\(\varepsilon\)- NIELSEN COINCIDENCE POINT THEORY

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Abstract
Let \(f, g : X \to X\) be maps of a compact connected Riemannian manifold, with or without boundary. For \(\varepsilon > 0\) sufficiently small, we introduce an \(\varepsilon\) – Nielsen coincidence number \(N^\varepsilon(f, g)\) that is a lower bound for the number of coincidence points of all self – maps that are \(\varepsilon\) - homotopic to \(f\) and \(g\). We prove that there is always maps \(f_1, g_1 : X \to X\) that is \(\varepsilon\) – homotopic to \(f\) and \(g\) such that \(f_1\) and \(g_1\) have exactly \(N^\varepsilon(f, g)\) coincidence points.

Introduction
The Nielsen coincidence point theory applied to study of the calculation by computer of multiple solutions of systems of polynomials equations, using a Nielsen coincidence number to obtain a lower bound for the number of distinct solution [5].

Because machine accuracy is finite, the solution procedure requires approximations, but the information is still applicable to the original problem. The reason is that sufficiently close functions on well - behaved spaces are homotopic and Nielsen coincidence number is a homotopy invariant. Although the homotopy between two sufficiently close maps are through maps that are close to both, no limitation on the homotopies employs. The purpose of this paper is to introduce a type of Nielsen coincidence point theory that does assume that a specified to-lerance for error must be respected.

If distortion is limited to a pre-assigned amount, then it may not be possible, without exceeding the limit, to deform maps \(f\) and \(g\) so that it has exactly \(N(f, g)\) coincidence points. For very simple example, consider the maps \(f, g : I \to I = [0,1]\) such that

\[
\begin{align*}
    f(0) &= f(2/3) = 1, \\
    f(1/3) &= f(1) = 0 \\
    g(0) &= g(2/3) = 0, f(1/3) = f(1) = 0 \quad \text{and} \\
    g(1/3) &= g(1) = 1 .
\end{align*}
\]

If the maps \(f_2\) and \(g_1\) have \(N(f, g) = 1\) coincidence point, then there must be some \(t\in I\) such that \(|f(t) - f_1(t)| < 1/3\) and \(|g(t) - g_1(t)| < 1/3\).

This example suggests a concept of the geometric minimum (coincident-ce point) number of maps \(f, g : X \to X\) different from the one, \(MF[f, g]\), that is the focus of Nielsen coincidence point theory, namely,

\[
MF[f, g]=\min\{#\text{coin}(f, g) : f_1, g_1\text{ homotopic to } f, g \text{ respectively }\}, \quad \text{..................(1)}
\]

where \(#\text{coin}(f, g)\) denotes the cardinality of the coincidence point set. The distance \(d(f, g)\) between maps \(f, g : Z \to X\), where \(Z\) is compact and \(X\) is a metric space with distance function \(d\), is defined by

\[
d(f, g) = \max\{d(f(z), g(z)) : z \in Z\}. \quad \text{........(2)}
\]

Given \(\varepsilon > 0\), a homotopy \(\{h_t\} : Z \to X\) is an \(\varepsilon\) – homotopy if \(d(h_t, h_{t'}) < \varepsilon\) for all \(t, t' \in I\).

For a given \(\varepsilon > 0\), we define the \(\varepsilon\) - minimum (coincidence point) number \(MF^\varepsilon(f, g)\) of maps \(f, g : X \to X\) of a compact metric space by

\[
MF^\varepsilon(f, g) = \min\{#(f, g) : f_1, g_1\text{ is }\varepsilon\text{- homotopic to } f, g \text{ respectively}\}, \quad \text{..................(3)}
\]

Note that the concept of \(\varepsilon\)-homotopic maps does not give an equivalence relation

The notation \(MF[f, g]\) for the minimum number incorporates the symbol \([f, g]\), generally used to denote the homotopy class of \(f\) and \(g\), because \(MF[f, g]\) is a homotopy invariant. We do note use the corresponding notation for the \(\varepsilon\)-minimum number because it is not invariant on the homotopy class of \(f\) and \(g\). For instance, although constants maps \(K_1\) and \(K_2\) of \(I\) are homotopic to maps \(f\) and \(g\) of the example, for which \(MF^\varepsilon(f, g) = 3\) for any \(\varepsilon \leq 1/3\), obviously \(MF^\varepsilon(K_1, K_2) = 1\) for any choice of \(\varepsilon\).
Let \( f, g : X \to X \) be maps of a compact manifold. Just as the Nielsen coincidence number \( N(f, g) \) has property \( N(f, g) \leq MF(f, g) \), in the next section we will introduce the \( \varepsilon \)-Nielsen coincidence number \( N^\varepsilon(f, g) \), for \( \varepsilon \) sufficiently small, that has the property \( N^\varepsilon(f, g) \leq MF^\varepsilon(f, g) \). My main result proven in section 3, is a "minimum coincidence theorem": give \( f, g : X \to X \), there exists \( f_1 \) and \( g_1 \) with \( d(f, f_1) < \varepsilon \) and \( d(g, g_1) < \varepsilon \) such that \( f_1 \) and \( g_1 \) have exactly \( N^\varepsilon(f, g) \) coincidence points.

**The \( \varepsilon \)-Nielsen coincidence number**

Let \( X \) be a compact, connected differentiable manifold, possibly with boundary. We introduce a Riemannian metric on \( X \) and denote the associated distance function by \( d \). If the boundary of \( X \) is nonempty, we choose a product metric on a tubular neighborhood of the boundary and then use a partition of unity to extend to a metric for \( X \). There is an \( \varepsilon > 0 \) small enough so that, if \( p, q \in X \) with \( d(p, q) < \varepsilon \), then there is a unique geodesic \( c_{pq} \) connected them. This choice of \( \varepsilon \) is possible even though the manifold may have a nonempty boundary because the metric is a product on a neighborhood of the boundary for the rest of this paper, \( \varepsilon > 0 \) will always be small enough so that points within a distance of \( \varepsilon \) are connected by a unique geodesic. We view the geodesic between \( p \) and \( q \) as a path \( c_{pq}(t) \) in \( X \) such that \( c_{pq}(0) = p \) and \( c_{pq}(1) = q \). The function that takes the pair \( (p, q) \) to \( c_{pq} \) is continuous. If \( x \in c_{pq} \) then \( d(p, x) \leq d(p, q) \) because \( c_{pq} \) is the shortest path from \( p \) to \( q \) (see [7, corollary 10.8 on page 62]).

If \( f, g, f_1, g_1 : Z \to X \) are maps with \( d(f, f_1) < \varepsilon \) and \( d(g, g_1) < \varepsilon \), then setting \( h_1(z) = c_{f_1(z), f_1}(t) \) and \( h_1(z) = c_{g(z), g_1(z)}(t) \) defines an \( \varepsilon \)-homotopy between \( f \) and \( f_1 \), \( g \) and \( g_1 \) respectively. Thus an equivalent definition of the \( \varepsilon \)-minimum coincidence number of \( f, g : X \to X \) is

\[
MF^\varepsilon(f, g) = \min \left\{ \#(\text{Coin}(f_1, g_1)) : d(f, f_1) < \varepsilon \text{ and } d(g, g_1) < \varepsilon \right\}.
\]

For maps \( f, g : X \to X \), let

\[
\Delta^\varepsilon(f, g) = \{ x \in X : d(f(x), g(x)) < \varepsilon \}
\]

Theorem (1): The set \( \Delta^\varepsilon(f, g) \) is open in \( X \).

**Proof:** Let \( \mathbb{R}^+ \) denote the subspace of \( \mathbb{R} \) of non-negative real numbers. Define \( D_{f, g} : X \to \mathbb{R}^+ \) by \( D_{f, g}(x) = d(f(x), g(x)) \). Since \( [0, \varepsilon) \) is open in \( \mathbb{R}^+ \), it follows that \( \Delta^\varepsilon(f, g) = D_{f, g}^{-1}([0, \varepsilon)) \) is an open subset of \( X \).

For maps \( f, g : X \to X \), define an equivalence relation on \( \text{Coin}(f, g) \) as follows: \( x, y \in \text{Coin}(f, g) \) are \( \varepsilon \)-equivalent, if there is a path \( w : I \to X \) from \( x \) to \( y \) such that \( d(f \circ w, g \circ w) < \varepsilon \).

The equivalence classes will be called the \( \varepsilon \)-coincidence point classes or, more briefly, the \( \varepsilon \)-cpc of \( f \) and \( g \).

**Theorem (2):** Coincidence points \( x, y \) of \( f, g : X \to X \) are \( \varepsilon \)-equivalent if and only if there is a component of \( \Delta^\varepsilon(f, g) \) that contains both of them.

**Proof:** Suppose \( x, y \in \text{Coin}(f, g) \) are \( \varepsilon \)-equivalent and let \( w \) be a path in \( X \) from \( x \) to \( y \) such that \( d(f \circ w, g \circ w) < \varepsilon \). Thus for each \( s \in I \) we have \( d(f(w(s)), g(w(s))) < \varepsilon \), so \( w(I) \subset \Delta^\varepsilon(f, g) \). Since \( w(I) \) is connected it is contained in some component of \( \Delta^\varepsilon(f, g) \). Conversely, suppose \( x, y \in \text{Coin}(f, g) \) are in the same component of \( \Delta^\varepsilon(f, g) \). The components of \( \Delta^\varepsilon(f, g) \) are pathwise connected so there is a path \( w \) in it from \( x \) to \( y \). Since \( w \) is in \( \Delta^\varepsilon(f, g) \) that means \( d(f \circ w, g \circ w) < \varepsilon \) and thus \( x \) and \( y \) are \( \varepsilon \)-equivalent. \( \square \)

Theorems (1) and (2) imply that the \( \varepsilon \)-cpc are open in \( \text{Coin}(f, g) \), so there are finitely
many of them $F^\varepsilon_1, F^\varepsilon_2, \ldots, F^\varepsilon_r$. We denote the component of $\Delta^\varepsilon(f, g)$ that contains $F^\varepsilon_j$ by $\Delta_j^\varepsilon(f, g)$. An $\varepsilon$–cpc, $F^\varepsilon_j = \text{Coin}(f, g) \cap \Delta_j^\varepsilon(f, g)$ is essential if the coincidence point index $i((f, g), \Delta_j^\varepsilon(f, g)) \neq 0$. The $\varepsilon$–Nielsen coincidence number of $f$ and $g$, denoted by $N^\varepsilon(f, g)$, the number of essential $\varepsilon$–cpc.

**Theorem (3):**
If the coincidence points $x$ and $y$ of $f, g : X \to X$ are $\varepsilon$–equivalent, then $x$ and $y$ are in the same (Nielsen) coincidence point class. Therefore each coincidence point class is a union of $\varepsilon$–cpc and $N^\varepsilon(f, g) \geq N(f, g)$.

**Proof:**

If $x$ and $y$ are $\varepsilon$–equivalent by means of a path $w$ between them such that $d(f \circ w, g \circ w) < \varepsilon$ then $h_t(z) = c_{f \circ w(z), g \circ w(z)}(t)$ defines a homotopy, relative to the endpoints, between $f \circ w$ and $g \circ w$ so $x$ and $y$ are in the same coincidence point class. Therefore a coincidence point class $F$ of $f$ and $g$ is the union of $\varepsilon$–cpc. If $F$ is essential, the additively property of coincidence point index implies that at least one of the $\varepsilon$–cpc it contains must be an essential $\varepsilon$–cpc, thus $N^\varepsilon(f, g) \geq N(f, g)$. □

The $\varepsilon$–Nielsen coincidence number is a local Nielsen coincidence number in the sense of [4], [1], specifically $N^\varepsilon(f, g) = n((f, g), \Delta^\varepsilon(f, g))$.

However, in the local Nielsen coincidence theory, the domain $U$ of the local Nielsen $n((f, g), U)$ is the same for all the maps considered whereas $\Delta^\varepsilon(f, g)$ depends on $f$ and $g$.

**Theorem (4):**

Let $f, g : X \to X$ be maps then $N^\varepsilon(f, g) \leq MF^\varepsilon(f, g)$.

**Proof:**

Given maps $f_1, g_1 : X \to X$ with $d(f, f_1) < \varepsilon$ and $d(g, g_1) < \varepsilon$, let $\{h_t \}$: $X \to X$ be the $\varepsilon$–homotopy with $h_0 = f$ and $h_1 = f_1$, $h'_0 = g$ and $h'_1 = g_1$ defined by $h_t(x) = c_{f(x), f_1(x)}(t)$ and $h'_t(x) = c_{g(x), g_1(x)}(t)$ respectively. Theorem (1) implies that $d(f(x), g(x)) \geq \varepsilon$ for all $x$ in the boundary of $\Delta^\varepsilon(f, g)$. Thus for $x$ in the boundary of $\Delta^\varepsilon_f(f, g)$ and $t \varepsilon I$ we have $d(h_t(x), h'_t(x)) + d(h_t(x), h'_t(x)) \geq d(f(x), g(x)) \geq \varepsilon$. (6)

Since $\{h_t \}$ and $\{h'_t \}$ are an $\varepsilon$–homotopy, $d(h_t(x), f(x)) = d(h_t(x), h_0(x)) < \varepsilon$ and $d(h'_t(x), h'_0(x)) = d(h'_t(x), h'_0(x)) < \varepsilon$ so $d(h_t(x), h'_t(x)) > 0$, that is $h_t$ and $h'_t$ have no coincidence points on the boundary of $\Delta^\varepsilon_f(f, g)$.

Therefore, the homotopy property of coincidence point index implies that $i((f, g), \Delta^\varepsilon_f(f, g)) = i((f_1, g_1), \Delta^\varepsilon_f(f, g))$ ......... (7)

Consequently, if $\ell^\varepsilon = \text{Coin}(f, g) \cap \Delta^\varepsilon_f(f, g)$ is an essential $\varepsilon$–cpc, then $i((f_1, g_1), \Delta^\varepsilon_f(f, g)) \neq 0$ so $f_1$ and $g_1$ have a coincidence point in $\Delta^\varepsilon_f(f, g)$. We conclude that $f_1$ and $g_1$ have at least $N^\varepsilon(f, g)$ coincidence points. □

**The minimum coincidence theorem:**

The main result in this section is to prove the minimum coincidence theorem, but before that we need the following theorem.

**Lemma (5):**

Let $F$ be a closed subset of a compact manifold $X$ and let $U$ be an open, connected subset of $X$ that contains $F$, then there is an open, connected subset $V$ of $X$ containing $F$ such that the closure of $V$ is contained in $U$.

**Proof:**

Since $F$ and $X$ are disjoint compact sets, there is an open set $W$ containing $F$ such that the closure of $W$ is contained in $U$. There are finitely many components $W_1, \ldots, W_r$ of $W$ that contain points of the compact set $F$. Let $\alpha_1$ be a path in $U$ from $x_1 \in W_1 \cap F$ to $x_2 \in W_2 \cap F$ and let $A_1$ be an open subset of $U$ containing $\alpha_1$ such that the closure of $A_1$ is in $U$.

Since $\alpha_1$ is connected, we may assume $A_1$ is also connected. Continuing in this manner, we let
Let $F^j = \text{coin}(f,g) \cap \Delta^j(f,g)$ be an $\epsilon$ – cpc. By Lemma 5, there is an open, connected subset $V_j$ of $\Delta^j(f,g)$ containing $F^j$ whose closure $\text{cl}(V_j)$ is in $\Delta^j(f,g)$. For the map $D_{f,g}: X \to \mathbb{R}^n$ defined by $D_{f,g}(x) = d(f(x), g(x))$, we see that $D_{f,g}\left(\text{cl}(V_j)\right) = [0, \delta_j]$ where $\delta_j < \epsilon$. Choose $\alpha_j > 0$ small enough so that $\delta_j + 2\alpha_j < \epsilon$.

**Theorem (6) (Minimum Coincidence Theorem):**

Given $f, g: X \to X$, there exists $f_1, g_1: X \to X$ with $d(f, f_1) < \epsilon$ and $d(g, g_1) < \epsilon$ such that $f_1$ and $g_1$ have exactly $N^e(f, g)$ coincidence points.

**Proof:**

We will define $f_1$ and $g_1$ outside $\Delta^j(f, g)$ to be a simplicial approximation of $f$ and $g$ respectively such that $d(f, f_1) < \alpha$ and $d(g, g_1) < \alpha$, where $\alpha$ denotes the minimum of the $\alpha_j$. The proof then consists of describing $f_1$ and $g_1$ on each $\Delta^j(f, g)$ so to simplify notation, we will assume for now that $\Delta^j(f, g)$ is connected and thus we are able to suppress the subscript $j$. Triangulate $X$ and take a subdivision of such small mesh that if $u$ and $v$ are a simplicial approximation to $f$ and $g$ respectively with respect to that triangulation, then $d(u, f) < \alpha/2$ and $d(v, g) < \alpha/2$ and, for $\sigma$ a simplex that intersects $X \setminus \text{int}(V)$, we have $u(\sigma) \cap \sigma = \phi$ and $v(\sigma) \cap \sigma = \phi$. By the Hopf construction, we may modify $u$ and $v$, moving no point more than $\alpha/2$, so that it has finitely many coincidence points, each of which lies in a maximal simplex in $V$ and therefore in the interior of $X$ (see [2, Theorem 2 on page 118]). We will still call the modified maps $u$ and $v$, so we now have maps $u$ and $v$ with finitely many coincidence points and it has the property that $d(u, f) < \alpha$ and $d(v, g) < \alpha$. Define the triangulation of $X$ so that the coincidence points of $u$ and $v$ are vertices. Since $V$ is a connected $n$–manifold, we may connect the coincidence points of $u$ and $v$ by paths in $V$, let $P$ be the union of all these paths. With respect to a sufficiently fine subdivision of the triangulation of $X$, the star neighbor–hood $S(P)$ of $P$, which is a finite, connected polyhedron, has the property that the derived neighborhood of $S(p)$ lies in $V$. Let $T$ be a spanning tree for the finite connected graph that is the $1$–skeleton of $S(P)$, then $T$ contains $\text{coin}(u,v)$. Let $R(T)$ be a regular neighborhood of $T$ in $V \cap \text{int}(X)$, then, since $T$ is collapsible, $R(T)$ is the $n–$ ball by [8, Corollary 3.27 on page 41].

Thus we have a subset $W = \text{int}(R(T))$ of $V$ containing $\text{coin}(u,v)$ and a homeomorphism $\varphi: W \to \mathbb{R}^n$. We may assume that $\varphi(\text{coin}(u,v))$ lies in the interior of the unit ball in $\mathbb{R}^n$, which we denote $B_1$. Set $\varphi^{-1}(B_1) = B_1^\ast$. If $x \in B_1^\ast$, then $\varphi^{-1}\left(\varphi(\sigma)\phi\right) \subset B_1$. Define a retraction $\rho: B_1^\ast \to \partial B_1$, the boundary of $B_1$ by

$$\rho(\sigma) = \varphi^{-1}\left(\frac{1}{|\varphi(\sigma)|}\varphi(\sigma)\right).$$

Define $K: B_1^\ast \times [0,t_0] \to W$ by setting $K(0^*, t) = 0^*$ for all $t$ and, otherwise let

$$K(x, t) = \varphi^{-1}\left(\varphi(\sigma)\phi\right)\left(H(\rho(x), t)\right).$$

The function $K$ is continuous because $\varphi(H(\partial B_1 \times I))$ is a bounded subset of $\mathbb{R}^n$. Now define $D_R: B_1^\ast \times [0, t_1] \to \mathbb{R}^n$ by $D_R(x, t) = d(x, K(x, t))$. Since $D_R^{-1}(\{0\})$ is an open subset of $B_1^\ast \times [0,t_0]$ containing $B_1^\ast \times \{0\}$, there exists $0 < t_1 < t_2 < t_0$ such that $d(x, K(x, t_2)) < \alpha$ and $d(x, K(x, t_2)) < \alpha$. Define
Next we extend \( h_1 \) and \( h_2 \) to the set \( B_2^* \) consisting of \( x \in W \) such that \( 0 \leq |\varphi(x)| \leq 2 \) by letting

\[
\begin{align*}
    h_1(x) &= C_{u(x),v(x)}((1 - t_1)|\varphi(x)| + 2t_1 - 1) \\
    h_2(x) &= C_{u(x),v(x)}((1 - t_2)|\varphi(x)| + 2t_2 - 1)
\end{align*}
\]

where \( 1 \leq |\varphi(x)| \leq 2 \). Noting that \( h_1(x) = u(x) \) and \( h_2(x) = v(x) \) if \( \varphi(x) = 2 \), we extend \( h_1 \) and \( h_2 \) to all \( X \) by setting \( h_1 = u \) and \( h_2 = v \) outside \( B_2^* \).

The maps \( h_1 \) and \( h_2 \) have a single coincidence point at \( 0^* \). If

\[
i \left( (f, g), \Delta^x(f, g) \right) = 0,
\]

we let

\[
f_t = h_1, \quad g_t = h_2 : X \to X.
\]

If \( i \left( (f, g), \Delta^x(f, g) \right) = 0 \), by [2, Theorem 4 on page 123], there are maps \( f_t, g_t : X \to X \), identical to \( u \) and \( v \) respectively outside of \( B_1^* \), such that \( f_t \) and \( g_t \) have no coincidence point in \( B_1^* \) and \( d(f_t, u) < \alpha \) and \( d(g_t, v) < \alpha \). We claim that \( d(f, f_t) < \epsilon \) and \( d(g, g_t) < \epsilon \). For \( x \in B_2^* \), we defined \( f_t(x) = u(x) \) and \( g_t(x) = v(x) \) where \( d(u, f) < \alpha < \epsilon \) and \( d(v, g) < \alpha < \epsilon \).

If \( x \in B_2^* - B_1^* \), then \( f_t(x) = h_1(x) e^{C_{u(x),v(x)}} \) and \( g_t(x) = h_2(x)e^{C_{u(x),v(x)}} \). Let

\[
\begin{align*}
    d(h_1(x), u(x)) &\leq d(v(x), u(x)) \\
    d(h_2(x), v(x)) &\leq d(u(x), v(x))
\end{align*}
\]

Therefore,

\[
\begin{align*}
    d(f_t(x), f(x)) &\leq d(h_1(x), f(x)) + d(h_2(x), f(x)) \\
    &\leq d(u(x), f(x)) + d(v(x), f(x)) \\
    &\leq \left(d(v(x), f(x)) + d(u(x), f(x))\right) + d(v(x), f(x)) \\
    &< \delta + 2\alpha < \epsilon.
\end{align*}
\]

Now suppose \( x \in B_1^* \). If

\[
i \left( (f, g), \Delta^x(f, g) \right) \neq 0,
\]

\[
f_t(x) = h_1(x) = K(x, t_1) \text{ and } g_t(x) = h_2(x) = K(x, t_2),
\]

\[
d(f_t(x), f(x)) = d(K(x, t_1), f(x)) \\
\leq d(K(x, t_1), u(x)) + d(u(x), f(x)) \\
< \alpha + \delta < \epsilon.
\]

If \( i \left( (f, g), \Delta^x(f, g) \right) = 0 \), then

\[
d(f_t(x), f(x)) = d(f_t(x), h_1(x)) + d(h_1(x), f(x)) \\
< \alpha + \delta < \epsilon.
\]

Which completes the proof that \( d(f, f_t) < \epsilon \) and \( d(g, g_t) < \epsilon \).

We return now to the general case, in which \( \Delta^x(f, g) \) may be not connected. Applying the construction above to each \( \Delta^x(f, g) \) gives us maps \( f_t, g_t : X \to X \) with exactly \( N^x(f, g) \) coincidence points. For \( x \in \Delta^x(f, g) \) we defined \( f_t \) and \( g_t \) to be a simplicial approximation with \( d(f, f_t) < \alpha < \epsilon \) and \( d(g, g_t) < \alpha < \epsilon \). For \( x \in \Delta^x(f, g) \), the argument just concluded proves that

\[
\begin{align*}
    d(f, f_t) &\leq 2\alpha + \delta_j < \epsilon \\
    d(g, g_t) &\leq 2\alpha + \delta_j < \epsilon
\end{align*}
\]

because \( \alpha \) is the minimum of the \( \alpha_j \), so we know that \( d(f, f_t) < \epsilon \) and \( d(g, g_t) < \epsilon \). □

Theorem 6 throws some light on the failure of the Wecken property for surfaces [3]. For instance, consider the celebrated example of
Jiang [6], of maps $f$ and $g$ of the paths surface with $N(f, g) = 0$ but $MF(f, g) = 2$. The coincidence point set of $f$ and $g$ consists of three points, one of them of index zero. The other two coincidence points, $y_1$ and $y_2$ are of index $1$ and $-1$ respectively and Jiang described a path, call it $\sigma$ from $y_1$ to $y_2$ such that $g \circ \sigma$ is homotopic to $f \circ \sigma$ relative to the endpoints. Suppose $\varepsilon > 0$ is small enough so that points in the pants surface that are within $\varepsilon$ of each other are connected by a unique geodesic. If there were a path $\tau$ from $y_1$ to $y_2$ such that $g \circ \tau$ and $f \circ \tau$ were $\varepsilon$ - homotopic, then $N^\varepsilon(f, g) = 0$ and therefore, by theorem 6, there would be a coincidence point free maps homotopic to $f$ and $g$. Since Jiang proved that no maps homotopic to $f$ and $g$ can be coincidence point free, we conclude that no such path $\tau$ exists. In other words, for any paths $\tau$ from $y_1$ to $y_2$ that is homotopic to $g \circ \tau$ and $f \circ \tau$ relative to the endpoints, it must be that $d(g \circ \tau, f \circ \tau) > \varepsilon$.

References

الخلاصة
دال $f, g: X \rightarrow X$ لثنائ، لتشعب ريمان المتصل المتراص، مع أو بدون حد لليك $\varepsilon > 0$ الصغيرة بشكل كافى سوف نقدم $\varepsilon$ - عدد نيلسون المتطلب $N^\varepsilon(f, g)$ أيحدد الأولى نعدد النقاط المتطلبة لكل الدوال التي تكون $\varepsilon$ - هموموتوبيا لـ $f$ و $g$. سوف نبرهن في هذا البحث أن هناك $f_1, g_1: X \rightarrow X$ الدوال التي تكون $\varepsilon$ - هموموتوبيا لـ $f$ و $g$ وعدد النقاط المتطلبة لها هي $N^\varepsilon(f, g)$ بالضبط.