) : X\right) \) is said to be an S-classical solution (semi-classical solution) to the nonlinear initial value problem defined in (3), if \( v(t) \) has the following form:

\[
v_w(t) = T(t)x_0 + \int_0^t T(t-s)(Bw)(s) + f(s, v_w(s)) + \int_0^s h(s-t)g(s, v_w(t))dt \, ds
\]

\( \forall w \in L^p((0, r) : O) \)

Satisfies the following conditions: \( v_w(0) = x_0 \),

\[
dv_w(t) \text{ is continuous on } (0, r), \quad v_w(t) \in D(A)
\]

for all \( t \in (0, r) \) and \( v_w(.) \) satisfies equation (3) on \( (0, r) \).

Throughout this paper \( X \) will be a Banach space equipped with the norm \( || \cdot || \) and the operator \( A : D(A) \subseteq X \rightarrow X \) will be the infinitesimal generator of an analytic semigroup of bounded linear operators \( \{T(t)\}_{t \geq 0} \) on \( X \). For the theory of analytic semigroup, refer to Pazy [4], Jerome [5] and Balachandran [6]. The books of Pazy [4], Krien [7] and Fitzgibbon [8] contained therein, give a good account of important results. We mention here only some notation and properties essential to our purpose. In particular, we assume that \( \{T(t)\}_{t \geq 0} \) is an analytic semigroup generated by infinitesimal generator \( A \) and \( 0 \in \rho(A) \), \( \rho(A) \) stands for resolvent set. In this case it is possible to define the fractional power \( (-A)^\alpha \), for \( 0 < \alpha < 1 \), as a closed linear operator with domain \( D((-A)^\alpha) \) dense in \( X \) and the expression

\[
\|x\|_\alpha = \left\|(-A)^\alpha x\right\|_X
\]

defines a norm on \( D((-A)^\alpha) \). Hereafter we represent by \( X_\alpha \) the space \( D((-A)^\alpha) \) endowed with the norm \( \|\cdot\|_\alpha \).

Preliminaries
Definition:
A family \( \{T(t)\}_{t \geq 0} \) of bounded linear operators on a Banach space \( X \) is called a semigroup on \( X \) if it satisfies the following conditions:

\[
T(t+s) = T(t)T(s), \quad \forall t, s \geq 0 \quad T(0) = I, \quad (T(0) \text{ is the identity operator on } X).
\]

Definition:
A family \( \{T(t)\}_{t \geq 0} \) is said to be an analytic semigroup if the following conditions are satisfied:

(i) \( t \rightarrow T(t) \) is analytic in some sector \( \Delta \), where \( \Delta \) is a sector containing the nonnegative real axis.

(ii) \( T(0) = I \) and

\[
\lim_{t \downarrow 0} \|T(t)x - x\|_X = 0, \quad \forall x \in X.
\]

(iii) \( T(t+s) = T(t)T(s), \quad \forall t, s \geq 0 \).

Definition:
A semigroup \( \{T(t)\}_{t \geq 0} \) on a Banach space \( X \) is called strongly continuous semigroup of bounded linear operators or (C_0 semigroup) if

\[
T(0) = I \quad \text{in } L(X), \quad \forall t \geq 0.
\]

Definition:
If \(-A\) is the infinitesimal generator of bounded analytic semigroup then the fractional power \( A^{-\alpha} \) exist for \( \alpha > 0 \).

Definition:
Let \(-A\) be the infinitesimal generator of an analytic semigroup \( T(t) \) if \( 0 \in \rho(A) \), then:

(a) \( T(x) : X \rightarrow D(A^\alpha) \), for every \( t > 0 \) and \( \alpha \geq 0 \).

(b) For every \( x \in D(A^\alpha) \), we have \( T(t)A^\alpha x = A^\alpha T(t)x \).

(c) For every \( t \geq 0 \), the operator \( A^\alpha T(t) \) is bounded and \( \|A^\alpha T(t)\| \leq M_\alpha t^{-\alpha} \).

(e) Let \( 0 < \alpha \leq 1 \) and \( x \in D(A^\alpha) \) then

\[
\|T(t)x - x\| \leq C_\alpha t^{\alpha} \left\|A^\alpha x\right\|_X,
\]

where \( C_\alpha \) is the positive constant depend on \( \alpha \).
Definition:
Suppose X is a Banach space. A mapping T: X→X is said to be strict contraction with strict contraction constant L, if ∥Tx−Ty∥X ≤ L∥x−y∥X, ∀ x, y ∈ X, where 0 < L < 1.

Definition Banach contraction principle:
Let M is a closed nonempty set in the Banach space X over k, where k are a scalar field and the operator T: M→M is strict contraction operator then T has a unique fixed point.

Definition:
Let I be an interval, A function f: I→X, where X is a Banach space is said to be Hölder continuous with exponent θ, 0 < θ < 1 on I, if there is a constant L such that ∥f(t)−f(s)∥X ≤ L|t−s|θ, for s, t ∈ I.

Main Result
It should be notice that the local existence and uniqueness of S-classical solution (Semi-classical Solution) of the semilinear initial value control problem defined in (3) developed, by assuming the following assumptions:
A. A be the infinitesimal generator of bounded analytic semigroup {S(t)}t≥0 and 0 ∈ ρ(A), where the operator A defines from D(A)⊂ X into X, (X is a Banach space).
B. Let U be an open subset of [0, r)×Xα, for 0<r<∞, where Xα is a Banach space being dense in X.
C. For every (t, x)∈U, there exists a neighborhood G⊂ U of (t, x), the nonlinear maps f, g:[0,r)×Xα→X satisfy the locally Lipschitz condition with respect to second argument,
  ∥f(t, u)−f(s, v)∥X ≤ L0∥v−u∥Xα
  ∥g(t, u)−g(s, v)∥X ≤ L1∥v−u∥Xα
  for all (t, u) and (s, v) ∈ G.
D. For t''>0, ∥f(t,v)∥X ≤ B1, ∥g(t,v)∥X ≤ B2, for 0≤t≤t'' and for every v ∈ Xα.
E. For t">0, ∥S(t)−I∥∥Aαu0∥ ≤ δ', Where δ' < δ, 0 ≤ t ≤ t".
F. h is continuous function which at least h∈L1([0,r);R). Where R is the real number.
G. w(.) be the arbitrary control function is given in Lp([0,r];O), a Banach space of control functions with O as a Banach space and here B is a bounded linear operator from O into X and ∥w(t)∥O ≤ k1, for 0 ≤ t < r.

H. Let t1>0 such that t1 = min { t', t", t*, r}, satisfy the condition
  t1 ≤ ([K0K1+(δL0+B1)] +
  (H.i)
  h1((δL1+B2)]Cα1(1−α)(δ−δ')−1
  ⇒
  h1((δL1+B2)]Cα1(1−α)(δ−δ')−1
  ≤ [K0K1+(δL0+B1)] +
  (H.ii)
  h1((δL1+B2)]Cα1(1−α)(δ−δ')−1

I. There exist C2≥0 and 0<ξ<1 such that:
  ∥h(t)−h(s)∥O ≤ C3|t−s|ξ, for all t, s ∈ [0, t1].
J. There exist R0≥0 and 0<ξ<1 such that:
  ∥w(t)−w(s)∥O ≤ R0|t−s|ξ, for all t, s ∈ [0, t1].

Main Theorem:
Assume that hypotheses (A)-(J) are hold, then for every u0∈Xα, there exists a fixed number t1, 0 < t1 < r, such that the initial value control problem defined in (3) has a unique S-classical solution u ∈ C([0, t1]; X), for every control function w(.) ∈ Lp([0, t1]:O).

Proof:
Without loss of generality, we may suppose r<∞, because we are concerned here with the local existence only.

For a fixed point (0,u0) in the open subset U of [0,r)×Xα, we choose δ>0 such that the neighborhood G of the point (0,u0) define as follow:

G= {(t, x)∈U: 0≤t≤t', ∥v−v0∥α ≤ δ}⊂ U {since U is an open subset of [0,r)×Xα}.

It is clear that ∥AαS(t)∥ ≤ Cαt−α, for t>0, see {theorem (1.8.7) in [4]}.

Where Cα is a positive constant depending on α and assume
\[ h = \int \| h(s) \| ds. \]

Set \( Y = C([0,t_1];X) \), then \( Y \) is a Banach space with the supremum norm: \( \| y \| = \sup_{0 \leq t \leq 1} \| y(t) \|_x \).

Let \( S_w \) be the nonempty subset of \( Y \), define as follow:

\[ S_w = \{ x_w \in Y : x_w(0) = A^\alpha x_0, ||x_w(t) - A^\alpha x_0||_x \leq \delta, 0 \leq t \leq t_1 \}, \quad (4) \]

To prove the closedness of \( S_w \) as a subset of \( Y \). Let \( x^n_w \in S_w \), such that

\[ x^n_w \xrightarrow{P.C.} x_w \text{ as } n \to \infty, \text{ we must prove that } x^n_w \in S_w, \text{ where } (P.C) \text{ stands for point wise convergence.} \]

Since \( x^n_w \in S_w \Rightarrow x^n_w \in Y, x^n_w(0) = A^\alpha x_0 \) and \( ||x^n_w(t) - A^\alpha x_0||_x \leq \delta, 0 \leq t \leq t_1. \)

Since \( x^n_w \xrightarrow{U.C.} x_w \), hence \( x_w \in Y \), where \( (U.C) \) stands for the uniform convergence, and also since \( x^n_w \xrightarrow{U.C.} x_w \),

\[ \Rightarrow ||x^n_w - x_w||_Y \to 0, \text{ as } n \to \infty \]

\[ ||x^n_w - x_w||_{Y} = \sup_{0 \leq t \leq 1} ||x^n_w(t) - x_w(t)||_x \to 0, \text{ as } n \to \infty, \text{ (By } ||y|| = \sup_{0 \leq t \leq 1} ||y(t)||_x \text{).} \]

Implies that \( ||x^n_w(t) - x_w(t)||_x \to 0, \text{ as } n \to \infty, \) for every \( 0 \leq t \leq t_1, \)

i.e., \( \lim_{n \to \infty} x^n_w(t) = x_w(t), \forall \ 0 \leq t \leq t_1 \), \( (5) \)

\[ \Rightarrow \lim_{n \to \infty} x^n_w (0) = x_w(0) \quad \{ \text{by } (5) \} \]

\[ \Rightarrow \lim_{n \to \infty} A^\alpha x_0 = x_w(0) \{ \text{since } x^n_w \in S \} \]

\[ \Rightarrow A^\alpha x_0 = x_w(0) \]

Notice that:

\[ ||x_w(t) - A^\alpha x_0||_x = \lim_{n \to \infty} x^n_w (t) - A^\alpha x_0 ||_x, \text{ by } (5) \]

\[ = \lim_{n \to \infty} \frac{X^n_w (t) - A^\alpha x_0}{x_w(t)} = \lim_{n \to \infty} \frac{X^n_w (t) - A^\alpha x_0}{x_w(t)} \]

\[ \Rightarrow ||x_w(t) - A^\alpha x_0||_x \leq \lim_{n \to \infty} \delta \}

\{ \text{since } x^n_w \in S_w \}

\[ \Rightarrow ||x_w(t) - A^\alpha x_0||_x \leq \delta, \text{ for } 0 \leq t \leq t_1. \]

We have got \( S_w \) is closed subset of \( Y \).

Now, define a map \( F_w: S_w \to Y \), given by:

\[ (F_w x_w)(t) = \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]

\[ \quad \lim_{n \to \infty} \alpha \ A^\alpha S(t-s) f(s - \alpha x_w(s)) + \int_{s=0}^{t} h(s - \alpha) g(t - \alpha x_w(t)) ds + \int_{s=0}^{t} \alpha \ A^\alpha S(t-s) (Bw)(s) ds \]
By using the properties $\|x\|_k = \|A^\alpha x\|_K$, we get:

$$
\|(F_w x_w)(t) - A^\alpha x_0\|_k \leq \delta' + \int_0^1 C_\alpha (t-s)^{-\alpha} K_0 K_1 \ ds + \\
\int_0^1 C_\alpha (t-s)^{-\alpha} L_0 \|A^{-\alpha} x_w(s) - x_0\|_2 \ ds \\
+ \int_0^1 C_\alpha (t-s)^{-\alpha} h_{t_1} L_0 \|A^{-\alpha} x_w(s) - x_0\|_2 \ ds \\
+ \int_0^1 C_\alpha (t-s)^{-\alpha} [B_1 + h_{t_1} B_2] \ ds
$$

Let $x'_w, x''_w \in S_w$, then:

$$
\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_K = \|S(t)A^\alpha x_0 + \\
\int_0^1 A^\alpha S(t-s) (Bw)(s) ds + \\
\int_0^1 A^\alpha S(t-s) \left( f(s, A^{-\alpha} x'(s)) - f(s, x_0) \right) ds + \\
\int_0^1 A^\alpha S(t-s) \left( \int_0^s \left[ g(\tau, A^{-\alpha} x'(\tau)) - g(\tau, x_0) \right] d\tau \right) ds + \\
\int_0^1 \|f(s, x_0)\| + \int_0^1 \|h(s - \tau)\| \|g(\tau, x_0)\| d\tau \ ds
$$

After a series of simplifications and using the conditions C, D and E we have got:

$$
\|(F_w x_w)(t) - A^\alpha x_0\|_k \leq \delta' + \int_0^1 C_\alpha (t-s)^{-\alpha} K_0 K_1 \ ds + \\
\int_0^1 C_\alpha (t-s)^{-\alpha} L_0 \|A^{-\alpha} x_w(s) - x_0\|_2 \ ds \\
+ \int_0^1 C_\alpha (t-s)^{-\alpha} h_{t_1} L_0 \|A^{-\alpha} x_w(s) - x_0\|_2 \ ds \\
+ \int_0^1 C_\alpha (t-s)^{-\alpha} [B_1 + h_{t_1} B_2] \ ds
$$

Thus, we have that $F_w : S_w \rightarrow S_w$.

Now, to show that $F_w$ is a strict contraction on $S_w$, this will ensure the existence of a unique classical solution to the semilinear initial value control problem.

Let $x'_w, x''_w \in S_w$, then:

$$
\|(F_w x''_w)(t) - (F_w x'_w)(t)\|_K = \|S(t)A^\alpha x_0 + \\
\int_0^1 A^\alpha S(t-s) (Bw)(s) ds + \\
\int_0^1 A^\alpha S(t-s) \left( f(s, A^{-\alpha} x'(s)) - f(s, x_0) \right) ds + \\
\int_0^1 A^\alpha S(t-s) \left( \int_0^s \left[ g(\tau, A^{-\alpha} x'(\tau)) - g(\tau, x_0) \right] d\tau \right) ds - S(t)A^\alpha x_0
$$

So one can select $t_1 > 0$, such that:

$$
t_1 = \min \{ t', t'', t''' , r, \} \\
\left[ K_0 K_1 + (\delta L_0 + B_1) + h_{t_1} (\delta L_1 + B_2) \right] C_\alpha (1 - \alpha)^{-1} t_1^{1-\alpha}
$$

Thus, we have that $F_w : S_w \rightarrow S_w$.
\[ \int_{t_0}^{t} A^\alpha(t-s) \left[ f(s, A^{-\alpha}x'_w(s)) + \int_{t_0}^{s} h(s-\tau)g(\tau, A^{-\alpha}x'_w(\tau))d\tau \right] ds \]
\[ - \int_{t_0}^{t} A^\alpha(t-s) \left[ f(s, A^{-\alpha}x'_w(s)) + \int_{t_0}^{s} h(s-\tau)g(\tau, A^{-\alpha}x'_w(\tau))d\tau \right] ds \]
\[ \| (F_w x''_w)(t) - (F_w x'_w)(t) \| \leq \frac{1}{\delta} [ \delta L_0 + \delta ] \]
\[ h_1 L_1 + K_0 K_1 + B_1 + h_1 B_2 + C_0 (1 - \alpha)^{-1} t_1^{1-\alpha} \]
\[ \parallel \parallel x_w''(t) - x'_w(t) \parallel \parallel Y \]
\[ \| (F_w x''_w)(t) - (F_w x'_w)(t) \| \leq \frac{1}{\delta} [ K_0 K_1 + (\delta L_0 + B_1) + h_1 (\delta L_1 + B_2) ] C_0 (1 - \alpha)^{-1} \parallel x''_w - x'_w \parallel Y t_1^{1-\alpha} \]
\[ \parallel (F_w x''_w)(t) - (F_w x'_w)(t) \parallel \leq \left( 1 - \frac{\delta'}{\delta} \right) \parallel x''_w - x'_w \parallel Y \]

By using the condition C and the properties
\[ \| A^\alpha(t) \| \leq C \alpha t^{-\alpha} \] with
\[ h_r = \int_{0}^{t} \parallel h(s) \parallel ds, \]
We have got:
\[ \| (F_w x''_w)(t) - (F_w x'_w)(t) \| \leq \]
\[ \int_{s_0}^{t} C_0 (1-s)^{-\alpha} L_0 \| A^{-\alpha} x''_w(s) - A^{-\alpha} x'_w(s) \| ds \]
\[ + \int_{s_0}^{t} C_0 (1-s)^{-\alpha} h_1 L_1 \| A^{-\alpha} x''_w(s) - A^{-\alpha} x'_w(s) \| ds \]
\[ \| (F_w x''_w)(t) - (F_w x'_w)(t) \| \leq C_0 (1-\alpha)^{-1} L_0 \]
\[ h_1 t_1 \parallel x''_w(t) - x'_w(t) \parallel \parallel x''_w - x'_w \parallel Y \]
\[ \| (F_w x''_w)(t) - (F_w x'_w)(t) \| \leq C_0 (1-\alpha)^{-1} h_1 t_1 \parallel x''_w - x'_w \parallel Y \]
\[ \| (F_w x''_w)(t) - (F_w x'_w)(t) \| \leq \left( 1 - \frac{\delta'}{\delta} \right) \parallel x''_w - x'_w \parallel Y \]

Thus, \( F_w \) is a strict contraction map from \( S_w \) to itself and therefore by the Banach contraction principle there exist a unique fixed point \( x_w \) of \( F_w \) in \( S_w \), i.e., there is a unique \( x_w \in S_w \) such that \( F_w x_w = x_w \). This fixed point satisfies the integral equation:
\[ x_w(t) = S(t)A^\alpha x_0 + \int_{s_0}^{t} A^\alpha s(t-s) f(s, A^{-\alpha} x_w(s)) + h(s-\tau)g(\tau, A^{-\alpha} x_w(\tau))d\tau \]
\[ \int_{t_0}^{1} h(s-\tau)g(\tau, A^{-\alpha} x_w(\tau))d\tau \]
\( x_u(t) = S(t)A^\alpha x_0 + \)
\[ \int_{s=0}^{t} A^\alpha S(t-s) \left[ \tilde{f}(s) + \int_{\tau=0}^{s} h(s-\tau) \tilde{g}(\tau) d\tau \right] ds + \]
\[ \int_{s=0}^{t} A^\alpha S(t-s) Bw(s) ds , \quad \text{for } 0 \leq t \leq t_1 , \]
\( \forall w(.) \in L^2((0,t_1) ; \mathcal{O}) \) ............................................ (9)

To show that \( \tilde{t}(t) \) is locally H\ölder continuous on \((0, t_1)\).

We first show that \( x_u(t) \) given by (9) is locally H\ölder continuous on \((0, t_1)\).

Notice that, from the theorem (IV.7) in [4], it follows that for every \( 0 < \beta < 1 - \alpha \) and every \( 0 < h < 1 \), we have:
\[
\| (S(h) - I)A^\alpha S(t-s) \| \leq C_{\beta} h^\beta \| A^{\alpha+\beta} S(t-s) \| \leq Ch^{\beta}(t-s)^{(\alpha+\beta)} \quad \text{.................................. (10)}
\]

Which is useful for proving \( x_u(t) \) given by (9) is locally H\ölder continuous on \((0, t_1)\).

Next, we have for \( 0 < t < t + h \leq t_1 \)
\[
\| x_u(t+h) - x_u(t) \| \leq \| S(t+h)A^\alpha x_0 + \int_{s=0}^{t+h} A^\alpha S(t+h-s) \left[ \tilde{f}(s) + \int_{\tau=0}^{s} h(s-\tau) \tilde{g}(\tau) d\tau \right] ds + \int_{s=0}^{t+h} A^\alpha S(t+h-s) Bw(s) ds \| X + \]
\[ \int_{s=0}^{t+h} A^\alpha S(t+h-s) \left[ \tilde{f}(s) + \int_{\tau=0}^{s} h(s-\tau) \tilde{g}(\tau) d\tau \right] ds - \int_{s=0}^{t} A^\alpha S(t-s) Bw(s) ds \| X \]
\[
\| x_u(t+h) - x_u(t) \| = \| S(t+h)A^\alpha x_0 - S(t)A^\alpha x_0 + \int_{s=0}^{t+h} A^\alpha S(t+h-s) \left[ \tilde{f}(s) + \int_{\tau=0}^{s} h(s-\tau) \tilde{g}(\tau) d\tau \right] ds + \int_{s=0}^{t+h} A^\alpha S(t+h-s) Bw(s) ds \| X + \]
\[ \int_{s=0}^{t+h} A^\alpha S(t+h-s) \left[ \tilde{f}(s) + \int_{\tau=0}^{s} h(s-\tau) \tilde{g}(\tau) d\tau \right] ds - \int_{s=0}^{t} A^\alpha S(t-s) Bw(s) ds \| X \]

\( I_1 \leq M_1 h^\beta \), where \( M_1 = C \| x_0 \| t^{(\alpha+\beta)} \) depends on \( t \) for \( 0 \leq t \leq t_1 \).

\( I_2 \leq \int_{s=0}^{t} (B_1 + h_1 B_2)Ch^{\beta}(t-s)^{-(\alpha+\beta)} ds \),

\( \text{by using equation (10) and the condition D with } h_t = \frac{1}{h} |h(s)| ds \),

\[
I_2 \leq \frac{(B_1 + h_1 B_2)Ch^{\beta}}{1 - (\alpha + \beta)} t^{-(\alpha+\beta)+1} \leq \frac{(B_1 + h_1 B_2)Ch^{\beta}}{1 - (\alpha + \beta)} \]

\( I_2 \leq M_2 h^\beta \), where \( M_2 = \frac{(B_1 + h_1 B_2)Ch^{\beta}t^{-(\alpha+\beta)+1}}{1 - (\alpha + \beta)} \) is independent of \( t \) for \( 0 \leq t \leq t_1 \).

\( I_3 = \int_{s=0}^{t} \| (S(h) - I)A^\alpha S(t-s) \| X \| Bw(s) \| X ds \)

\[
I_3 \leq \int_{s=0}^{t} Ch^{\beta}(t-s)^{-(\alpha+\beta)} K_0 K_1 ds \leq \]
\[ Ch^{\beta} K_0 K_1 \int_{s=0}^{t} (t-s)^{-(\alpha+\beta)} ds \]
\[
\| \tilde{f}(t) - \hat{f}(s) \|_X = \| f(t, A^{-\alpha}x_w(t)) - f(s, A^{-\alpha}x_w(s)) \|_X
\]
\[
\| \tilde{f}(t) - \hat{f}(s) \|_X \leq L_0 \| t - s \|^\theta + \| A^{-\alpha}x_w(t) - A^{-\alpha}x_w(s) \|_\alpha
\]
for \( 0 < \theta \leq 1 \) by using the condition C,
\[
\| \tilde{f}(t) - \hat{f}(s) \|_X \leq L_0 \| t - s \|^\theta + \| x_w(t) - x_w(s) \|_\alpha
\]
by using the properties \( \|x\|_\alpha = \|A^{-\alpha}x\|_X \),
\[
\| \tilde{f}(t) - \hat{f}(s) \|_X \leq L_0 \| t - s \|^\theta + C_1 |t - s|^\beta
\]
by using equation (12)
\[
\| \tilde{f}(t) - \hat{f}(s) \|_X \leq L_0 (1 + C_1) |t - s|^\gamma
\]
\[
\| \tilde{f}(t) - \hat{f}(s) \|_X \leq C_2 |t - s|^\psi,
\]
Where \( C_2 = L_0 (1 + C_1) \) is a positive constant.

Let \( \tilde{h}(t) = \tilde{f}(t) + \int_{t_0}^t h(t - \tau) \hat{g}(\tau) d\tau + Bw(t) \)

To show that \( \tilde{h}(t) \) is locally Hölder continuous on \((0, t_1]\). For \( t > s \), we have:
\[
\| \tilde{h}(t) - \tilde{h}(s) \|_X = \| \tilde{f}(t) + \int_{t_0}^t h(t - \tau) \hat{g}(\tau) d\tau + \hat{f}(s) - \int_{t_0}^s h(s - \tau) \hat{g}(\tau) d\tau + \]
\[
+ |Bw(t) - Bw(s)|
\]
\[
\| \tilde{h}(t) - \tilde{h}(s) \|_X \leq \| \tilde{f}(t) - \tilde{f}(s) \|_X + \int_{t_0}^t |h(t - \tau) - h(s - \tau)| \| \hat{g}(\tau) \|_X d\tau + \]
\[
\| B(w(t) - w(s)) \|_X
\]

After a series of simplifications and using the conditions I, D and J with equation (13), we have got:
\[
\| \tilde{h}(t) - \tilde{h}(s) \|_X \leq C_2 |t - s|^{\gamma} + C_3 B_2 |t - s|^{\psi} t_1 + K_0 R_0 |t - s|^{\xi}
\]
\[
\| \tilde{h}(t) - \tilde{h}(s) \|_X \leq C_2 |t - s|^{\gamma} + C_3 B_2 |t - s|^{\psi} t_1 + K_0 R_0 |t - s|^{\xi}, \text{ where } \gamma = \min \{ \gamma, \beta, \xi \}
\]
\[
\| \tilde{h}(t) - \tilde{h}(s) \|_X \leq \{ C_2 + C_3 B_2 t_1 + K_0 R_0 \} |t - s|^{\psi}
\]
This show that \( \tilde{h}(t) \) is locally Hölder continuous on \((0, t_1]\).
From the theorem (2.4.1) in [5], we infer that the function:

\[ v_w(t) = S(t)x_0 + \int_0^t S(t-s)\hat{f}(s)ds + Bw(t). \]

where

\[ \hat{f}(t) = \int_0^t f(t)g(s)ds + Bw(t), \]

\[ v_w(t) = S(t)x_0 + \int_0^t S(t-s)\hat{f}(s)ds + Bw(s) \]

For 0 < t \leq t_1, \ \forall \ \omega \in L^p((0, t_1]:O).

is \ X_a \text{-valued}, that the integral terms in (14) are functions in \ C^i((0, t_1]; X) \text{and that}

\[ v_w(t) \in D(A), \ \forall \ t \in (0, t_1]. \]

Operating on both sides of equation (14) with \ A^a, we have got:

\[ A^a v_w(t) = S(t)A^a x_0 + \int_0^t S(t-s)A^a \left[ f(s, A^{-\alpha}x_u(s)) + \int_0^s h(s-\tau)g(\tau, A^{-\alpha}x_u(\tau))d\tau + Bw(s) \right] ds \]

(15)

From equation (8), implies that \ A^a v_w(t) = x_u(t), i.e., \ v_w(t) = A^{-\alpha} x_u(t),

For 0 < t \leq t_1, \ \forall \ \omega(.) \in L^p((0, t_1]:O), and hence that \ v_w(t) is a \ C^i \text{ function on } [0, t_1]. \text{So we have got a unique S-classical solution } v_w \in C([0, t_1]: X).

References