ON $g$, $g^*$, $g^{**}$-COMPACT FUNCTIONS

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Abstract
In this paper we introduce and study $g$, $g^*$, $g^{**}$- compact functions and we study the relation of compact functions with this types of the functions. Finally, we study further theorems and properties on $g$, $g^*$, $g^{**}$- compact functions.

Keywords: $g$- Compact set, Generalized closed set, Compact function.

1-Introduction
The objective of present paper is to introduce certain classes of sets namely $g$-compact sets and certain types of $g$, $g^*$, $g^{**}$-compact functions. Various properties of such functions have been discussed.

A space $X$ means a topological spaces $(X, \tau)$ on which no separation axioms are assumed, unless explicitly stated. The interior and the closure of any subset $A$ of $X$ will be denoted by $\text{Int}(A)$ and $\text{cl}(A)$ respectively. A function $f : X \to Y$ is said to be a compact if $f^{-1}(K)$ is a compact subset of $X$, whenever $K$ is a compact subset of $Y$ [1]. A subset $K$ of a space $X$ is called generalized compact, (briefly $g$-compact) if for every $g$-open cover of $K$ has a finite subcover. Every $g$-compact set is compact, but the converse is not true in general as in the following example illustrate.

Example (2.2): Let $R$ be the real line , $N$ be the subset of $R$ and $\zeta = \{ U \subseteq R \mid U = R \text{ or } U \cap N = \emptyset \}$. It is clear that $(R, \zeta)$ is a topological space. Put $U_i = N^c \cup \{ i \} = \{ R - N \} \cup \{ i \}$, $i = 1, 2, \ldots$. $U_i$ is not open subset of $R$, where $i \in N$. Since $U_i \cap N = \{ i \}$, $i = 1, 2, \ldots$.

Now to show that $U_i$ is $g$-open subset of $R$. Since the only open set which is cover $N$ is $U=R$ and so $\emptyset \subset U^0$ this implies to $U_i$ is $g$-open for each $i = 1, 2, \ldots$.

Hence the family $\{ U_i \}_{i=1}^{\infty}$ forms a $g$-open cover to $R$; that is, $\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} (R - N) \cup \{ i \} = R$, but this cover can not reducible into finite subcover. Therefore $R$ is not $g$ - compact.

To show that $R$ is compact. Since the only open set which is cover $N$ is $U=R$ and so every open cover to $R$ must be contains $U=R$. This means every open cover to $R$, we can choose finite subfamily $\{ R \}$ cover to $R$. Therefore $R$ is compact.

Now we introduce the following definitions:

2-Certain Types of Compact Functions:

Definition (2.1), [4]:
A subset $K$ of a space $X$ is said to be generalized compact, (briefly $g$-compact) if for every $g$-open cover of $K$ has a finite subcover.

Every $g$-compact set is compact, but the converse is not true in general as in the following example illustrate.

Example (2.2): Let $R$ be the real line , $N$ be the subset of $R$ and $\zeta = \{ U \subseteq R \mid U = R \text{ or } U \cap N = \emptyset \}$. It is clear that $(R, \zeta)$ is a topological space. Put $U_i = N^c \cup \{ i \} = \{ R - N \} \cup \{ i \}$, $i = 1, 2, \ldots$. $U_i$ is not open subset of $R$, where $i \in N$. Since $U_i \cap N = \{ i \}$, $i = 1, 2, \ldots$.

Now to show that $U_i$ is $g$-open subset of $R$. Since the only open set which is cover $N$ is $U=R$ and so $\emptyset \subset U^0$ this implies to $U_i$ is $g$-open for each $i = 1, 2, \ldots$.

Hence the family $\{ U_i \}_{i=1}^{\infty}$ forms a $g$-open cover to $R$; that is, $\bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} (R - N) \cup \{ i \} = R$, but this cover can not reducible into finite subcover. Therefore $R$ is not $g$ - compact.

To show that $R$ is compact. Since the only open set which is cover $N$ is $U=R$ and so every open cover to $R$ must be contains $U=R$. This means every open cover to $R$, we can choose finite subfamily $\{ R \}$ cover to $R$. Therefore $R$ is compact.

Now we introduce the following definitions:
Definition (2.3):
A function \( f : X \to Y \) is said to be g-compact if \( f^{-1}(K) \) is g-compact subset in \( X \), whenever \( K \) is a compact subset in \( Y \).

Every g-compact function is compact, but the converse is not true in general as in the following example illustrate.

Example (2.4):
The identity function \( I_R : (R, \zeta) \to (R, \zeta) \) is compact but not g-compact function.

The compact sets in \( R \) is the set that contain \( N \). Since any covering to this sets must be contain \( R \), then we can choose \( \{ R \} \) to cover this sets. Hence \( I_R \) is compact function, since for any compact subset \( K \) of \( R \), \( I_R^{-1}(K) = K \).

But \( I_R \) is not g-compact function, since \( R \) is compact but \( I_R^{-1}(R) = R \) is not g-compact (example (2.2))

Definition (2.5):
A function \( f : X \to Y \) is said to be g*-compact if \( f^{-1}(K) \) is g-compact subset in \( X \), whenever \( K \) is g-compact subset in \( Y \).

Every compact function is g*-compact but the converse is not true in general as in the following example illustrate.

Example (2.6):
Let \( I : (R, \tau_d) \to (R, \zeta) \), where \( I \) be the identity function. \( I \) is g*-compact function since the g-compact sets in \( \zeta \) are the only finite sets which their inverse images are compact sets in \( \tau_d \). But \( I \) is not compact function since \( (R, \zeta) \) is compact (example(2.2)) hence \( I^{-1}(R) = R \), but \( (R, \tau_d) \) is not compact, which implies \( I \) is not compact function.

Definition (2.7):
A function \( f : X \to Y \) is said to be g**-compact if \( f^{-1}(K) \) is g- compact subset in \( X \), whenever \( K \) is g- compact subset in \( Y \).

Every g-compact function is g**-compact, which they g*-compact, but the converses is not true in general.

Definition (2.8), [5]:
A function \( f : X \to Y \) is said to be point inversely compact (briefly p.i.compact) if \( f^{-1}(y) \) is compact subset in \( X \), for every \( y \in Y \).

We introduce the following concept.

Definition (2.9):
A function \( f : X \to Y \) is said to be point inversely generalized compact (briefly p.i.g-compact) if \( f^{-1}(y) \) is g-compact subset in \( X \), for every \( y \in Y \).

Remarks (2.10):
Every compact function is p.i.compact but the converse is not true in general, every p.i.g-compact function is p.i. compact but the converse is not true in general and every g-compact function is p.i. compact and p.i.g- compact but the converse is not true in general.

Example (2.11):
Let \( I_R : (R, \tau_d) \to (R, \tau_u) \), where \( I_R \) is the identity function.

To show that \( I_R \) is p.i.compact and p.i.g-compact.

Since \( I_R^{-1}(x) = \{ x \} \), for all \( x \in R \) and every finite set in every topological space is compact and g-compact.

But it is clear \([0,1]\) is compact in usual topology, while \( I_R^{-1}([0,1]) = [0,1] \) is not compact in the discrete topology and hence it is not g-compact.

Therefore \( I_R \) is neither compact nor g-compact function.

Theorem (2.12):
Let \( f : X \to Y \) be g-closed function and p.i.compact, then \( f \) is g*-compact.

Proof:
Let \( K \) be g-closed subset of \( Y \) and \( \{ U_\alpha \}_{\alpha \in \Omega} \) be an open cover of \( f^{-1}(K) \), where \( \Omega \) is the index set. \( T \) be a family of all finite subset of \( \Omega \) set. \( \{ U_\alpha \}_{\alpha \in \Omega} \) be an open cover of \( f^{-1}(K) \), where \( \Omega \) is the index set. \( T \) be a family of all finite subset of \( \Omega \) set. \( U_i = \bigcup_{\alpha \in \Omega} U_\alpha \), where \( i \in T \). Since \( f \) is p.i. compact, for every \( k \in K \), implies \( f^{-1}([k]) \) is compact set and contained in \( U_i \), where \( i \in T \). Hence \( K \subseteq Y - f(X - U_i) \).

So \( K \subseteq \bigcup_{i \in T} (Y - f(X - U_i)) \).
But \( Y - f(X - U_t) \) is g-open, then there exist
\[ t_1, t_2, \ldots, t_n \in T \text{ s.t } K \subseteq \bigcup_{i=1}^{n} (Y - f(X - U_{t_i})) \]
(since \( K \) is g-compact) and so
\[ f^{-1}(K) \subseteq \bigcup_{i=1}^{n} f^{-1}(Y - f(X - U_{t_i})) \]
\[ = \bigcup_{i=1}^{n} (X - f^{-1}(f(X - U_{t_i}))), \]
since \( f^{-1}(Y) = X \)
\[ \subseteq \bigcup_{i=1}^{n} (X - (X - U_{t_i})) \]
\[ = \bigcup_{i=1}^{n} U_{t_i} = \bigcup_{\alpha \in \Omega_0} U_{\alpha}, \]
where \( \Omega_0 = t_1 \cup t_2 \cup \ldots \cup t_n \).
So \( f^{-1}(K) \) is compact in \( X \). Therefore \( f \) is g'-compact function.

**Theorem (2.13):**
Every g'-closed subset of g-compact space is g'-compact.

**Proof:**
Let \( K \) be a g'-closed subset of g-compact space \( X \). Let \( \{G_{\alpha}\}_{\alpha \in \Omega} \) be a g-open cover of \( K \), that is; \( K \subseteq \bigcup_{\alpha \in \Omega} G_{\alpha} \). But \( X - K \) is g-open so,
\[ X = (X - K) \cup \bigcup_{\alpha \in \Omega} G_{\alpha}. \]
Since \( X \) is g-compact then \( X = (X - K) \cup \bigcup_{\alpha \in \Omega} G_{\alpha} \), and
\[ K \subseteq \bigcup_{\alpha \in \Omega} G_{\alpha}. \]
Therefore \( K \) is g'-compact.

**Remark (2.14):**
Every finite set is g'-compact.

**Theorem (2.15):**
A continuous function from g-compact space into \( T_2 \) space is g'-closed.

**Proof:**
Let \( F \) be a closed subset of \( X \) which is g-compact, so \( X \) is compact. Then \( F \) is compact in \( X \).
Since \( f \) is continuous function, then \( f(F) \) is compact in \( Y \) which is \( T_2 \)-space, then \( f(F) \) is closed, so it is g-closed in \( Y \).
Therefore \( f \) is g'-closed function.

**Theorem (2.16):**
Let \( f : X \rightarrow Y \) be g-compact function and \( A \) is a closed subset of \( X \), then
\[ f|A : A \rightarrow Y \] is also g'-compact.

**Proof:**
Let \( K \) be a compact subset of \( Y \), then \( f^{-1}(K) \) is g-compact subset in \( X \), but \( A \) is closed in \( X \), so \( A_1 f^{-1}(K) \) is closed in \( f^{-1}(K) \). Hence it is g-closed in \( f^{-1}(K) \).

Therefore by theorem (2.13) \( A_1 f^{-1}(K) \) is g-compact. But \( f^{-1}|_A(K) = A_1 f^{-1}(K) \),
then \( f|A \) is g'-compact.

**Definition (2.17),[5]:**
Let \( f : X \rightarrow Y \) be a function and \( T \) be a subset of \( Y \), we define \( f_T : f^{-1}(T) \rightarrow T \) by:
\[ f_T(x) = f(x) \text{ for all } x \in f^{-1}(T). \]

**Theorem (2.18):**
If \( f : X \rightarrow Y \) is g-compact continuous function and \( T \) is closed subset of \( Y \), then
\[ f_T : f^{-1}(T) \rightarrow T \] is also g'-compact.

**Proof:**
Let \( G \) be a compact subset of \( T \), then it is compact in \( Y \) and so \( f^{-1}(G) \) is g-compact in \( X \).
Since \( f^{-1}(T) \) is closed in \( X \), then \( f^{-1}(T) \cup f^{-1}(G) \) is closed in \( f^{-1}(G) \), which implies it is g-closed, then by theorem (2.13) \( f^{-1}(T) \cup f^{-1}(G) \) is g-compact. But \( f_T^{-1}(G) = f^{-1}(T) \cup f^{-1}(G) \); that is, \( f_T \) is g-compact.

**Theorem (2.19):**
Let \( f : X \rightarrow Y \) be bijective function. Then the g''-continuous image of g-compact set is g-compact.

**Proof:**
Let \( G \) be a g-compact subset of \( X \), and \( \{V_{\alpha}\}_{\alpha \in \Omega} \) be a g-open cover of \( f(K) \); that is,
\[ f(K) = \bigcup_{\alpha \in \Omega} V_{\alpha}. \]
So \( K = f^{-1}(f(K)) = f^{-1} \bigcup_{\alpha \in \Omega} V_{\alpha} = \bigcup_{\alpha \in \Omega} f^{-1}(V_{\alpha}). \)
Since \( f \) is \( g^{**} \)-continuous, then \( f^{-1}(V_a) \) is g-open set for all \( \alpha \in \Omega \), so \( \{f^{-1}(V_a)\}_{\alpha \in \Omega} \) is a g-open cover of \( K \), which is g-compact, so

\[
K = \bigcup_{i=1}^{n} f^{-1}(V_{a_i}) ,
\]

then

\[
f(K) = f \left( \bigcup_{i=1}^{n} f^{-1}(V_{a_i}) \right) = \bigcup_{i=1}^{n} f f^{-1}(V_{a_i}) = \bigcup_{i=1}^{n} V_{a_i} .
\]

Therefore \( f(K) \) is g-compact.

**Theorem (2.20):**

Let \( f_1 : X \rightarrow Y \) and \( f_2 : Y \rightarrow Z \) be functions then

1. If \( f_1 \) is g-compact and \( f_2 \) is \( g^{*} \)-continuous, then \( f_2 \circ f_1 \) is \( g^{**} \)-compact.
2. If \( f_1 \) is \( g^{*} \)-compact and \( f_2 \) is g-compact, then \( f_2 \circ f_1 \) is g-compact.
3. If \( f_1 \) is \( g^{**} \)-compact and \( f_2 \) is \( g^{*} \)-compact, then \( f_2 \circ f_1 \) is g-compact.
4. If \( f_1 \) and \( f_2 \) are \( g^{**} \)-compact, then \( f_2 \circ f_1 \) is g-compact.
5. If \( f_1 \) is g-compact and \( f_2 \) is g-compact, then \( f_2 \circ f_1 \) is g-compact.
6. If \( f_1 \) is g-compact and \( f_2 \) is \( g^{*} \)-compact, then \( f_2 \circ f_1 \) is g-compact.
7. If \( f_1 \) is \( g^{**} \)-compact and \( f_2 \) is \( g^{*} \)-compact, then \( f_2 \circ f_1 \) is g-compact.

**Proofs:**

1. Let \( K \) be a g-compact subset of \( Z \), then \( f_2^{-1}(K) \) is compact subset of \( Y \), so \( f_1^{-1}(f_2^{-1}(K)) \) is g-compact in \( X \). But

\[
f_1^{-1}(f_2^{-1}(K)) = f_1^{-1} \circ f_2^{-1}(K) = (f_2 \circ f_1)^{-1}(K) .
\]

Therefore \( f_2 \circ f_1 \) is \( g^{**} \)-compact.

In the same way we can prove the others.

**Theorem (2.21):**

Let \( f_1 : X \rightarrow Y \) and \( f_2 : Y \rightarrow Z \) be functions then

1. If \( g \circ f \) is \( g^{*} \)-compact and \( f \) is surjective \( g^{*} \)-continuous function, then \( g \) is g-compact.
2. If \( g \circ f \) is \( g^{*} \)-compact and \( g \) is one to one \( g^{*} \)-continuous function, then \( f \) is g-compact.
3. If \( g \circ f \) is \( g^{**} \)-compact and \( g \) is one to one \( g^{**} \)-continuous function, then \( f \) is \( g^{*} \)-compact.

**Proofs:**

1. Let \( M \) be a compact subset of \( Z \), then \( (g \circ f)^{-1}(M) \) is g-compact in \( X \), so

\[
f(g \circ f)^{-1}(M) \text{ is g-compact in } Y .
\]

But

\[
f(g \circ f)^{-1}(M) = f(f^{-1}(g^{-1})(M)) = f(f^{-1}(g^{-1}(M))) = g^{-1}(M) .
\]

Therefore \( g \) is g-compact.

2. Let \( M \) be g-compact subset of \( Y \), then by theorem (2.19) we obtain \( g(M) \) is g-compact in \( Z \).

Hence \( (g \circ f)^{-1}(g(M)) \) is compact in \( X \). But

\[
(g \circ f)^{-1}(g(M)) = (f^{-1} \circ g^{-1})(g(M)) = f^{-1}(g^{-1}(M)) = f^{-1}(M) .
\]

Therefore \( f \) is \( g^{*} \)-compact.

In the same way we can prove (3).

**Theorem (2.22):**

If \( A \) is a closed subset of a space \( X \), then the inclusion function of \( A \) is \( g^{**} \)-compact.

**Proof:**

Let \( i : A \rightarrow X \) be an inclusion function and let \( K \) be a g-compact subset of \( X \). Since \( i^{-1}(K) = A \), \( K \) is closed in \( K \), so it is g-compact. Hence \( A \) is g-compact that is, \( i^{-1}(K) \) is g-compact.

**Theorem (2.23):**

Let \( f : X \rightarrow Y \) be a homeomorphism then if \( K \) is a g-compact subset of \( X \) so \( f(K) \) is also g-compact.

**Proof:**

Let \( \{V_a\}_{\alpha \in \Lambda} \) be a g-open cover of \( f(K) \) and let \( F \) be closed subset of \( f^{-1}(V_a) \), for some \( \alpha_0 \in \Lambda \). Let \( F \subset V_{a_0} \) which is g-open set. Then

\[
f(F) \subset V_{a_0}^0 ,
\]

so \( f^{-1}(V_{a_0}) \) is a g-open set, but \( K = \bigcup_{\alpha \in \Lambda} f^{-1}(V_a) \)

so \( K = \bigcup_{i=1}^{n} f^{-1}(V_{a_i}) \), implies

\[
f(K) = f \left( \bigcup_{i=1}^{n} f^{-1}(V_{a_i}) \right) = \bigcup_{i=1}^{n} f f^{-1}(V_{a_i}) = \bigcup_{i=1}^{n} V_{a_i} .
\]

Therefore \( f(K) \) is g-compact set.

**Theorem (2.24):**

Let \( f_1 : X_1 \rightarrow Y_1 \) and \( f_2 : X_2 \rightarrow Y_2 \) be functions, then if
$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is $g^*$-compact then, $f_1$ and $f_2$ are also $g^*$-compact.

**Proof:**
To prove $f_1$ is $g^*$-compact.
Let $K$ be $g^*$- compact subset of $Y_1$ , also \{y_2\} is a $g^*$-compact subset of $Y_2$, where $y_2 \in Y_2$.
But $K \times \{y_2\} \equiv K$ (by theorem 2.23), which is $g^*$-compact.

$$(f_1 \times f_2)^{-1} (K \times \{y_2\})$$ is compact subset of $X_1 \times X_2$, but

$$(f_1 \times f_2)^{-1} (K \times \{y_2\}) = (f_1^{-1} \times f_2^{-1}) (K \times \{y_2\})$$

$$= (f_1^{-1} (K)) \times f_2^{-1} (\{y_2\}).$$
Hence $f_1^{-1} (K)$ is compact subset of $X_1$, therefore $f_1$ is $g^*$-compact.
In the same way, we can prove $f_2$ is $g^*$-compact.

**References**