On Principally Generalized Lifting Modules

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Abstract
In this paper we introduce principally generalized lifting as a generalization of principally lifting modules and we prove under certain conditions some relations between Mj-projective (quasi-discrete) and PGD1. [DOI: 10.22401/JNUS.20.4.14]

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δ1 Introduction
Let R be an associative ring with identity and let M be a unital R-module. A sub module L of an R-module M is called small for (short L ≪ M), if K + L ≠ M for any proper sub module K of M. A module M is called hollow, if every proper submodule of M is small in M [1]. A non zero module M is called so- semi hollow, if each proper finitely generated sub module is small in M, and a non zero module M is so-called P-hollow, if each proper cyclic sub module is small in M [5]. It is clear that every hollow is semi hollow and every semi hollow is P- hollow. A module M is called lifting (or has the condition D1), if for every sub module L of M, there is a decomposition M = N ⊕ S such that N ≤ L and S ∩ L ≪ M [2]. It was introduced in [3] that a module M is principally lifting module (or has PD1), if for all m ∈ M, M has a decomposition M = N ⊕ S with N ≤ mR and mR ∩ S ≪ M. M is said to have condition (D2) in case, if B is a submodule of M with M / B is isomorphic to summand of M then B is a summand of M [4]. A module M is called a discrete module, if it has the condition (D1) and (D2). M is said to have the condition (D3) just in case of if M1 and M2 are summand. Such that M1 + M2 = M then M1 ∩ M2 is a summand of M. A module M is called so-called quasi-discrete module, if it has the condition (D1) and (D3). [4]

A module M is so-called a generalized lifting module, if every submodule L of M, there is a decomposition M = M1 ⊕ M2 such that M1 ≤ L and M2 ∩ L ≤ Rad(M). As a generalization of Principally lifting module we introduce a principally generalized lifting module (for short PGD1). Where Rad (M) is the Jacobson radical of M. It is known that Rad (M) equal the sum of all small submodules of M. [4]. In this paper we study the relation between PD1 and PGD1 modules and prove some properties of a PGD1.

δ2 P-hollows and the condiion (PGD1)
In this section we introduce PGD1 module as a generalization of PD1, that appeared in [3] and we prove results on PGD1 module.

We start by the following.

Lemma (2.1) [5,2.15]:
Let M be a module then
1. If M is semi- hollow, then each factor modul is semi-hollow.
2. If B ≪ M and M / B is semi-hollow then M is semi-hollow.
3. M is semi-hollow if and only if M is local or Rad(M) = M”.

Proposition (2.2) [3]:
The following are equivalent for a module M.
1. M is P- hollow.
2. B ≪ M when M / B is a non Zero cyclic module “.

Remark (2.3):
1- P- hollow modules need no hollow just as is explained in [5] by considering the set Q of all rational as Z- module (Q / Z) is no hollow while is no cyclic for all that proper sub modul K of Q.
2- "hollow module are indecomposable modules then the direct sums of hollow module are not hollows, while according to lemma (2.1), if M = ⊕ i∈I P,where P, are non-cyclical P-hollows for all i∈I, then M is P – hollow".
Remark (2.4):
Every hollow module is lifting [6].

Definition (2.5):-[5]
A module M is called Principally lifting (or has (PD_1)) if for all m ∈ M, M has a decomposition M = N ⊕ S with N ≤ mR and mR ∩ S ≪ M.

As generalization of definition (2.5) we introduce the following:

Definition (2.6):-
M is principally generalized lifting (or has PDG_1), If for all m ∈ M, M has a decomposition M = A ⊕ B with A ≤ mR and mR ∩ B ≤ Rad(M).

Note:-
hollow module → lifting module → principally lifting module → principally generalized lifting module.

Example (2.7):-
1. \( \mathbb{Z}_p^\infty \) is (PGD_1).
2. \( \mathbb{Z}_4 \) as \( \mathbb{Z}- \) module is (PGD_1).
3. \( \mathbb{Z}_p \), p is prim number is PGD_1.
4. \( \mathbb{Z} \) as \( \mathbb{Z} - \) module is not PGD_1.

Proposition (2.8):-
The condition (PGD_1) is inherited by sum ands.

Proof:
Suppose that M have the condition PGD_1, also K ≤ ⊕ M, if k ∈ K, when M has a decomposition M = A ⊕ B with A ≤ mR and kR ∩ B ≤ Rad(M), it follows that K = A ⊕ (K ∩ B) and kR ∩ (K ∩ B) ≤ kR ∩ B ≤ Rad(M), so kR ∩ (K ∩ B) ≤ Rad(K)(due to K ≤ ⊕ M). Therefore K has (PGD_1).

Lemma (2.9):-
The following are equivalent for an indecomposable module M.
1- M has (PGD_1).
2- M is a P-hollow module.

Proof:
(1) ⇒ (2) Suppose that 0 ≠ m ∈ M, Rm is proper submodule of M, then by (1) there exist decomposable M = N ⊕ S, with N ≤ Rm and Rm ∩ S ≤ Rad (M), because M is indcomposable.

Then either S = 0 or N = 0, if S =0 then M = N, hence M = Rm (Contradiction) (since Rm is proper), hence N = 0. Thus M= S therefor Rm ∩ S = Rm ∩ M = Rm ≤ Rad(M) thus Rm ≤ Rad(M) hence mε Rad(M), Rm ≪ M.[11].

(2)⇒ (1) Since M is P- hollow then for each proper cyclic sub module mR of M, mR ≪ M. thus M = 0 ⊕ M and 0 ≤ mR, mR ∩ M = mR ≤ Rad (M).

The following definition appeared in [7]

Definition (2.10) :-
[7] Suppose that M is an R-module, if N,L ≤ M and M = N + L, then L is so- called generalized supplement of N just is case N ∩ L ≤ Rad(L). M is called generalized supplemented or (briefly GS) in case each submodule N has a generalized supplement in M.

Example (2. 11):-
[8] Suppose that M is a GS and Rad(M) be Noetherian or M satisfy A.C.C on small sub module, then M is a supplemented module.

Lemma (2.12):-
Suppose that M has (PGD_1), then each cyclic submodule mR has a generalized supplemented S whichever is a summand of M.

Proof:
Let mR ≤ M then there exist N ≤ mR with M = N ⊕ S and mR ∩ S ≤ Rad(M), hence M = mR + S and mR ∩ S ≤ Rad(M), hence S is a GS of M and S ≤ ⊕ M.

Lemma (2.13):-
"The following are equivalent for a module M."
1- M has PGD_1
2- Every one cyclic submodule K of M can be written as K = N⊕ S with N ≤ ⊕ M and S ≤ Rad (M).
3- Each m ∈ M there exist principal ideals I and J of R such that mR = mI ⊕ mJ, where mI ≤ ⊕ M and mJ ≤ Rad(M).

Proof:
(1) ⇒ (2) clear.
(2) ⇒ (1) Let K be a cyclic submodul of M then by(2) K = N ⊕ S with N ≤ ⊕ M and
S ≤ Rad (M). Write M = N ⊕ N¬, it follow
that K = N ⊕ K ∩ N¬.

Let π : N ⊕ N¬ → N¬ be the natural projection, we have K ∩ N¬ = π(K) = π(N ⊕ S) = π(S) ≤ Rad(M), hence M has PGD1.

(2) ⇔ (3)

§3 Results on Mj- projective (quasi-discrete) and PGD1 modules.

In this section we prove under certain conditions some relations between Mj-projective (quasi-discrete) and PGD1 module.

We need the definition:

Definition (3.1)[12]:
Let M = ⊕i∈j Hi, then Hj is Hj-projective for each i ≠ j, if every supplement C of Hj in M is a direct summand.

Lemma (3.2)[9, corollary 4.50]:
Let M = ⊕Mi, where Mi is hollow and Mj-projective whenever i ≠ j. Then M is a quasi-discrete module.

"It is known that each quasi – discrete module is a direct sum of hollow sub module unique up to isomorphism and is fully relatively projective”.

Proposition (3.3):
Suppose that M = ⊕j∈j Hj, where each Hj is a hollow module and is Hj –projective (j ≠ i). Then M has (PGD1).

Proof:
Suppose that K is a cyclic sub module of M, and there exists a finite subset F of I that K = ⊕i∈F Hi. By lemma (3.2), ⊕i∈F Hi is quasi discrete, thus K can be written as K = N ⊕ S wherever N ≤ ⊕i∈F Hi, hence N ≤ ⊕i∈F M and S ≤ Rad(⊕i∈F Hi). Therefore by lemma (2.13) M has PGD1.

Proposition (3.4):
Suppose that M is module with PGD1, if M = V + W such that W ≤ ⊕M and V ∩ W is cyclic, then W contains generalized supplemented of V in M.

Proof:
Because M has PGD1 and V ∩ W is cyclic we have by lemma (2.13) V ∩ W= N ⊕ S, where N ≤ ⊕ M and S ≤ Rad(M), Since W ≤ ⊕ M, we have S ≤ Rad(W). Write W = N ⊕ N1. It follows that V ∩ W = N ⊕ (V ∩ W ∩ N1) = N ⊕ (V ∩ N1).

Let π : N ⊕ N1 → N be that natural projection. It follows that V ∩ N1 = π(N ⊕ (V ∩ N1) = π(V ∩ W) = π(N ⊕ S) = π(S), hence π(S) ≤ Rad(M), hence V ∩ N1 ≤ Rad(M) such that M = V + N + N1= V + N1. Therefore N1 is generalized supplemented of V in M that is contained in W.

Corollary (3.5):
Suppose that M is a module with PGD1 over a principally "ideal ring", if M = V + mR, then mR contains a generalized supplemented of V in M.

Proof:
By lemma(2.13) we have mR = N ⊕ S, wherever N ≤ ⊕ M and S ≤ Rad(M), it follows that M = V + N, hence by lemma (2.13) N is cyclic summand of M, hence V ∩ N is a cyclic submodule of M and thus apply proposition (3.4).

Lemma (3.6):
Suppose that M is module such that PGD1, then each indcomposable cyclic submodule C of M is either small in M or a sum and of M.

Proof:
"by lemma (2.13) we have C = N ⊕ S with N ≤ ⊕ M and S ≤ Rad(M), since C is indecomposable either C = S" or C = N, if C = S, then C ≤ Rad (M) since C is cyclic, then C = Rx ≤ Rad(M), hence x ∈ Rad (M) implies C = Rx is small in M. If C = N, then C ≤ ⊕ M.

Definition (3.7):
[4] "A module M is said to be π – projective, if for every two submodule U, V of M with M= U + V, there exist f ∈ End(M) with Imf ≤ U and Im(1- f) ≤ V".

Lemma (3.8):
[9, 4.47][10, 3.2] let M = M1 ⊕ M2. "Then following are equivalent."

1. M1 is M2- projective.
2. If M = N ⊕ M2, and N ∩ M2 ≤ ⊕ N hence M = N1 ⊕ M2, wherever N1 ≤ N.
**Proposition (3.9):**

Let $M = \bigoplus_{i=1}^{n} P_i$, where the $P_i$ are local modules for all $i$, if $M$ has(D3),"then the following are equivalent".

1- $M$ has PGD$_1$
2- "$M$ is a quasi-discrete module".

**Proof:**

(1) $\Rightarrow$ (2) Because PGD$_1$ and D$_3$ are inherited by summand, we have $p_i \oplus p_j$ has PGD$_1$ and D$_3$ for all $i,j$ ($i \neq j$).

If $p_i \oplus p_j = K + p_j$, then $p_i \cong (p_i \oplus p_j)/p_j \cong (K + p_j)/p_j \cong K$ (K is cyclic module).

Thus form some $m \in p_i \oplus p_j$,

$$K = mR + (K \cap p_j).$$

By PGD$_1$ for $p_i \oplus p_j$ and by lemma (2.13) we get $mR = N \oplus S$ with $N \leq p_i \oplus p_j$, so $S \leq \text{Rad}(p_i \oplus p_j)$ hence $p_i \oplus p_j = K \oplus p_j = (N \oplus S) + (K \cap p_j) + p_j = N + p_j$ and by (D$_3$) for $p_i \oplus p_j$ we have $p_i \oplus p_j = N + p_j$ with $N \leq K$. Hence by lemma (3.8) $p_i$ is $p_j$-projective for all $i \neq j$, therefor by lemma (3.2), $M$ is quasi-discrete.

(2) $\Rightarrow$ (1) it is obvious.

**Proposition (3.10):**

Suppose that $M$ is a module over a local ring $R$. If $M$ has PGD$_1$, then a cyclic submodule of $M$ is either small in $M$ or a summand of $M$.

**Proof:**

"The proof follows from lemma (3.6) and the fact that every cyclic module over a local ring is a local module".

**Definition (3.11)[3]:**

Suppose that $M_1$ and $M_2$ be R-modules $M_1$ is said to be P-projective relative to $M_2$ (or $M_1$ is $M_2$- P-projective), if for each $m_2 \in M_2$ epimorphism $g: m_2R \rightarrow m_2R/K$ and each homomorphism $\phi: M_1 \rightarrow m_1R/K$, there exists a homomorphism $f: M_1 \rightarrow m_2R$ with $g \circ f = \phi$.

**Remark (3.12) [3]:**

Clery every M- projective module is M- P projective, if M is a cyclic module then each M- P projective modul is M - projective module, there are R-modules $M_1$ and $M_2$, where $M_1$ is $M_2$- P projective whilst $M_1$ is no $M_2$-projective. Example $M_1 = Q$ (the set of all rational number) $R= Z$ and $M_2 = \bigoplus_{i=1}^{n} Z$, where $f: \bigoplus_{i=1}^{n} Z \rightarrow Q$ is an epimorphism (as $Q$ is a homomorphic image of a free $Z$-module). Clearly Q is $\bigoplus_{i=1}^{n} Z$- projective for every finite subset $F$ of I, hence Q is $(\bigoplus_{i=1}^{n} Z)$-P projective, while Q is not $(\bigoplus_{i=1}^{n} Z)$–projective, since f does not split (due to Q not a projective $Z$-module).

**Lemma (3.13):**

Let $M = M_1 \oplus M_2$ be an R-module. Then the following are equivalent".

1- $M_1$ is $M_2$ –Pprojective
2- $M_1$ is $m_2R$- projective for all that $m_2 \in M_2$

For all $m_2 \in M_2$, if $M_1 \oplus m_2R = m_2R + Y$, there is $L \leq Y$ such that $M_1 \oplus m_2R = L \oplus m_2R$.

**Proof:**

(1)$\Rightarrow$ (2) by definition of relative Pprojective

(2)$\Rightarrow$ (3) by lemma (3.8)

(3)$\Rightarrow$ (1) by lemma(3.8)

**Corollary (3.14):**

Let $M = M_1 \oplus M_2$ a module over local ring R- module $M_1$ and $M_2$ are relatively Pprojective in that case M has PGD$_1$, if and only if every one $M_1$ and $M_2$ have PGD$_1$.

**Proof:**

$\Leftarrow$) Suppose that C are arbitrary cyclic submodule of M then C =$\langle m_1 + m_2R \rangle$, where $m_1 \in M_1$, $m_2 \in M_2$, since $M_1$ and $M_2$ have PGD$_1$, then we have nothing to prove either $m_1 = 0$ or $m_2 = 0$.

Now to avoid triviality we may consider C is not a small submodule of M since $C = \langle m_1 + m_2R \rangle$ if $m_1 \in M_1$, $m_2 \in M_2$, we have $m_1R$ or $m_2R$ is not small in M. Without loss of generality we may assume $m_1R$ is no small in M, hence it is not small in $M_1$ by pro position (3.10), $m_1R$ is a summand of $M_1$ and hence $m_1R$ is $M_2$-Pprojective hence $m_1R$ is $m_2R$-projective.

Since $m_1R \oplus m_2R = (m_1 + m_2R) + m_2R$, we have by lemma (3.13) that there is $N \leq (m_1 + m_2R)$ with $m_1R \oplus m_2R = N \oplus m_2R$.It follows that $\langle m_1 + m_2R \rangle = N \oplus \langle m_1 + m_2R \rangle \cap m_2R$.

"Since C is a local module and $m_2R$ is not contained in C, we have that $C = N$.To show that N is a summand of M."
It is clear that $m_1R \oplus M_2 = N + M_2$ and hence $N \cap M_2 = N \cap (N \oplus m_2R) \cap M_2 = (m_1R \oplus m_2R) \cap M_2 \cap N = m_2R \cap N = 0$ (since $N = C$). As $m_1R \leq \oplus M_1$, where $N \oplus M_2 = m_1R \oplus M_2 \leq \oplus M \leq N \oplus \oplus M$. Therefore $C \oplus L = M$. The converse follows from proposition (2.8).

References