HYPERCYCLICITY AND SUPERCYCLICITY FOR SOME CLASSES OF OPERATORS

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Abstract
In this paper, we prove that, if \( T \) is the quotient of a decomposable on a separable Banach space (M–hyponormal operator on a real Hilbert space), then \( T \) is hypercyclic operators. We also show that these classes of operators are supercyclic operators.

Key words and phrases: Hypercyclic operator, supercyclic operator, decomposable operator, M–hyponormal operator, single valued extension property (SVEP), Bishop’s property (\( B \)), Dunford’s property (\( G \)), decomposition property (\( D \)).

Introduction
Let \( X \) be a complex Banach space, and \( \mathcal{L}(X) \) be the set of all bounded linear operators on \( X \), we also denote as usual the spectrum of \( T \) by \( \sigma(T) \). If \( T \in \mathcal{L}(X) \), then a part of \( T \) is a bounded operator obtained by restricting \( T \) to an invariant closed subspace \( M \), say \( T|_M \), a part of the spectrum of \( T \) is denoted by \( \sigma(T|_M) \), where \( M \) is an invariant closed subspace of \( T \).

An operator \( T \in \mathcal{L}(X) \) is called hypercyclic if there is a vector \( x \in X \) with dense orbit \( \{x, Tx, T^2x, ...\} \), and is called supercyclic if there is a vector \( x \in X \) \( \{cT^n x : n \geq 0, c \in \mathbb{C}\} \), is dense in \( X \), see[3]. \( T \in \mathcal{L}(X) \) is said to be decomposable if every open cover \( U = U \cup V \) of the complex plane \( \mathbb{C} \) by two open sets \( U \) and \( V \) effects a splitting of the spectrum \( \sigma(T) \) and of the space \( X \), in the sense that there exist \( T \)-invariant closed linear subspaces \( Y \) and \( Z \) of \( X \) for which \( \sigma(T|_Y) \subseteq U \), \( \sigma(T|_Z) \subseteq V \), and \( X = Y \oplus Z \), for example, all normal operators on a Hilbert space, compact operators and generalized scalar operators on Banach spaces are decomposable, see[6].

Also, for a \( T \)-invariant closed subspace \( M \) of \( T \), let \( T/M \in \mathcal{L}(X/M) \) denote the operator induced by \( T \in \mathcal{L}(X) \) on the quotient space \( X/M \) and called it the quotient of operator, It is known that every quotient space is Banach space, if \( X \) is Banach space, see [6].

Following to[2], let \( H \) be a complex Hilbert space, \( T \in \mathcal{L}(H) \), \( T \) is said to be \( M \)-hyponormal operator, if there exists a constant number \( M > 0 \) such that \( \| (T - \lambda I)^* x \| \leq M \| (T - \lambda I)x \| \) for each complex number \( \lambda \). It is known that every hyponormal operator and every normal operator are \( M \)-hyponormal operators. The purpose of the present paper is to study the quotient of a decomposable on a complex Banach space and the \( M \)-hyponormal operator on a real Hilbert space to be hypercyclic or supercyclic under sufficient conditions.

Preliminaries
An operator \( T \in \mathcal{L}(X) \) is said to be have single valued extension property (SVEP) at \( \lambda_0 \) if for every open set \( U \subseteq \mathbb{C} \) containing \( \lambda_0 \), the only analytic solution \( f : U \rightarrow X \) of the equation

\[
(T - \lambda I)f(\lambda) = 0 \quad (\lambda \in U)
\]

is the zero function., an operator \( T \) is said to have SVEP if \( T \) has SVEP at every \( \lambda \in \mathbb{C} \), ([4], [6]).

Given \( T \in \mathcal{L}(X) \), the local resolvent set \( \rho_T(x) \) of \( T \) at the point \( x \in X \) is defined as the
union of all open subsets $U \subseteq \mathbb{C}$ for which there is an analytic function $f: U \to X$ such that
\[(T - \lambda I)f(\lambda) = x \quad (\lambda \in U).\]

The local spectrum $\sigma_T(x)$ of $T$ at $x$ is then defined as
\[\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)\]

For $T \in \mathcal{L}(X)$, we define the local (resp. glocal) spectral subspaces of $T$ as follows. Given a set $F \subseteq \mathbb{C}$ (resp. a closed set $G \subseteq \mathbb{C}$).

\[X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}\]

(resp.
\[X_T(G) = \{x \in X : \text{there exists an analytic function } f: \mathbb{C} \setminus G \to X \text{ such that } (T - \lambda I)f(\lambda) = x \text{ for all } \lambda \in \mathbb{C} \setminus G\}\]

Note that $T$ has SVEP if and only if $X_T(F) = X_T(F)$ for all closed sets $F \subseteq \mathbb{C}$. [6, Proposition (3.3.2)].

An operator $T \in \mathcal{L}(X)$ has Dunford's property (C) if the local spectral subspace $X_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$. We also say that $T$ has Bishop's property (G) if for every sequence $f_n: U \to X$ such that $(T - \lambda I)f_n(\lambda) \to 0$ uniformly on compact subsets in $U$, it follows that $f_n \to 0$ uniformly on compact subsets in $U$. It is well known [6, 7] that Bishop's property (G) $\implies$ Dunford's property (C) $\implies$ SVEP.

Moreover, an operator $T \in \mathcal{L}(X)$ has decomposition property (G) if
\[X = X_T(U) + X_T(V)\]
for every open cover $(U, V)$ of $\mathbb{C}$.

As shown in [1], an operator $T \in \mathcal{L}(X)$ has property (G) iff it is the quotient of a decomposable operator. Moreover properties (G) and (G) are dual to each other, in the sense that an operator $T \in \mathcal{L}(X)$ has property (G) iff its adjoint has property (\tilde{G}), and conversely, $T$ has property (G) iff its adjoint has property (G).

The following result from Feldman, Miller and Miller [3], gives the relation between parts of the spectrum and the local spectra of an operator with Dunford's property (C).

**Proposition (2.1):**

If $T \in \mathcal{L}(X)$ has Dunford's property (C), then $\sigma_T(x) = \sigma(T|_{X_T(F)})$, whenever $F = \sigma_T(x)$ for some nonzero $x \in X$.

The following result from Feldman, Miller and Miller [3], gives sufficient condition for an operator to be hypercyclic, we denote the interior and exterior of the unit circle by $\mathbb{D}$, $\mathbb{C} \setminus \overline{\mathbb{D}}$ respectively.

**Corollary (2.2):**

Let $X$ be a complex Banach space and suppose that $T \in \mathcal{L}(X)$ has the decomposition property (G). If $\sigma_T(x^*) \cap \mathbb{D} = \emptyset$ and $\sigma_T(x^*) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) = \emptyset$ for every nonzero $x^* \in X^*$. Then $T$ is hypercyclic.

If $A$ is a compact set in the complex plane and $\varepsilon > 0$, then $B(A, \varepsilon)$ denote the $\varepsilon$-neighborhood of $A$, that is, $B(A, \varepsilon) = \{z : \text{dist}(z, A) < \varepsilon\}$. For the proof of the following classic result see Newman [7], Corollary 1.

**Lemma (2.3):**

If $K$ is any compact set in the complex plane, $A$ is a component of $K$, and $\varepsilon > 0$, then there exists disjoint open sets $U, V$ such that $K \subseteq U \cup V$ and $A \subseteq U \subseteq B(A, \varepsilon)$.

If $\rho \geq 0$, we denote the circle $\{z \in \mathbb{C} : |z| = \rho\}$ by $\Gamma_\rho$. The interior and exterior of $\Gamma_\rho$ are the regions $\text{int}\Gamma_\rho = \{z \in \mathbb{C} : |z| < \rho\}$ and $\text{ext}\Gamma_\rho = \{z \in \mathbb{C} : |z| > \rho\}$. Recall that an operator $T$ of is said to be $\rho$-outer (outer with respect to $\Gamma_\rho$) or $\rho$-inner (inner with respect to $\Gamma_\rho$) provided that $T$ satisfies conditions either (a) $X_T(\text{int}\Gamma_\rho)$ is dense and for every $\varepsilon > 0$, $X_T(\text{int}\Gamma_{\rho - \varepsilon})$, or (b) $X_T(\text{ext}\Gamma_\rho)$ is dense and for every $\varepsilon > 0$, $X_T(\text{ext}\Gamma_{\rho - \varepsilon})$ is dense, respectively.

The following Theorem is a stronger form of a result due to Herrero [5, Proposition (3.1)].
Theorem (2.4)
If $T \in \mathcal{L}(X)$ is a supercyclic operator on a separable Banach space $X$, then there exists a circle $\Gamma_\rho$, $\rho \geq 0$, such that $\sigma(T^n|_{X^n}) \cap \Gamma_\rho = \emptyset$ for every nonzero weakly closed $T^n$-invariant subspace $M$ of $X^n$.

In particular, every component of the spectrum of $T$ intersects $\Gamma_\rho$.

If $T$ is a supercyclic operator, then any circle as in Theorem (2.4) will be called a supercyclicity circle for $T$.

The following result from Feldman, Miller and Miller [3], gives sufficient condition for an operator to be supercyclic

Corollary (2.5):
Let $X$ be a complex Banach space and assume that $T \in \mathcal{L}(X)$ has the decomposition property $\langle \gamma \rangle$. If there exists a circle $\Gamma_\rho$, $\rho \geq 0$, satisfying either:

(a) For every nonzero $x^n \in X^n$, $\sigma_T(x^n)$ intersects both $\Gamma_\rho$ and $\text{int} \Gamma_\rho$, or

(b) For every nonzero $x^n \in X^n$, $\sigma_T(x^n)$ intersects both $\Gamma_\rho$ and $\text{ext} \Gamma_\rho$.

Then $T$ is supercyclic.

Main Results For hypercyclicity

Proposition (3.1):
If $T$ is a quotient of a decomposable operator on a complex Banach space $X$, and $\sigma(T^n|_{X^n}) \cap D = \emptyset$ and $\sigma(T^n|_{X^n}) \cap (C \setminus D) = \emptyset$ for every hyperinvariant $M$ of $T$, then $T$ is hypercyclic.

Proof:
Let $T$ be a quotient of a decomposable operator on $X$, then $T$ has property $\langle \beta \rangle$. Hence, $T$ has property $\langle \beta \rangle$, and so $T^*$ has property $\langle C \rangle$. Since $\sigma(T^n|_{X^n}) \cap D = \emptyset$ and $\sigma(T^n|_{X^n}) \cap (C \setminus D) = \emptyset$ for every hyperinvariant $M$ of $T^*$, and $X^*(F)$ is hyperinvariant for every closed set $F \subseteq C$, then $\sigma(T^n|_{X^*(F)}) \cap D = \emptyset$ and $\sigma(T^n|_{X^*(F)}) \cap (C \setminus D) = \emptyset$. Since $T^*$ has property $\langle \beta \rangle$, then $\sigma(T^*(x^n)) = \sigma(T^n|_{X^*(F)})$ whenever $F = \sigma_T(x^n)$ for some nonzero $x^n \in X^n$ by Proposition (2.1), it follows that $\sigma_T(x^n) \cap D = \emptyset$ and $\sigma_T(x^n) \cap (C \setminus D) = \emptyset$ for every nonzero $x^n \in X^n$. Thus Corollary (2.2) applies to give that $T$ is hypercyclic.

Corollary (3.2):
If $T$ is a quotient of a decomposable operator on $X$, and $T$ is both inner and outer with respect to a circle $\Gamma_\rho$ (where inner and outer with respect to a circle $\Gamma_\rho$ is defined above), $\rho \geq 0$, then a multiple of $T$ is hypercyclic.

Proof:
If $T$ is both inner and outer with respect to $\Gamma_\rho$, then $\frac{1}{\gamma} T$ will be hypercyclic by Proposition (3.1).

We now present new results for $M$-hyponormal operator on a real Hilbert space which is needed, then later

Proposition (3.3):
If $T$ is $M$-hyponormal operator on a real Hilbert space $H$, then $T^*$ has Bishop's property $\langle \beta \rangle$.

Proof:
Let $U \subseteq C$ be an open set, and consider a sequence of analytic functions $f_n : U \rightarrow H$ for which $\langle (T^* - \lambda I)f_n(\lambda) \rightarrow 0 \rangle$ as $n \rightarrow \infty$ locally uniformly on $U$. We want to show that $f_n \rightarrow 0$ as $n \rightarrow \infty$, again locally uniformly on $U$. Since $T$ is $M$-hyponormal operator, then $T$ has property $\langle \beta \rangle$, by [5, Proposition (2.4.9)]. Hence $f_n \rightarrow 0$ as $n \rightarrow \infty$ uniformly on all compact subsets of $U$, for every sequence of analytic functions $f_n : U \rightarrow H$ for which $\langle (T - \lambda I)f_n(\lambda) \rightarrow 0 \rangle$ as $n \rightarrow \infty$ uniformly on all compact subsets of $U$, but we need $f_n \rightarrow 0$ as $n \rightarrow \infty$ locally uniformly on $U$, when $\langle (T^* - \lambda I)f_n(\lambda) \rightarrow 0 \rangle$ as $n \rightarrow \infty$ locally uniformly on $U$. Again, since $T$ is $M$-hyponormal
operator, then there exists a constant number $M > 0$ such that
\[ \| (T - \lambda I)^n x \| \leq M \| (T - \lambda I)^{n-1} x \| \quad \text{for all } \lambda \in \mathbb{R}, \]
\[ x \in H. \] Therefore we have
\[ \| (T - \lambda I)^n f_\lambda (x) \| \leq M \| (T - \lambda I)^{n-1} f_\lambda (x) \| \quad \text{for all } \lambda \in \mathcal{U}, \quad f_\lambda (x) \in H. \] So $(T - \lambda I)^n f_\lambda (x) \to 0$ as $n \to \infty$. Therefore $T^*$ has Bishop's property $(\mathfrak{B}_\rho)$.

**Remark (3.4):**
Proposition (3.3) is not true if $H$ is a complex Hilbert space.

Now we shall prove that every $M$–hyponormal operator on a real Hilbert space is hypercyclic.

**Proposition (3.5):**

*If $T$ is $M$–hyponormal operator on a real Hilbert space $H$, and $\sigma(T^*|_M) \cap \mathbb{D} = \emptyset$ and $\sigma(T^*|_{M^*}) \cap (\mathbb{C} \setminus \mathbb{D}) = \emptyset$ for every hyperinvariant $M$ of $T^*$, then $T$ is hypercyclic.*

**Proof:**

If $T$ is $M$–hyponormal operator on a real space $H$, then $T^*$ has property $(\mathfrak{B})$, by Proposition (3.3). Thus $T^*$ has property $(\mathfrak{C})$, and so $T$ has property $(\mathfrak{B})$. Now, since $\sigma(T^*|_M) \cap \mathbb{D} = \emptyset$ and $\sigma(T^*|_{M^*}) \cap (\mathbb{C} \setminus \mathbb{D}) = \emptyset$ for every hyperinvariant $M$ of $T^*$, and $H_T(F)$ is hyperinvariant for every closed set $F \subseteq \mathbb{R}$, then $\sigma(T^*|_{H_T(F)}) \cap \mathbb{D} = \emptyset$ and $\sigma(T^*|_{H_T(F)}) \cap (\mathbb{C} \setminus \mathbb{D}) = \emptyset$ Since $T^*$ has property $(\mathfrak{C})$, then $\sigma(T^*)(x) = \sigma(T^*|_{H_T(F)})(x)$ whenever $F = \phi_T(x)$ for some nonzero $x \in H$. by Proposition (2.1), it follows that $\sigma_T(x) = \sigma(T^*|_{H_T(F)})(x)$, for every nonzero $x \in H$. Therefore $T$ is hypercyclic by Corollary (2.2).

**Corollary (3.6):**

*If $T$ is $M$–hyponormal operator on a real Hilbert space $H$, and $T$ is both inner and outer with respect to a circle $\Gamma_\rho$ (where inner and outer with respect to a circle $\Gamma_\rho$ is defined above), then $\rho > 0$, then a multiple of $T$ is hypercyclic.*

**Proof:**

If $T$ is both inner and outer with respect to $\Gamma_\rho$, then $\frac{1}{\rho} T$ will be hypercyclic by Proposition (3.5).

**Main Results for Supercyclic**

**Proposition (4.1):**

*If $T$ is a quotient of a decomposable operator on a complex Banach space $X$, and there exists a circle $\Gamma_\rho$, $\rho \geq 0$, such that either:

a–For every hyperinvariant subspace $M^\ast$ of $T^*$, $\sigma(T^*|_{M^\ast}) \cap \mathbb{D} = \emptyset$ and $\mathbb{D} \setminus \mathbb{D}$ or

b–For every hyperinvariant subspace $M^\ast$ of $T^*$, $\sigma(T^*|_{M^\ast}) \cap \mathbb{D} \setminus \mathbb{D}$ intersects $\partial \mathbb{D}$. Then $T$ is supercyclic.*

**Proof:**

Since $T$ is a quotient of a decomposable operator on $X$, then $T$ has property $(\mathfrak{B})$. Hence $T^*$ has property $(\mathfrak{B})$, and so $T^*$ has property $(\mathfrak{C})$. If $\sigma(T^*|_{M^\ast})$ intersects $\Gamma_\rho$ and $\partial \mathbb{D}$, for every hyperinvariant subspace $M^\ast$ of $T^*$. And since $X^\ast = \{F \subset \mathbb{R} \mid F \text{ is hyperinvariant for every closed set } F \subset \mathbb{R} \}$, then $\sigma(T^*|_{M^\ast}) \cap \mathbb{D} \setminus \mathbb{D}$ intersects $\Gamma_\rho$ and $\partial \mathbb{D}$. Since $T^*$ has property $(\mathfrak{C})$, then $\sigma_T(x) = \sigma(T^*|_{M^\ast}) \cap \mathbb{D} \setminus \mathbb{D}$, whenever $F = \phi_T(x)$ for some nonzero $x \in X$, by Proposition (2.1), it follows that $\sigma_T(x)$ intersects both $\partial \mathbb{D}$ and $\mathbb{D} \setminus \mathbb{D}$, for every nonzero $x \in X$. Thus Corollary (2.5) applies to give that $T$ is supercyclic. Similarly, if $\sigma(T^*|_{M^\ast})$ intersects $\partial \mathbb{D}$, and $\mathbb{D} \setminus \mathbb{D}$, for every hyperinvariant subspace $M^\ast$ of $T^*$, then $T$ is supercyclic.

We say that an operator is *purely supercyclic*, if it pure (It mains the restriction of operator on any nontrivial invariant subspace is not normal), supercyclic and no multiple of it is hypercyclic.

**Corollary (4.2):**
If $T$ is a quotient decomposable operator on $X$, and $T$ is purely supercyclic, then $T$ has unique supercyclicity circle.

**Proof:**
If there are two supercyclicity circles, $\Gamma_{\rho_1}$ and $\Gamma_{\rho_2}$, with $0 \leq \rho_1 < \rho_2$, then every part of the spectrum of $T^\ast$ intersects both $\Gamma_{\rho_1}$ and $\Gamma_{\rho_2}$. Now choose a $\rho$ such that $\rho_1 < \rho < \rho_2$. An application of Lemma (2.3) and the fact that every part of $\sigma(T^\ast)$ must intersect both $\Gamma_{\rho_1}$ and $\Gamma_{\rho_2}$ imply that every part of $\sigma(T^\ast)$ will intersect $\Gamma_\rho$, as well as the interior and exterior of $\Gamma_\rho$. Thus, $T$ is both $\rho$–inner and $\rho$–outer, and the previous result implies that a multiple of $T$ is hypercyclic, contrary to our assumption.

**Corollary (4.3):**
If $\{T_n\}$ is a bounded sequence of quotient of decomposable operators such that for every $n$, $T_n$ is supercyclic, then $\bigcap_n T_n$ is supercyclic if and only if there is a common supercyclicity circle, $\Gamma_\rho$, $\rho \geq 0$, and $T_n$ is $\rho$–inner for every $n$ or $T_n$ is $\rho$–outer for every $n$.

**Proof:**
Let $T = \bigcap_n T_n$. If $T$ is supercyclic, then a supercyclicity circle for $T$ will be a supercyclicity circle for each $T_n$. Similarly, if $T$ is $\rho$–inner (or $\rho$–outer), then $T_n$ is $\rho$–inner (or $\rho$–outer) for each $n$. Conversely, suppose $\Gamma_\rho$ is a supercyclicity circle for each $T_n$ and each $T_n$ is $\rho$–outer. We need to check that if $M^\ast$ is a hyperinvariant subspace for $T^\ast$, then $\sigma(T^\ast|_{M^\ast})$ intersects both $\Gamma_\rho$ and $\text{ext} \Gamma_\rho$. However, since $M^\ast$ is hyperinvariant, it must be invariant under every coordinate projection. Thus $M^\ast = \bigcap_n M^\ast_n$, where $M^\ast_n$ is a hyperinvariant subspace of $T_n^\ast$. Thus, $\sigma(T^\ast|_{M^\ast}) \subseteq \sigma(T_n^\ast|_{M^\ast_n})$ for each $n$. So, if $n$ is such that $M^\ast_n = \{0\}$, then by assumption $\sigma(T_n^\ast|_{M^\ast_n})$ intersects both $\Gamma_\rho$ and $\text{ext} \Gamma_\rho$. Thus $\sigma(T^\ast|_{M^\ast})$ also intersects both $\Gamma_\rho$ and $\text{ext} \Gamma_\rho$. So, Theorem (4.1) implies that $T$ is supercyclic. If each $T_n$ is $\rho$–inner, then the proof is similar.

**Proposition (4.4):**
If $T$ is $M$–hyponormal operator on a real Hilbert space $H$, and there exists a circle $\Gamma_\rho$, $\rho \geq 0$, such that either:

- a– For every hyperinvariant subspace $M$ of $T^\ast$, $\sigma(T^\ast|_{M^\ast})$ intersects $\Gamma_\rho$ and $\text{int} \Gamma_\rho$ or

- b– For every hyperinvariant subspace $M$ of $T^\ast$, $\sigma(T^\ast|_{M^\ast})$ intersects $\Gamma_\rho$ and $\text{ext} \Gamma_\rho$.

Then $T$ is supercyclic.

**Proof:**
Since $T$ is $M$–hyponormal operator on a real Hilbert space $H$, then $T^\ast$ has property (C), by Proposition (3.3). Thus $T^\ast$ has property (C), and so $T$ has property (C). If $\sigma(T^\ast|_{M^\ast})$ intersects $\Gamma_\rho$ and $\Gamma_\rho$, for every hyperinvariant subspace $M$ of $T^\ast$. And since $H_{T^\ast}(F)$ is hyperinvariant for every closed set $F \subseteq \Omega$, then $\sigma(T^\ast|_{H_{T^\ast}(F)^\ast})$ intersects $\Gamma_\rho$ and $\text{int} \Gamma_\rho$. Now since $T^\ast$ has property (C), then $\sigma_{T^\ast}(x^\ast) = \sigma(T^\ast|_{H_{T^\ast}(F)^\ast})$ whenever $F = \sigma_{T^\ast}(x^\ast)$ for some nonzero $x^\ast \in H$ by Proposition (2.1), it follows that and $\sigma_{T^\ast}(x^\ast)$ intersects both $\Gamma_\rho$ and $\text{int} \Gamma_\rho$, for every nonzero $x^\ast \in H$. Thus Corollary (2.5) applies to give that $T$ is supercyclic. Similarly, if $\varphi(T^\ast|_{M^\ast})$ intersects $\Gamma_\rho$ and $\text{ext} \Gamma_\rho$ for every hyperinvariant subspace $M$ of $T^\ast$, then $T$ is supercyclic.

**Corollary (4.5):**
If $T$ is $M$–hyponormal operator on a real Hilbert space $H$, and $T$ is purely supercyclic, then $T$ has unique supercyclicity circle.

**Proof:**
If there are two supercyclicity circles, $\Gamma_{\rho_1}$ and $\Gamma_{\rho_2}$ with $0 \leq \rho_1 < \rho_2$, then every part of the
spectrum of $T^*$ intersects both $\Gamma^\rho_{R_1}$ and $\Gamma^\rho_{R_2}$. Now choose a $\rho$ such that $\rho_1 < \rho < \rho_2$. An application of Lemma (2.3) and the fact that every part of $\sigma(T^*)$ must intersect both $\Gamma^\rho_{R_1}$ and $\Gamma^\rho_{R_2}$, imply that every part of $\sigma(T^*)$ will intersect $\Gamma_{R_1}$, as well as the interior and exterior of $\Gamma_{R_2}$. Thus, $T$ is both $\rho$-inner and $\rho$-outer, and the previous result implies that a multiple of $T$ is hypercyclic, contrary to our assumption.

**Corollary (4.6):**

If $\{T_n\}$ is a bounded sequence of $M$-hyponormal operators on a real Hilbert space $H$ such that for every $n$, $T_n$ is supercyclic, then $\bigoplus_n T_n$ is supercyclic if and only if there is a common supercyclicity circle, $\Gamma^\rho$, $\rho \geq 0$, and $T_n$ is $\rho$-inner for every $n$ or $T_n$ is $\rho$-outer for every $n$.

**Proof:**

Let $=\bigoplus_n T_n$. If $T$ is supercyclic, then a supercyclicity circle for $T$ will be a supercyclicity circle for each $T_n$. Similarly, if $T$ is $\rho$-inner (or $\rho$-outer), then $T_n$ is $\rho$-inner (or $\rho$-outer) for each $n$. Conversely, suppose $\bar{T}$ is a supercyclicity circle for each $T_n$ and each $T_n$ is $\rho$-outer. We need to check that if $M$ is a hyperinvariant subspace for $T^*$, then $\sigma(T^*|_{M})$ intersects both $\Gamma^\rho_{R_1}$ and $\text{ext} \Gamma^\rho_{R_2}$. However, since $M$ is hyperinvariant, it must be invariant under every coordinate projection. Thus $M = \bigoplus_n M_n$ where $M_n$ is a hyperinvariant subspace of $T_n^*$. Thus, $\sigma(T^*|_{M_n}) \supset \sigma(T^*|_{M_n})$ for each $n$. So, if $n$ is such that $M_n = \{0\}$, then by assumption $\sigma(T^*|_{M_n})$ intersects both $\Gamma_{R_1}$ and $\text{ext} \Gamma_{R_2}$. Thus $\sigma(T^*|_{M_n})$ also intersects both $\Gamma_{R_1}$ and $\text{ext} \Gamma_{R_2}$. So, Theorem (4.4) implies that $T$ is supercyclic. If each $T_n$ is $\rho$-inner, then the proof is similar.

**References**