**ADOMAIN DECOMPOSITION METHOD FOR SOLVING SYSTEMS OF MULTI-DIMENSIONAL LINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND**

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**Abstract**

The aim of this work is to use Adomian decomposition method to solve Systems of multi-dimensional linear Fredholm integral equations of the second kind.

1-Introduction

Adomain decomposition method was first introduced by Adomain G. in 1980. This method is used to solve differential equations, [1], [2]. The convergence of Adomian decomposition method applied to the one-dimensional integral equations is discussed in [5]. Moreover this method is used to solve systems of the one-dimensional Volterra integral equations of first kind, [4], systems of linear equations and systems of the one-dimensional Volterra integral equations of second kind, [3], systems of the one-dimensional Fredholm integral equations of the second kind, [7] and systems of fractional differential equations, [6]. Here we use this method to solve systems of the multi-dimensional linear Fredholm integral equations of the second kind:

\[ u_i(x_1, x_2, \ldots, x_n) = f_i(x_1, x_2, \ldots, x_n) + \int \cdots \int \sum_{j=1}^{n} k_{i,j}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \]

where \( f_i \) is a known function of \( x_1, x_2, \ldots, x_n \), \( k_{i,j} \) is a known function of \( x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \), \( \{a_j\}_{j=0}^{n} \), \( \{b_j\}_{j=0}^{n} \) are known constants such that \( a_i \leq x_i \leq b_i \), \( i=1,2,\ldots,n \) and \( u_1, u_2, \ldots, u_n \) are the unknown functions that must be determined.

2-Adomain Decomposition Method Applied to System(1)

Consider the system of the multi-dimensional linear Fredholm integral equations of the second kind given by equation (1).

We rewrite this equation as a canonical form of Adomian's equation by letting

\[ N_i(x_1, x_2, \ldots, x_n) = \int \cdots \int \sum_{a=1}^{m} k_{i,a}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \]

\[ u_i(y_1, y_2, \ldots, y_n)dy_1dy_2\ldots dy_n \]

To solve equation(2) by Adomian's decomposition method, we let

\[ u_j(x_1, x_2, \ldots, x_n) = \sum_{m=0}^{\infty} A_{j,m}(x_1, x_2, \ldots, x_n) \]

and

\[ N_i(x_1, x_2, \ldots, x_n) = \sum_{m=0}^{\infty} A_{i,m}(x_1, x_2, \ldots, x_n) \]

where \( A_{j,m}, m=0,1,\ldots \) are polynomial depending on \( u_{i,0}, u_{i,1}, \ldots, u_{i,m}, u_{j,0}, u_{j,1}, \ldots, u_{j,m} \) and they are called Adomian polynomials. Hence, equation (2) can be rewritten as:

\[ \sum_{m=0}^{\infty} u_{i,m}(x_1, x_2, \ldots, x_n) = f_i(x_1, x_2, \ldots, x_n) + \sum_{m=0}^{\infty} A_{i,m}(u_{i,0}, u_{i,1}, \ldots, u_{i,m}, u_{n,0}, u_{n,1}, \ldots, u_{n,m}) \]

(3)

From equation (3) we define :

\[ u_{i,0}(x_1, x_2, \ldots, x_n) = f_i(x_1, x_2, \ldots, x_n), \]

\[ u_{i,m+1}(x_1, x_2, \ldots, x_n) = A_{i,m}(u_{i,0}, u_{i,1}, \ldots, u_{i,m}, u_{n,0}, u_{n,1}, \ldots, u_{n,m}), \]

\[ i = 1, 2, \ldots, n, m = 0, 1, \ldots \]

(4)

To determine Adomian polynomials, we consider the expansions:
\[ u_{i,m}(x_1, x_2, \ldots, x_n) = \sum_{m=0}^{\infty} \lambda^m u_{1,m}(x_1, x_2, \ldots, x_n), \]  

\[ N_{i,m}(x_1, x_2, \ldots, x_n) = \sum_{m=0}^{\infty} \lambda^m A_{1,m} \]  

where \( \lambda \) is a parameter introduced for convenience. From equation (6) we obtain:

\[ A_{1,m} = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} N_{i,m}(u_1, u_2, \ldots, u_n) \right] \bigg|_{\lambda=0} \]  

and from equations (*), (5) and (7) we have:

\[ A_{1,m} = \frac{1}{m!} \left[ \prod_{j_1 \neq j_2} \int \cdots \int k_{i,j_1}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \prod_{n=1}^{m} \left( \sum_{y_n} \frac{1}{m!} \lambda^m u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \lambda^m \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

So, the solution of the system given by equation(1) will be as follows:

\[ u_{i,0}(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n), \]

\[ u_{i,m+1}(x_1, x_2, \ldots, x_n) = \int \cdots \int k_{i,j}(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n) \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \bigg|_{\lambda=0} \]

\[ = \frac{1}{m!} \left[ \frac{d^m}{d\lambda^m} \prod_{n=1}^{m} \left( \sum_{y_n} u_{1,n}(y_1, y_2, \ldots, y_n) dy_1 \cdots dy_n \right) \right] \bigg|_{\lambda=0} \]

3-Numerical Example

In this section we give two examples of systems of multi-dimensional linear Fredholm integral equations of the second kind with their approximated solutions via Adomain decomposition method.

**Example (1):**
Consider the tow-dimensional linear Fredholm integral equation of the second kind:

\[ u(x, y) = xy^2 - \frac{1}{8} x \]

and

\[ u_{v1}(x, y) = \int \int xmu_r(z, m)dzdm, r = 1, 2, \ldots \]

For the first iteration, we have:

\[ u_1(x, y) = \int \int xmu_0(z, m)dzdm \]

\[ = \frac{3}{32} x. \]

Therefore the approximated solution of this example with two terms is:

\[ Q_2(x, y) = u_0(x, y) + u_1(x, y) \]

\[ = xy^2 - \frac{1}{8} x + \frac{3}{32} x \]

\[ = xy^2 - \frac{1}{8} x + \frac{3}{32} x \]

For the second iteration, we have:

\[ u_2(x, y) = \int \int xmu_1(z, m)dzdm \]

\[ = \frac{1}{32} x. \]

Therefore the approximated solution of this example with three terms is:

\[ Q_3(x, y) = u_0(x, y) + u_1(x, y) + u_2(x, y) \]

\[ = xy^2 - \frac{1}{8} x + \frac{3}{32} x + \frac{1}{32} x \]

\[ = xy^2 \]

Not that \( Q_3(x, y) = xy^2 \) is the exact solution of this example.

**Example (2):**
Consider the system of the two-dimensional linear Fredholm integral equations:

\[ u_i(x, y) = \frac{7}{12} xy + \int \int xy(u_i(z, m) + u_j(z, m))dzdm \]

\[ u_j(x, y) = \frac{17}{24} x^2y + \int \int x^2yz(u_i(z, m) + u_j(z, m))dzdm \]

This example is constructed such that the exact solution of it is \( u_1(x, y) = xy \) and \( u_2(x, y) = x^2y \).
Here we use Adomain decomposition method to find the solutions $u_i, u_2$ of this example. To do this we use the following Adomain scheme:

$$u_{i,0}(x, y) = \frac{7}{12} xy$$

$$= 0.5833xy$$

$$u_{2,0}(x, y) = \frac{17}{24} x^2 y$$

$$= 0.7083x^2y$$

and

$$u_{i,r}(x, y) = \int_0^1 \int_0^1 xy(u_{i,r-1}(z, m) + u_{2,r-1}(z, m))dzdm, r = 1, 2,...$$

For the first iteration, we have:

$$u_{1,1}(x, y) = \int_0^1 \int_0^1 xy(u_{1,0}(z, m) + u_{2,0}(z, m))dzdm$$

$$= \frac{9}{72} xy$$

$$= 0.2639xy.$$  

$$u_{2,1}(x, y) = \int_0^1 \int_0^1 x^2yz(u_{1,0}(z, m) + u_{2,0}(z, m))dzdm$$

$$= \frac{107}{576} x^2y$$

$$= 0.1858x^2y.$$  

Therefore the approximated solutions of this example with two terms are:

$$Q_{1,1}(x, y) = u_{1,0}(x, y) + u_{1,1}(x, y) + u_{2,1}(x, y)$$

$$= \frac{7}{12} xy + \frac{19}{72} xy$$

$$= 0.5833xy + 0.2639xy$$

$$= 0.8472xy.$$  

$$Q_{2,1}(x, y) = u_{1,0}(x, y) + u_{2,1}(x, y) + u_{2,2}(x, y)$$

$$= \frac{17}{24} x^2y + \frac{107}{576} x^2y$$

$$= 0.7083x^2y + 0.1858x^2y$$

$$= 0.8941x^2y.$$  

For the second iteration, we have:

$$u_{1,2}(x, y) = \int_0^1 \int_0^1 xy(u_{1,1}(z, m) + u_{1,2}(z, m))dzdm$$

$$= \frac{335}{3456} xy$$

$$= 0.0969xy.$$  

$$u_{2,2}(x, y) = \int_0^1 \int_0^1 x^2yz(u_{1,1}(z, m) + u_{2,1}(z, m))dzdm$$

$$= \frac{929}{13824} x^2y$$

$$= 0.0672x^2y.$$  

Therefore the approximated Solutions of this example with three terms are:

$$Q_{1,2}(x, y) = u_{1,0}(x, y) + u_{1,1}(x, y) + u_{2,1}(x, y) + u_{1,2}(x, y)$$

$$= \frac{7}{12} xy + \frac{19}{72} xy + \frac{335}{3456} xy$$

$$= \frac{3263}{3456} xy$$

$$= 0.9442xy.$$  

$$Q_{2,2}(x, y) = u_{1,0}(x, y) + u_{2,1}(x, y) + u_{2,2}(x, y) + u_{2,3}(x, y)$$

$$= \frac{17}{24} x^2y + \frac{107}{576} x^2y + \frac{929}{13824} x^2y$$

$$= \frac{13289}{13824} x^2y$$

$$= 0.9613x^2y.$$  

In the same way, the components $Q_{1,k}(x, y)$ and $Q_{2,k}(x, y)$ can be calculated for $k=4, 5, ...$ The solutions with ten terms are given as:

$$Q_{1,10}(x, y) = \sum_{i=0}^{9} u_{1,i}(x, y)$$

$$= \frac{47544222539155}{47552535724032} xy$$

$$= 0.9998xy.$$  

$$Q_{2,10}(x, y) = \sum_{i=0}^{9} u_{2,i}(x, y)$$

$$= \frac{63395395174439}{63403380965376} x^2y$$

$$= 0.9999x^2y.$$  

4- Conclusion

As seen before Adomain decomposition method have been successfully employed to obtain the approximated solutions of systems of the multi-dimensional linear Fredholm integral equations of the second kind. More
accurate results can be obtained by increasing the number of iteration. On the other hand, finding the approximated solutions of systems of the multi-dimensional nonlinear Fredholm integral equations of the second kind by using Adomian decomposition method is a good subject for further research.

5-References